Research Announcement

THE TANGENT STARS OF A SET AND EXTENSIONS OF SMOOTH FUNCTIONS

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Let X be a closed subset of a C^k manifold M. We establish a necessary and sufficient condition for a continuous function $f : X \to \mathbf{R}$ to possess a C^k extension to M. This solves a problem left open by Whitney [2].

This condition is expressed in terms of the k–th order tangent star, $\mathrm{Tan}^k(M, X)$, of the pair (M, X), which is defined in the following way. Let $C^k(M)$ denote the Frechet algebra of all C^k real valued functions on M, $C^k(M)^*$ its dual, and $I_k(X)$ the ideal of functions in $C^k(M)$ that vanish on X; for $a \in M$, we write $I_k(\{a\})$ as $I_k(a)$. The space of k–th order tangents to (M, X) at a is the set

$$\mathrm{Tan}^k(M, X, a) = C^k(M)^* \cap I(X)^{\perp} \cap (I(a)^{k+1})^{\perp}.$$

This is a topological vector space of finite dimension over $\mathbf{R}$, and a module over a finite dimensional algebra. The k–th order tangent star of the pair (M, X) is given by

$$\mathrm{Tan}^k(M, X) = \bigcup_{a \in M} \mathrm{Tan}^k(M, X, a).$$

If Y is a closed subset of a C^k manifold N, and $b \in Y$, then a C^k–map $F : M \to N$ such that $F(X) \subseteq Y$ and $F(a) = b$ induces a continuous linear map from $\mathrm{Tan}^k(M, X, a)$ to $\mathrm{Tan}^k(N, Y, b)$, which is also a module homomorphism, and a

* Supported by EOLAS grant SC/90/070

function $F_* : \mathrm{Tan}^k(M, X) \to \mathrm{Tan}^k(N, Y)$ which is a morphism of stars. The associations $(M, X) \to \mathrm{Tan}^k(M, X)$ and $F \mapsto F_*$ yield a covariant functor from the category of pairs to the category of stars.

In particular, let G denote the graph of f and $\tilde{a}$ the point $(a, f(a))$ of G. The projection $\pi : M \times \mathbf{R} \to M$, defined by $\pi(x, y) = x$, induces a map from $\mathrm{Tan}^k(M \times \mathbf{R}, G, \tilde{a})$ to $\mathrm{Tan}^k(M, X, a)$ for each $a \in X$, and a morphism

$$\pi_* : \mathrm{Tan}^k(M \times \mathbf{R}, G) \to \mathrm{Tan}^k(M, X).$$

Theorem. *The function f has a C^k extension to M if and only if the map π_* is a bijection.*

The stars $\mathrm{Tan}^k(M, X)$ may be explicitly calculated. First order tangent stars are related to the classical tangents of Denjoy, Whitney and Zariski, and higher order tangent stars are related to the higher order tangent bundles of Pohl and to the paratangent spaces of Glaeser. The details will appear in [1].

References

[1] A. G. O'Farrell and R. O. Watson, *The tangent stars of a set, and extensions of smooth functions,* J. Reine Angew. Math. (1992) (To appear).

[2] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets,* Trans. Amer. Math. Soc. **36** (1934), 63–89.

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