IRISH MATHEMATICAL SOCIETY

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, in March and December. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files if possible, in order to ensure speedier processing.

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8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.
9. Please send the completed application form with one year's subscription fee to

The Treasurer, I.M.S.
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Minutes of Meetings of the Irish Mathematical Society

Ordinary Meeting

April 12 1990

A general meeting of the Irish Mathematical Society was held at 12.15 p.m. on Thursday 12 April 1990. There were eight present. The President, F. Gaines, was the in the Chair. Apologies were received from R. Timoney.

1. **Minutes**
   The Minutes of the meeting of 22 December 1989 were read, approved and signed.
2. **Bulletin**
   The Bulletin is now at the printers and will be ready in about 3 weeks.
   It was noted that the Celtic Studies section of the DIAS is getting a phototypesetter, which might be of use in producing the Bulletin. It is the policy of the Society to produce the Bulletin within Ireland if possible.
3. **Euromath**
   A report from Richard Timoney was circulated.
   Tim Murphy and Tony O'Farrell both expressed reservations about the structured document editor, GRIF, which has been adopted by Euromath.
4. **September meeting**
   The mathematical meeting of the IMS is being held at DCU on (probably) Wednesday 5 and Thursday 6 September 1990. Alistair Wood is organising it. On Monday and Tuesday of that week, there will be a meeting of SEFI, and on the Friday there will be a meeting about cooperation between 3rd Level Education and Industry. Alistair Wood said it might be necessary to move the Education/Industry meeting to the

Wednesday, and to move the IMS meeting to the Thursday and Friday. He will know for definite when he gets replies from the speakers.
It was agreed to hold the September 1991 IMS meeting in UCG.

5. **Correspondence**
(a) EOLAS will provide travelling expenses for one speaker at the September 1990 IMS meeting, and for S. Sternberg at the TCD meeting on 24 May 1990. Tony O'Farrell congratulated the organisers of the TCD meeting on obtaining such a distinguished speaker.
(b) A letter from Richard Timoney was read, giving details of the TCD Mathematics survey.

6. **A.O.B.**
(a) Fergus Gaines mentioned that a recent UCD graduate is studying for a Ph.D. at TCD, whilst being employed by Hitachi at their TCD laboratory.
(b) Alistair Wood and David Simms have arranged for the The Royal Society Pop Maths Roadshow to come to the RDS from 23 to 29 October 1990. The RDS is supplying space free of charge. The admission to the Roadshow is free.
Volunteers are needed to give popular lectures. (Lectures for schoolchildren in the afternoons, and adult lectures in the evenings).
There is also a need for people (research students) to look after the exhibits. This work would be paid.
(c) Tony O'Farrell suggested holding an annual walk along the Canal from Dunsink on the 16th October to commemorate the discovery of the Quaternions.

Ordinary Meeting

21 December 1990

The Irish Mathematical Society held an Ordinary Meeting at 12:45 on Friday, 21 December 1990, in the DIAS. Fourteen members were present. The President, F. Gaines, was in the chair. The Secretary sent his apologies, and A. O'Farrell took minutes.

1. **Minutes**
The minutes of the meetings of 12 April 1990 and 6 & 7 September 1990 were approved and signed.

2. **Matters arising**
R. Timoney promised a report on the TCD survey of Mathematics Graduates for the Society's Bulletin.
Satisfaction was expressed with the increased level of the Society's activities during the year, especially in relation to the visit of S. Sternberg and the September Meeting at DCU.
The proposed visit of the Royal Society Pop Math Roadshow had to be cancelled for lack of sponsors.
The first Quaternion anniversary walk went well. Professor Wayman and the Dunsink staff arranged a sherry reception and exhibit at the start, and the Academy staff arranged a glass of wine at the terminus. The weather was exceptionally fine. Eight mathematicians are known to have made the walk, and it is expected to grow as the years go by. The sesquicentenary of the invention comes in 1993, and Professor Wayman promises champagne.

3. **Bulletin**
R. Ryan, who is stepping down as editor, was thanked for his valuable serviced. J. Ward is to take over to issue number 26 (Easter 1991). In the meantime F. Gaines plans to look after number 24 (Easter 1990), which is not completed, and number 25 (Christmas 1990). It was agreed that
(a) the editorial and secretarial/technical functions should be separated,

(b) submissions in TEX should be encouraged now and required eventually, and
(c) there should be a new section for Research Announcements in the Bulletin.
F. Gaines expressed the hope that number 24 might be available by February. He called for contributions to number 25 and subsequent issues. R. Timoney has offered to assist with number 24.

4. **The European Mathematical Society**
B. Goldsmith, who has been appointed our official representative to EMS, reported. The inaugural meeting of the EMS was held in Poland in October. The IMS is a founding member, in Class 1. Members will be encouraged to become individual members of the EMS. There will be an article in the Bulletin with further information.

5. **Euromath**
R. Timoney, Chairman of the Irish Euromath National Coordinating Committee, reported. Phase 11, the final phase is now starting. He attended an advisory Board meeting in Luminy, in July 1990, and held an NCC meeting on December 10th. It would be a mistake to underestimate what has been done. The Euromath people have 1) identified the UNIX workstation and X-windows as reasonable standards to aim for, 2) used collective bargaining power to get discounts on DEC equipment and the FIZ database (-Zentralblatt), 3) almost completed the Directory of Mathematicians. He would however question whether Euromath will succeed in making a seamless integrated system to meet all the information-technology needs of mathematicians.
A. O'Farrell, the Society's representative at the General Assembly of the European Mathematical Trust, expressed 1) the hope that some good might come of Euromath, now that R. Timoney has been invited to join the Executive Committee, and 2) the opinion that conflicts of interest in the organisation were still an unholy mess, that the research funds allocated to Euromath would be better spent on real

mathematical research, that the structural flaws in Euromath could not be cured, so that the only reasonable and honourable way for the Society was to pull out. Nevertheless, he recommended, and it was agreed, that the Society should take no action on this for a further year, to see whether things improve.

6. **Elections**
The following were proposed, seconded and elected unopposed:

	Proposed by	Seconded by
President		
R. Timoney,	S. Tobin,	S. Dineen.
Vice-President		
B. Goldsmith	T. J. Laffey	M. Newell.
Members of the committee:		
F.Holland	S. Dineen	A. O'Farrell.
B. McCann	P. Barry	M. Ó Searcóid.
F. Gaines	A. O'Farrell	M. Newell.
M. Ó Searcóid	F. Gaines	R. Timoney.

G. Enright, A. O'Farrell, D. Simms and R. Watson continue on the Committee, and the Bulletin editor R. Ward will be coopted as usual. The Treasurer, D. Tipple, and the Secretary G. Ellis, also continue in office for a further year.
F. Gaines was warmly thanked for his invaluable services as President.

7. **Treasurer's Report**
D. Tipple presented his report. It was adopted on the proposal of R. Timoney, seconded by A. O'Farrell. It was agreed to appoint A. Pierce as auditor, and to authorise the Treasurer to fix official membership rates in other currencies, as he sees fit.

8. **September meeting**
R. Ryan is organising the September 1991 meeting, which is to be held in UCG on 5th and 6th September. J.-L. Loday has agreed to speak. Suggestions for other invited speakers, and offers of contributed talks are welcome. It was agreed that if necessary the Society would underwrite the full cost of the meeting form its own funds.
Members were reminded that invitations to host the September 1992 meeting should come before the Easter 1991 meeting. There was an indication that Waterford RTC might wish to host the meeting.
Whether or not the Society will continue with the fourth meeting of the past two years is conditional on the continuation of support form EOLAS.
9. **Other Business**
(a) The terms of the new IMS-IMTA reciprocity agreement were approved. These provide that IMS members who wish to join the IMTA may do so by contacting the IMTA and paying 50% of the usual subscription to that association. Details will appear in the Bulletin.
(a) The question of whether a uniform national points system for university entry should include extra points for Leaving Certificate Honours Mathematics, was discussed at some length. S. Tobin expressed some reservations about the propriety of giving extra points to Mathematics for faculties like Arts, and suggested that it might be better to focus on reducing the overloaded syllabus. T. Laffey quoted the remarkable statistical study done by A. Moran, which demonstrated that Mathematics deserves extra weight, in view of its exceptional value as a predictor of success, even in Arts faculties. He also stressed the likely effect of removing the Honours Mathematics bonus on the numbers taking the subject. He was especially concerned that girls' schools might stop offering Honours Mathematics, which would be severely damaging to the prospects of women. S. Dineen pointed out that the universities don't and can't control the overloading of the syllabus, and do control points. R. Timoney said that the

mathematics course should be heavier than others. M. Newell agreed, and said that Mathematics should count double. He thought it essential that mathematicians pull together on this one. A. O'Farrell stressed the value of Mathematics grades as a predictor of success, and the possible impact on the technological development of the whole country if the numbers taking Honours Mathematics fell substantially. M. Ó Searcóid proposed a PR campaign to promote Mathematics. He supported the view that the Mathematics course should be substantial, quoting the British situation. S. Dineen suggested that the RTC's should also operate a bonus for Honours Mathematics. S. Tobin was impressed by the evidence brought forward in the discussion, and agreed that a common front was appropriate on this one.

Graham Ellis,
University College,
Galway.

Conference Announcements

GROUPS IN GALWAY 1992

This annual conference will be held on Friday and Saturday, 15th and 16th May 1992 in University College, Galway. The speakers will include A. Christofides (UCG), B. Hartley (Manchester), K. Hutchinson (UCD) and H. Smith (Bucknell, Pennsylvania and Cardiff, Wales).

Further details may be obtained from Rex Dark, University College, Galway, e-mail MATDARK@BODKIN.UCG.IE.

NASECODE VIII

The Eighth International Conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits will be held at the City Club, Vienna, Austria on 18–22 May 1992. Further details may be obtained either from the Nasecode Secretariat, 26 Temple Lane, Dublin 2, or (on scientific matters) from Professor J. Miller, Telephone (01) 679-7655, e-mail JMILLER@VAX1.TCD.IE, or (on all other matters) from Paulene McKeever, Telephone (01) 452081.

BAIL VI

The Sixth International Conference on Boundary and Interior Layers — Computational and Asymptotic Methods will be held in Summit City, Colorado, USA on 17–21 August 1992. Further details may be obtained from the Bail Secretariat, 26 Temple Lane, Dublin 2, or from Professor J. Miller or Paulene McKeever as in the previous announcement.

IMS SEPTEMBER 1992 MEETING

The Fifth September Meeting of the Irish Mathematical Society will take place in Waterford Regional Technical College on Thursday and Friday, 3rd and 4th September 1992. The speakers will include D. Armitage (QUB), R. Brown (Bangor, Wales), E. deLeastar (Waterford RTC), P. Fitzpatrick (UCC), D. Ince (Open University), J. Lewis (DIAS), J. McDermott (UCG), M. Stynes (UCC). Further details may be obtained from Brendan McCann or Michael Brennan, Dept of Physical and Quantitative Science, Waterford RTC.

Research Announcement

THE TANGENT STARS OF A SET AND EXTENSIONS OF SMOOTH FUNCTIONS

A.G. O'Farrell* and R.O. Watson

Let X be a closed subset of a C^k manifold M. We establish a necessary and sufficient condition for a continuous function $f : X \to \mathbf{R}$ to possess a C^k extension to M. This solves a problem left open by Whitney [2].

This condition is expressed in terms of the k–th order tangent star, $\mathrm{Tan}^k(M, X)$, of the pair (M, X), which is defined in the following way. Let $C^k(M)$ denote the Frechet algebra of all C^k real valued functions on M, $C^k(M)^*$ its dual, and $I_k(X)$ the ideal of functions in $C^k(M)$ that vanish on X; for $a \in M$, we write $I_k(\{a\})$ as $I_k(a)$. The space of k–th order tangents to (M, X) at a is the set

$$\mathrm{Tan}^k(M, X, a) = C^k(M)^* \cap I(X)^{\perp} \cap (I(a)^{k+1})^{\perp}.$$

This is a topological vector space of finite dimension over $\mathbf{R}$, and a module over a finite dimensional algebra. The k–th order tangent star of the pair (M, X) is given by

$$\mathrm{Tan}^k(M, X) = \bigcup_{a \in M} \mathrm{Tan}^k(M, X, a).$$

If Y is a closed subset of a C^k manifold N, and $b \in Y$, then a C^k–map $F : M \to N$ such that $F(X) \subseteq Y$ and $F(a) = b$ induces a continuous linear map from $\mathrm{Tan}^k(M, X, a)$ to $\mathrm{Tan}^k(N, Y, b)$, which is also a module homomorphism, and a

* Supported by EOLAS grant SC/90/070

function $F_* : \mathrm{Tan}^k(M, X) \to \mathrm{Tan}^k(N, Y)$ which is a morphism of stars. The associations $(M, X) \to \mathrm{Tan}^k(M, X)$ and $F \mapsto F_*$ yield a covariant functor from the category of pairs to the category of stars.

In particular, let G denote the graph of f and $\tilde{a}$ the point $(a, f(a))$ of G. The projection $\pi : M \times \mathbf{R} \to M$, defined by $\pi(x, y) = x$, induces a map from $\mathrm{Tan}^k(M \times \mathbf{R}, G, \tilde{a})$ to $\mathrm{Tan}^k(M, X, a)$ for each $a \in X$, and a morphism

$$\pi_* : \mathrm{Tan}^k(M \times \mathbf{R}, G) \to \mathrm{Tan}^k(M, X).$$

Theorem. *The function f has a C^k extension to M if and only if the map π_* is a bijection.*

The stars $\mathrm{Tan}^k(M, X)$ may be explicitly calculated. First order tangent stars are related to the classical tangents of Denjoy, Whitney and Zariski, and higher order tangent stars are related to the higher order tangent bundles of Pohl and to the paratangent spaces of Glaeser. The details will appear in [1].

References

[1] A. G. O'Farrell and R. O. Watson, *The tangent stars of a set, and extensions of smooth functions,* J. Reine Angew. Math. (1992) (To appear).

[2] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets,* Trans. Amer. Math. Soc. **36** (1934), 63–89.

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Abstract of Doctoral Thesis

SYMMETRIC BANACH MANIFOLDS

Pauline Mellon

Banach manifolds are manifolds modelled locally on open subsets of complex Banach spaces. Symmetric Banach manifolds , or simply symmetric manifolds, are Banach manifolds with a norm on the tangent bundle and which have a high degree of symmetric structure. Namely, for every point of the manifold there is an involutive automorphism of the manifold which acts as a symmetry about that point. This structure is rich enough to allow a Riemann mapping type classification of the symmetric manifolds.

In finite dimensions the symmetric manifolds are exactly the Hermitian symmetric spaces. The Hermitian symmetric spaces were classified by Cartan in the 1930s , using Lie algebraic techniques and later by Koecher, Loos and others using Jordan algebraic techniques. There is a natural duality between the Hermitian symmetric spaces of compact and non-compact type. An analogue of this phenomenon also holds in infinite dimensions, even though the symmetric manifolds are then non-compact, as they are modelled on infinite dimensional Banach spaces.

Kaup gave an algebraic classification of the symmetric manifolds in the general case, by associating to each symmetric manifold a Banach space with an algebraic triple product, called a J^*-triple system or J^*-triple. He proved that the category of all simply-connected symmetric manifolds with base point is equivalent to the category of J^*-triple systems. These J^*-triples include in particular, all C^*-algebras, all JB^*- algebras and all J^*-algebras.

J^*-algebras are algebras of operators between Hilbert spaces which were introduced and studied by Harris. They give us a concrete setting in which to study J^*-triple phenomenon. The

techniques used by Harris are more function theoretic in nature, using spectral theory, the functional calculus *etc.*.

In this thesis we study two topics, both related to symmetric manifolds, one of which comes from the operator theoretic side of the subject and the other from the more abstract Jordan algebraic/Lie algebraic side.

Our aim in the first topic was to generalise results from J^*-algebras to J^*-triple systems, by replacing operator-theoretic techniques with Jordan algebraic techniques, while in the second, we adopted the opposite approach, by using some of the classical examples from operator theory to improve our understanding and to obtain results for the class of dual symmetric manifolds.

Our first topic deals with Schwarz-type inequalities for holomorphic mappings which were obtained by Ando, Fan and Włodarczyk in a series of papers culminating in various Julia-type lemmata and Wolff-type theorems for operator valued holomorphic mappings on J^*-algebras. Using Bergmann operators we obtained similar results for JB^*-triple systems. In realising these results for J^*-algebras in terms of the Jordan rather than the operator theoretic structure we appear to place the results in a more natural setting (even though the transition is not always smooth).

Our second topic is dual symmetric manifolds. These manifolds are non-compact, as they are modelled on infinite dimensional Banach spaces, but should intuitively behave like compact manifolds. To investigate this phenomenon we found it necessary to restrict ourselves to a certain class of JB^*-triple systems including in particular all commutative C^*-algebras of the form $C(X)$, for X a compact Hausdorff space.

If X is a compact Hausdorff space and U a JB^*-triple system then $C(X,U)$ is again a JB^*-triple system with pointwise defined triple product. If M with base point m_0 is the dual symmetric manifold of U then we show that the dual symmetric manifold of the JB^*-triple $C(X,U)$ is given by the universal covering manifold of

$$F_X(M) := \{ f \in C(X,M) : f \text{ is homotopic to the constant mapping } m_0 \text{ in } C(X,M) \}.$$

In particular for the commutative C^*-algebra $C(X)$ the dual symmetric manifold is the universal covering manifold of $F_X(\overline{C})$. We then show that the dual symmetric manifold of $C(X)$ displays the compact-type property of admitting only constant complex-valued holomorphic mappings. We conclude this section of the thesis by examining the concrete example $C(X)$ for X a compact subset of $\mathbb{R}$ and showing that the dual manifold in this case is given by $C(X, \overline{C})$.

The last chapter of the thesis examines the holomorphic curvature of the tangent norm of an arbitrary dual symmetric manifold (the tangent norm is a Finsler metric and holomorphic curvature is therefore different than in the Riemannian sense). We find that the dual manifolds have constant positive holomorphic curvature.

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Abstract of Doctoral Thesis

A NUMERICAL STUDY OF THE NON-LINEAR BAROTROPIC INSTABILITY OF FREE ROSSBY WAVES AND TOPOGRAPHICALLY FORCED PLANETARY WAVES

William M. O'Brien

This thesis was prepared in the NIHE, Limerick (University of Limerick) under the direction of Professor P. F. Hodnett and was submitted for the award of Ph.D. to the University of Limerick, July 1990.

The stability of free and forced planetary waves on a β-plane is investigated by integrating numerically the nonlinear quasi-geostrophic, barotropic vorticity equation on a grid-point model. It is shown that the exponential growth rate predicted by the linear model of Lorenz (1972) is accurate. However it is also shown that instability can occur for wave amplitudes $A < A_c$ in the non-linear case where A_c is calculated using the linear model. When stability occurs for large A the perturbation undergoes exponential growth followed by a bounded oscillating behaviour. For small A the perturbation follows an oscillating pattern of growing and subsiding slowly over a long time period. This appears to confirm the analysis of Deininger (1982) and Deininger and Loesch (1982).

The effect of boundary conditions on stability is investigated by comparing the instability for a Rossby wave on a doubly periodic domain with the instability of exactly the same wave in channel. It is found that there is no significant effect on Rossby wave stability. The effect of changing the y dependence on the stability of a Rossby-Haurvitz in a channel is also investigated.

The nonlinear instability of topographic planetary waves on a doubly periodic *beta*-plane as well as in a channel is examined. In the former case an example of a system going form one equilibrium

state to another is found. Instability is studied as a function of various parameters. For a certain perturbation behaviour similar to that predicted by Deininger (1981) was found.

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DERIVATIONS AND COMPLETELY BOUNDED MAPS ON C*-ALGEBRAS

A Survey

Martin Mathieu

The present paper summarises a series of lectures delivered at the Department of Mathematics of University College Cork in early spring 1990 which were supported by the ERASMUS programme. Aimed at the non-specialist, we intend to provide a general survey of the theory of completely bounded linear operators on C^*-algebras with a closer view of their relations to derivations. Most of the details we have omitted can be found in Paulsen's fine treatise [26], in fact the reader may use this paper as a guide to [26] under the particular aspect of applications to derivations on C^*-algebras. A more comprehensive state-of-the-art overview on completely bounded operators is given in the recent paper by Christensen and Sinclair [7], while Effros' address to the ICM 86 [11] emphasises the connections with cohomology theory of operator algebras.

Since the mid 1970's it emerged that the classes of completely bounded and completely positive operators are among the most important classes of (multi-)linear mappings on C^*-algebras, as they are intimately related to a number of structural properties, and several open questions can be phrased in terms of these operators. Here, we shall mainly concentrate on how the problem of innerness of derivations naturally leads to consider completely bounded maps. On the way we will also add some remarks on the role these operators play in the operator algebraic approach to quantum theory. Occasionally, proofs are outlined in order to illustrate the typical techniques.

1. Prerequisites on C^*-algebras

This section is of a preparatory nature; we will compile several facts from C^*-algebra theory that will be needed in the sequel.

Throughout H, K denote Hilbert spaces over the complex field $\mathbb{C}$ and $L(H,K)$ is the Banach space of all bounded linear operators from H into K endowed with the operator norm, $\|T\| = \sup\{\|T\xi\| \mid \xi \in H, \ \|\xi\| \leq 1\}$. The Banach algebra $L(H) = L(H,H)$ carries a natural involution, $T \mapsto T^*$ where T^* is the adjoint of T, and the algebraic and the metric structures are related by the important identity $\|T^*T\| = \|T\|^2$.

A ***C^*-algebra*** A is a closed *-subalgebra of $L(H)$. Due to the work of Gelfand and Naimark, which was completed by several other mathematicians, there is the following abstract characterisation of C^*-algebras: every Banach algebra with an involution * (i.e. an anti-multiplicative conjugate-linear bijection of order two) satisfying $\|x^*x\| = \|x\|^2$ for all x is (isometrically *-isomorphic to) a C^*-algebra. To have both pictures of a C^*-algebra is advantageous: often C^*-algebras arise without specification of an *a priori* Hilbert space, but to realise immediately a C^*-algebra concretely as operators facilitates many arguments. If a given C^*-algebra A is *non-unital*, i.e. does not contain a multiplicative identity 1, we can embed A as a closed ideal into the unital C^*-algebra $\dot{A} = \{a + \lambda 1_{L(H)} \mid a \in A, \lambda \in \mathbb{C}\}$ where $A \subseteq L(H)$.

Here are some of the basic examples of C^*-algebras:

$M_n = L(\mathbb{C}^n)$, the complex $n \times n$-matrices, provides the simplest non-commutative C^*-algebra (for $n > 1$) and will play an extraordinary role in the following;

$K(H)$, the compact operators on an infinite dimensional Hilbert space H, is a simple non-unital C^*-algebra;

$C(H) = L(H)/K(H)$, the Calkin algebra, closely related to Fredholm operators;

$C_0(X)$, the continuous complex-valued functions on a locally compact Hausdorff space X vanishing at infinity. This is the prototype of a commutative C^*-algebra, as, by the Gelfand-Naimark theorem from 1941, every commutative C^*-algebra A is isometrically *-isomorphic to $C_0(\hat{A})$, $\hat{A}$ being the set of all homomorphisms from A onto $\mathbb{C}$ with the weak* topology.

Of course, there are many other fundamental examples and a number of methods to obtain new C^*-algebras from given ones. The following is the most important for our purposes.

Definition 1.1. Let $A \subseteq L(H)$ be a C^*-algebra. For each $n \in \mathbb{N}$ the set $M_n(A)$ of all $n \times n$-matrices with entries from A is a *-subalgebra of $L(H^n)$ under the canonical operations and thus can be normed with the operator norm. From

$$\max_{1\leq i,j\leq n} \|a_{ij}\| \leq \|(a_{ij})\| \leq \sum_{i,j=1}^{n} \|a_{ij}\|$$

for all $(a_{ij}) \in M_n(A)$ we see that $M_n(A)$ is complete, hence a C^*-algebra on H^n. If we change the faithful representation of A we obtain an isometrically *-isomorphic matrix algebra over A, thus $M_n(A)$ can be considered as an abstract C^*-algebra, too.

Examples. $M_n(K(H)) = K(H^n)$, $M_n(C_0(X)) = C_0(X, M_n)$.

We can also view $M_n(A)$ as a tensor product. If $\{u_{ij} \mid 1 \leq i,j \leq n\}$ denotes the canonical matrix units in M_n, the mapping

$$(a_{ij}) \mapsto \sum_{i,j=1}^{n} u_{ij} \otimes a_{ij}, \qquad M_n(A) \to M_n \otimes A$$

is a *-isomorphism. Let α be any C^*-cross norm on $M_n \otimes A$ and $M_n \otimes_\alpha A$ be its completion. Since every *-isomorphism between C^*-algebras is an isometry it follows that $M_n(A)$ and $M_n \otimes A$ are isometrically *-isomorphic, in particular, $M_n \otimes A = M_n \otimes_\alpha A$. (This argument shows in addition that all C^*-cross norms on $M_n \otimes A$ coincide, i.e. M_n is *nuclear*, see Section 4.)

The process of iterating matrix algebras is simplified by the *canonical shuffle*. If $n, m \in \mathbb{N}$, then, as a consequence of associativity and commutativity of tensor products, we have that

$$\begin{aligned} M_m(M_n(A)) &= M_m \otimes (M_n \otimes A) \\ &= M_n \otimes (M_m \otimes A) \\ &= M_n(M_m(A)) \end{aligned}$$

which amounts to a permutation of the entries (cf. [26]).

Let A be a C^*-algebra. Decomposing $x \in A$ into its real and imaginary parts shows that A is the topological direct sum of A_{sa} and iA_{sa} where $A_{sa} = \{x \in A \mid x = x^*\}$ is the *real* Banach space of all self-adjoint elements in A. The latter becomes an ordered Banach space by putting

$$x \leq y \quad \text{if } y - x \in A_+ \qquad (x, y \in A_{sa})$$

where $A_+ = \{x \in A_{sa} \mid \text{all spectral values of } x \text{ are non-negative}\}$ is the proper closed generating cone of positive elements in A. By the Fukamiya-Kaplansky-Kelley-Vaught theorem, there is the following important intimate interrelation between the order and the algebraic structure: $A_+ = \{x^*x \mid x \in A\}$. If A is unital, the identity 1 also serves as an order unit and thus A_{sa} will be an *order unit space*. As a consequence, the unit ball A_1 of A can be described as $A_1 = \{x \in A \mid xx^* \leq 1\}$.

This last observation can be used to derive the following criterion for positivity of certain 2×2-matrices which will turn out to be crucial in the sequel.

Lemma 1.2. *Let a be an element in a unital C^*-algebra A. Then $\|a\| \leq 1$ if and only if $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \geq 0$.*

Proof. If $\|a\| \leq 1$ then $1 - aa^* \geq 0$. Take $x \in A_{sa}$ such that

$x^2 = 1 - aa^*$. Then

$$\begin{aligned} 0 &\leq \begin{pmatrix} x & 0 \\ a^* & 1 \end{pmatrix}^* \begin{pmatrix} x & 0 \\ a^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} x & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ a^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} x^2 + aa^* & a \\ a^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \end{aligned}$$

proving the "only if"-part.

Conversely, if $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} = x^2$ is positive and $A \subseteq L(H)$, then, for all $\xi, \eta \in H$, we have

$$\begin{aligned} &|(a\xi \mid \eta)|^2 \\ &= \left| \left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} \middle| \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) \right|^2 = \left| \left(x \begin{pmatrix} 0 \\ \xi \end{pmatrix} \middle| x \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) \right|^2 \\ &\leq \left(x \begin{pmatrix} 0 \\ \xi \end{pmatrix} \middle| x \begin{pmatrix} 0 \\ \xi \end{pmatrix} \right) \left(x \begin{pmatrix} \eta \\ 0 \end{pmatrix} \middle| x \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} \middle| \begin{pmatrix} 0 \\ \xi \end{pmatrix} \right) \left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix} \middle| \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right) \\ &= \|\xi\|^2 \|\eta\|^2 \end{aligned}$$

which implies that $\|a\| \leq 1$. □

Finally we introduce the order preserving mappings.

Definition 1.3. Let A and B be C^*-algebras. A linear map $\phi: A \to B$ is called *positive* if $\phi(A_+) \subseteq B_+$, *n-positive* if $\mathrm{id} \otimes \phi: M_n \otimes A \to M_n \otimes B$ is positive, and *completely positive* if ϕ is n-positive for all $n \in \mathbb{N}$. The convex cone of all completely positive maps from A into B will be denoted by $CP(A, B)$.

Under the identification $M_n \otimes A \to M_n(A)$, the mapping $\mathrm{id} \otimes \phi$ becomes

$$(\mathrm{id} \otimes \phi)\big(\sum_{i,j} u_{ij} \otimes a_{ij}\big) = \sum_{i,j} u_{ij} \otimes \phi(a_{ij}) \mapsto \big(\phi(a_{ij})\big)_{1 \le i,j \le n}.$$

The mapping $(a_{ij}) \mapsto (\phi(a_{ij}))$, $M_n(A) \to M_n(B)$ is denoted by ϕ_n.

Note that, by a simple application of the uniform boundedness principle, every positive linear map is automatically bounded. For some more grounding on completely positive maps and C^*-algebras we refer to [35].

2. Derivations and homomorphisms of C^*-algebras

In this section some of the basic properties of derivations of C^*-algebras as well as their relations to homomorphisms will be studied, and one of the fundamental problems on derivations is phrased.

A linear map $\delta: A \to A$ where A is a C^*-algebra is called a *derivation of A* if

$$\delta(ab) = a\,(\delta b) + (\delta a)\,b \qquad (a, b \in A).$$

A derivation is automatically continuous, by a result due to Sakai, and only non-commutative C^*-algebras allow non-zero derivations, which is an observation of I. Singer. For each $x \in A$ the derivation

$$\mathrm{ad}\, x : a \mapsto ax - xa$$

is described as an *inner* derivation, and all non-inner derivations are called *outer*. There is a temptation to concentrate on inner derivations since they are given concretely and therefore their properties, e.g. their spectra, can be described more easily. However, if for instance $A = K(H)$ and p is an infinite dimensional projection on H, then the restriction of $\mathrm{ad}\,p$ to $K(H)$ will be outer. It is therefore important to know under which conditions derivations become inner.

Here are two well known and important answers to this question (see [31]: every derivation of a W^*-algebra and every derivation of a simple unital C^*-algebra is inner. Recall that a W^*-algebra is a C^*-algebra which is isometrically isomorphic to the dual of another Banach space which yields an additional weak* topology and thus many nice structural properties. Some necessary and sufficient conditions for derivations to be inner can be found in [27].

Before we proceed to a more general question, let us see where derivations arise in the applications.

In the operator algebraic approach to quantum theory one uses the self-adjoint part A_{sa} of a (suitable) C^*-algebra A for the collection of all observables of a specific physical system, and the *state space* $S(A)$ (the set of all normalised positive linear functionals on A) for the set of all physical states of the system. In the more traditional theory, A was $L(H)$ and the states were identified with unit vectors in H (*vector states*). The dynamics of the system is then given by a continuous one-parameter group of unitary operators $u(t)$, $t \in \mathbb{R}$, on H: while time passes from $t = 0$ to $t = t_0$, the system evolves from the state φ into the state $u(t_0)\,\varphi$. Of special interest is the infinitesimal generator h of $\{u(t) \mid t \in \mathbb{R}\}$ which is a self-adjoint operator (it corresponds to the energy). The identity $u(t) = e^{ith}$ yields the *Schrödinger equation*

$$\frac{d}{dt}\, u(t)\,\varphi = i\,h\,u(t)\,\varphi \qquad (\varphi \in H).$$

In the *Heisenberg picture*, the dynamics is on the observables rather than on the states and thus given by the one-parameter group of *-automorphisms $x \mapsto u(t)^*\,x\,u(t)$, $x \in L(H)$, $t \in \mathbb{R}$, whose generator is the inner derivation $x \mapsto i\,(xh - hx)$.

As it emerged that the algebra $L(H)$ is not always appropriate for the physical model one had to choose more general C^*-algebras. In many cases there is no *a priori* specified Hilbert space, whence the use of the Heisenberg picture and a one-parameter group of *-automorphisms $U(t)$ on A is more convenient; the generator of $\{U(t) \mid t \in \mathbb{R}\}$ will again be a derivation δ. Of course, the *Schrödinger picture* still exists and is equivalent: the adjoint of $U(t)$ will map $S(A)$ onto $S(A)$. In fact, since $U(t)$ is $*$-preserving, δ is a $*$-derivation, i.e. $\delta A_{sa} \subseteq A_{sa}$. (We are somewhat sloppy about the domain of definition of δ which depends on the continuity of $t \mapsto U(t)$. To be precise, we have to assume uniform continuity throughout.) Conversely, given a $*$-derivation δ of A, one defines a one-parameter group of $*$-automorphisms on A by $U(t) = e^{t\delta}$, $t \in \mathbb{R}$.

Now, if $U(t)$ is of the form $U(t)\,x = u(t)^*\,x\,u(t)$ for a group of unitaries $u(t)$ in A, then one has observability of the energy. And if δ is inner, $\delta = \operatorname{ad} ih$, then $U(t)$ will be inner with $u(t) = e^{ith}$, another reason for the interest in inner derivations. For example, Olesen proved in 1974 that every norm continuous group of $*$-automorphisms of A is inner in A^{**} (the enveloping W^*-algebra of A).

While groups of $*$-automorphisms are sufficient to describe reversible evolutions of quantum systems, irreversible evolutions may be described by semigroups of completely positive contractive operators. There are a lot of good reasons for the choice of completely positive contractions, both mathematical and physical ones. First of all, the adjoint operators have to fix the state space; thus they must be positive, hence the original ones also have to be. Secondly, an invertible completely positive contraction whose inverse is a completely positive contraction is a $*$-automorphism; this fails for general positive maps. And even more important is the fact that two interacting systems are usually described by the tensor product of the corresponding C^*-algebras whence the tensor product of the dynamical operators should give the joint dynamics. Complete positivity ensures this, while mere positivity doesn't.

One of the strategies to understand irreversible evolutions (*open quantum systems*) has been to try to 'embed' them into larger reversible systems (*Hamiltonian systems*). This is known as dilation theory (see e.g. [13], [20]).

The generators of norm continuous semigroups of completely positive operators can be described precisely: let $L\colon A \to A$ be a self-adjoint bounded linear operator (i.e. $L\,A_{sa} \subseteq A_{sa}$). Then $T(t) = e^{tL}$, $t \geq 0$, defines a semigroup of completely positive operators if and only if L is *conditionally completely positive* (see [13]). For a large class of *von Neumann algebras* (weakly closed unital $*$-subalgebras of $L(H)$) a more detailed description of conditionally completely positive maps is possible; they can be viewed as perturbations of completely positive maps by generalised inner derivations of a certain type. It is expedient to extend the notion of a derivation as follows.

Definition 2.1. Let A be a C^*-algebra and E be a Banach A-bimodule (i.e. E is a Banach space and an A-bimodule with continuous module multiplications). A linear map $\delta\colon A \to E$ satisfying

$$\delta(ab) = a\,(\delta b) + (\delta a)\,b \qquad (a, b \in A)$$

is called an *E-valued derivation* of A. Every such derivation is a bounded operator as proved by Ringrose [29]. Again, δ is said to be *(E-)inner* if $\delta = \operatorname{ad} x$ for some $x \in E$. A linear map $d\colon A \to E$ is said to be a *generalised inner derivation* if $d(a) = ax + ya$ for some $x, y \in E$ and all $a \in A$. In this case, we write $d = d_{x,y}$. Note that, if A is unital, $d_{x,y}$ is nothing but an additive perturbation of $\operatorname{ad} x$ by left multiplication with $x + y$.

The following situation often arises. A derivation of a C^*-algebra A is not inner in A but will become inner when A is regarded as a C^*-subalgebra of another C^*-algebra B and B is viewed as an A-bimodule. For example, we observed above that a derivation of a simple C^*-algebra A need not be inner in A, but it will be inner in the multiplier algebra $M(A)$ (another result by Sakai [32]). As, by the Gelfand-Naimark theorem, each C^*-algebra A can be considered as a C^*-subalgebra of some $L(H)$, the following question naturally arises.

Problem 2.2. *Let $A \subseteq L(H)$. Is every derivation $\delta\colon A \to L(H)$ inner?*

This problem can be considered as the major open question in the theory of (bounded) derivations of C^*-algebras. So far a number of important contributions have been made, and it is widely conjectured that, at least for von Neumann algebras, the answer is always yes. An affirmative answer for type I and for hyperfinite von Neumann algebras was given by Johnson and Ringrose in 1972, and for the properly infinite case by Christensen in [4]. Problem 2.2 also serves as the motivation for our exposition of the interrelations between derivations and completely bounded maps. Our final purpose is to present Christensen's equivalent formulation of Problem 2.2 in terms of completely bounded maps, and to relate it to a number of other important structural properties and questions (see below and Chapter 8 of [26]). Note that it is tantamount to ask whether every derivation $\delta: A \longrightarrow L(H)$ can be extended to a derivation on $L(H)$.

One of these applications is to a canonical decomposition of conditionally completely positive maps combining results by Lindblad from 1976 and Evans from 1977 [13].

Theorem 2.3. *The following conditions on a W^*-algebra A are equivalent.*

(a) *Whenever A is faithfully represented as a von Neumann algebra on a Hilbert space H, then every derivation $\delta: A \longrightarrow L(H)$ is inner.*

(b) *Whenever A is faithfully represented as a von Neumann algebra on a Hilbert space H, then every conditionally completely positive ultraweakly continuous self-adjoint linear map $L: A \longrightarrow L(H)$ can be decomposed as $L = \psi + d_{x,x^*}$ with $\psi: A \longrightarrow L(H)$ completely positive and $x \in L(H)$.*

Completely positive maps not only are important in the applications to mathematical physics but also play a central role in the theory of tensor products of C^*-algebras (see Section 4), non-commutative harmonic analysis, and non-commutative probability theory where they serve as transition operators of non-commutative stochastic processes.

In addition to the relation between derivations and homomorphisms given by exponentiation, $\delta \mapsto e^\delta$, there is a more algebraic connection which has also been known for a long time.

Suppose that A is a C^*-subalgebra of a C^*-algebra B and let $\delta: A \longrightarrow B$ be a derivation. Define a homomorphism $\rho: A \longrightarrow M_2(B)$ by

$$\rho(a) = \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}, \qquad (a \in A).$$

If A is unital, then ρ will be unital, but ρ need not be a *-homomorphism if δ is a *-derivation. Actually, the following is easily obtained [26].

Proposition 2.4. *Let $A \subseteq L(H)$ be a C^*-algebra. The derivation $\delta: A \longrightarrow L(H)$ is inner if and only if the canonically associated homomorphism $\rho: A \longrightarrow L(H^2)$ constructed above is similar to a *-homomorphism, i.e. there is an invertible operator $S \in L(H^2)$ such that $a \mapsto S^{-1}\rho(a)S$ defines a *-homomorphism.*

This result turns out to play a key role in an attack to solve Problem 2.2 (see the following section). The question how different homomorphisms of C^*-algebras can be from *-homomorphisms has been investigated by many authors. For example, a result due to Gardner [31] stating that two C^*-algebras which are isomorphic as algebras are in fact *-isomorphic yields a factorisation of an isomorphism $\rho: A \longrightarrow B$ between C^*-algebras A and B into a product of a *-isomorphism and an automorphism of the form e^δ, δ a derivation of A.

3. The similarity problem

In 1955 Kadison raised the question when a given homomorphism from a C^*-algebra A into $L(H)$ is similar to a *-homomorphism [19]. This was preceded by a related question whether a bounded representation of a topological group is similar to a unitary representation. The latter is certainly true for finite groups which is a classical result, and Dixmier [10] gave an affirmative answer for amenable groups. However, the result fails in general as was shown by Kunze and Stein in 1960. Kadison's question is still open, and in the present section we will develop the terminology to state a partial, but important answer due to Haagerup [16]. (In [16] the reader may find additional comments on the history of this problem.) From Haagerup's theorem a characterisation of inner derivations first given by Christensen in [5] is immediate (Theorem 3.3 below).

Let $\rho: A \to L(H)$ be a homomorphism. If there is a similarity $S \in L(H)$ such that $\pi(a) = S^{-1}\,\rho(a)\,S$ defines a *-homomorphism, then, since π is a contraction, ρ has to be bounded by $\|S^{-1}\|\,\|S\|$. Whether or not every homomorphism from a C^*-algebra is necessarily bounded was an open question since the beginning of the theory of C^*-algebras in the 1940's. Even in the commutative case the answer wasn't clear for many years, and had been one of the main stimuli in automatic continuity theory. One of the early answers is Gelfand's result stating that every homomorphism from a C^*-algebra into a semi-simple commutative Banach algebra is bounded, but it took some time until the assumption of semi-simplicity could be dropped (which was done by Laursen in 1987 for epimorphisms). The question for the case $A = C(X)$ was finally answered by Esterle in 1978. A good up-to-date account of this topic is given in Dales' paper [9].

If we extend ρ to $\rho_n: M_n(A) \to L(H^n)$, then $\rho((a_{ij})) = S_n\,\pi_n((a_{ij}))\,S_n^{-1}$ where S_n denotes the n-fold direct sum of S. Since π_n is a contraction and $\|S_n\| = \|S\|$, $\|S_n^{-1}\| = \|S^{-1}\|$ for all $n \in \mathbb{N}$, we still get that $\|\rho_n\| \leq \|S^{-1}\|\,\|S\|$. This stronger boundedness property motivates the following definition.

Definition 3.1. Let A and B be C^*-algebras, and for a linear map $\phi: A \to B$ let $\phi_n: M_n(A) \to M_n(B)$ be its extension as defined in Definition 1.3. Then ϕ is said to be *completely bounded* if $\sup_n \|\phi_n\| < \infty$, and in this case $\|\phi\|_{cb} = \sup_n \|\phi_n\|$ is called the *completely bounded norm* of ϕ. Moreover, ϕ is called *completely contractive*, respectively *completely isometric*, if $\|\phi\|_{cb} \leq 1$, respectively ϕ_n is an isometry for all $n \in \mathbb{N}$.

The set $CB(A, B)$ of all completely bounded linear maps from A into B is a Banach space under $\|\cdot\|_{cb}$, but is not complete under $\|\cdot\|$, in general; e.g. $(CB(A, L(H)), \|\cdot\|)$ is never complete and is topologically small, i.e. a rare subset, in $L(A, L(H))$ unless both A and H are finite dimensional [33].

The surprising result by Haagerup is that the complete boundedness of a homomorphism from a C^*-algebra not only is a necessary but also a sufficient condition for similarity to a *-homomorphism.

Theorem 3.2. (Haagerup 1983) *Let A be a unital C^*-algebra and $\rho: A \to L(H)$ be a unital homomorphism. Then ρ is similar to a *-homomorphism if and only if ρ is completely bounded. In this case, there exists a similarity S such that $a \mapsto S^{-1}\,\rho(a)\,S$ is a *-homomorphism and $\|\rho\|_{cb} = \|S^{-1}\|\,\|S\|$.*

This result was proved by Haagerup in [16]; a different proof given by Paulsen will be outlined in Section 5.

Suppose that $\delta: A \to L(H)$ is a derivation where $A \subseteq L(H)$. Since a derivation annihilates every central projection we may assume that A is unital whence the canonically associated homomorphism ρ: $A \to L(H^2)$ is unital. Using the canonical shuffle we easily obtain that $\|\delta_n\| \leq \|\rho_n\| \leq \|\delta_n\| + 2$ for all $n \in \mathbb{N}$, i.e. δ is completely bounded if and only if ρ is completely bounded. Combining Theorem 3.2 with Proposition 2.4 thus yields the following result (cf. [26]).

Theorem 3.3. (Christensen 1982) *Let $\delta: A \to L(H)$ be a derivation of a C^*-subalgebra A of $L(H)$. Then δ is inner if and only if δ is completely bounded.*

Christensen's original proof [5] rests on the ultrastrong continuity of a derivation defined on a properly infinite von Neumann algebra [4] as well as on an estimate relating the norm $\|\mathrm{ad}\, x_{|A}\|$ and the distance of $x \in L(H)$ to the commutant A'. It follows in particular that every derivation of an injective von Neumann algebra (for the terminology see Section 4) and of a C^*-algebra with cyclic vector is inner in $L(H)$. Both the arguments of Christensen and Haagerup use in some way Pisier's non-commutative Grothendieck inequality.

In the remainder of this section we will discuss some examples of completely bounded maps. The first result is a simple consequence of the fact that an element a in a C^*-algebra A is self-adjoint if $\varphi(a) \in \mathbb{R}$ for every state φ of A.

Proposition 3.4. *The following conditions on a unital homomorphism ρ between unital C^*-algebras are equivalent.*

(a) *ρ is contractive.*

(b) *ρ is completely contractive.*

(c) *ρ is a *-homomorphism.*

Together with the *-homomorphisms, the following are the prototypes of completely bounded maps. Let $a, b \in L(H,K)$. The mapping

$$M_{a^*,b}: L(K) \longrightarrow L(H), \quad x \mapsto a^*xb$$

is called a *two-sided multiplication*. Since $(M_{a^*,b})_n = M_{a_n^*,b_n}$, where $c_n \in L(H^n, K^n)$ is the n-fold direct sum of $c \in L(H,K)$, it is easily calculated that $M_{a^*,b}$ is completely bounded with

$$\|M_{a^*,b}\|_{cb} = \|a\|\,\|b\|.$$

In Section 4 we will discuss the representation theorems which state that every completely bounded (completely positive) linear map can be decomposed into a *-homomorphism and a (completely positive) two-sided multiplication. The completely positive multiplications can be described as follows.

Proposition 3.5. *The following conditions are equivalent.*

(a) *$M_{a^*,b}$ is positive.*

(b) *$M_{a^*,b}$ is completely positive.*

(c) *$M_{a^*,b} = M_{c^*,c}$ for some $c \in L(H,K)$.*

The proof given in [22] for the case $H = K$ is easily adopted to cover Proposition 3.5. Note in addition that the following polarisation identity holds which is useful in a deduction of Wittstock's decomposition theorem (Theorem 4.4 below)

(1) $$M_{a^*,b} = \tfrac{1}{4}\sum_{k=0}^{3} i^k\, M_{(b+i^k a)^*,\,b+i^k a}.$$

Some matrix calculations show that each bounded linear functional φ on a C^*-algebra is completely bounded with $\|\varphi\| = \|\varphi\|_{cb}$, and if φ is positive, then it is completely positive. As a result, bounded respectively positive linear mappings into commutative C^*-algebras are completely bounded respectively completely positive, and their norms coincide with the completely bounded norm (here, the identification $M_n(C(X)) = C(X, M_n)$ turns out to be useful). Likewise each positive linear map *from* a commutative C^*-algebra is completely positive which was already noted by Stinespring in 1955, however the corresponding result for bounded maps fails.

Finite-dimensionality has also its consequences on the behaviour of completely positive and completely bounded maps. For example, Choi proved that every n-positive linear map from M_n into a C^*-algebra is completely positive (cf. [26]), and Smith showed that $CB(A, M_n) = L(A, M_n)$ for every C^*-algebra A and that $\|\phi\|_{cb} = \|\phi_n\| \le n\|\phi\|$ for each $\phi \in L(A, M_n)$ (cf. [26]). However, as Haagerup [17] observed, there is in general no $m \in \mathbb{N}$ such that $\|\phi_m\| = \|\phi\|_{cb}$ if $\phi \in L(M_n, B)$.

The next result is not unexpected.

Proposition 3.6. *For all C^*-algebras A and B we have*

$$CP(A,B) \subseteq CB(A,B),$$

and the norm and the completely bounded norm of a completely positive map coincide.

This can be deduced nicely from Lemma 1.2. Assuming without restriction that A is unital we take $a \in M_n(A)$ with $\|a\| \le 1$ whence $\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right) \in M_{2n}(A)$ is positive. The complete positivity of ϕ yields that

$$\phi_{2n}\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} = \begin{pmatrix} \phi_n(1) & \phi_n(a) \\ \phi_n(a)^* & \phi_n(1) \end{pmatrix}$$

is positive which entails that

$$\|\phi_n(a)\| \le \|\phi_n(1)\| = \|\phi(1)\|.$$

Therefore, ϕ is completely bounded with $\|\phi\|_{cb} \leq \|\phi\|$, and the other inequality is obvious.

In particular, the linear span of the completely positive operators is contained in the completely bounded operators, and it was an open question for some time whether equality always holds. This is in fact true for a certain class of C^*-algebras which will be discussed in the next section, but for instance not the case for $A = B = C[0,1]$, as proved by Smith [33].

So far we haven't provided any concrete examples of positive respectively bounded maps that are *not* completely positive respectively completely bounded. The easiest positive mapping which is not 2-positive is the transpose map on M_2, and an infinite dimensional analogue on $L(\ell^2)$ gives a bounded not completely bounded map (for details see [26]).

4. Representation and extension theorems

Two important features of bounded linear functionals on C^*-algebras are the Jordan decomposition and, of course, the Hahn-Banach theorem. The former was established by Grothendieck in 1957 and generalises the fact that every bounded regular Borel measure on a compact Hausdorff space is a linear combination of four positive measures, while the latter is clearly an indispensable tool of the theory. In the present section we will discuss possible extensions of these results to arbitrary completely bounded maps.

In order to be able to formulate the problems, we have to extend the notions of complete positivity and complete boundedness as follows.

Definition 4.1. Every subspace M of a C^*-algebra A is called an *operator space*, with the understanding that, for each $n \in \mathbb{N}$, $M_n(M)$ is regarded as a subspace of $M_n(A)$. Every self-adjoint subspace S of a unital C^*-algebra which contains the identity is called an *operator system*. Note that the self-adjoint part S_{sa} of S is a real ordered normed space with generating cone $S_+ = \{x \in S \mid x \geq 0\}$ since

$$x = \tfrac{1}{2}(\|x\| + x) - \tfrac{1}{2}(\|x\| - x) \qquad (x \in S_{sa}).$$

Again, $M_n(S)$ is endowed with the order inherited from $M_n(A)$. If B is another C^*-algebra and $\phi\colon M \to B$ is a linear map, then the notions of ***complete boundedness***, ***complete contractivity***, ***complete isometry***, and ***n-positivity*** respectively ***complete positivity***, if $M = S$ is an operator system, are defined analogously to the case $M = A$.

An abstract characterisation of operator spaces which goes parallel with Banach's abstract characterisation of the subspaces of $C(X)$ was given by Ruan [30] as follows. Let M be a normed complex vector space, and suppose that for each $n \in \mathbb{N}$ norms are provided on the matrix spaces $M_n(M)$ satisfying

$$\|\alpha x\| \leq \|\alpha\|\,\|x\|, \quad \|x\alpha\| \leq \|x\|\,\|\alpha\|,$$

and

$$\|x \oplus y\| = \max\{\|x\|, \|y\|\}$$

for all $x \in M_n(M)$, $y \in M_m(M)$ and $\alpha \in M_n$. Then M is (completely isometric to) an operator space.

The following generalisation of the Hahn-Banach theorem was proved for the completely positive case by Arveson [1] in 1969, and for the completely bounded case independently by Haagerup [14], Paulsen [24] and Wittstock [37] several years later. Wittstock's original proof used a Hahn-Banach theorem for set-valued mappings into $L(H)$ while Haagerup elaborated techniques previously available for completely positive maps only for completely bounded maps. Paulsen's proof reduces the problem to the completely positive case via the "off-diagonal technique" described below, and the proof of Arveson's theorem can be divided into two steps: first consider the finite-dimensional situation and then extend the result to the general case by exploiting the compactness of closed bounded subsets of $CP(A, L(H))$ in the BW-topology.

Theorem 4.2. *Every completely bounded (completely positive) linear map from an operator space (operator system) in a unital C^*-algebra A into $L(H)$ can be extended to a completely bounded (completely positive) map on A under preservation of the cb-norm.*

A C^*-algebra B is called *injective* if every completely positive linear map from an operator system S in some C^*-algebra A into B can be extended to a completely positive map from A into B. Thus, Arveson's extension theorem states that $L(H)$ is injective. From this and a result by Tomiyama (see e.g. [13] or [35]), it is easily deduced that $B \subseteq L(H)$ is injective if and only if there exists a projection of norm one from $L(H)$ onto B (a *conditional expectation*).

Injectivity is related to a number of other important structural properties of C^*-algebras which are compiled in the next theorem, thus revealing the significance of completely positive operators. It is here where the real sorcerers in the field used all their magic.

Theorem 4.3. *The following conditions on a C^*-algebra A are equivalent.*

(a) *A is nuclear.*

(b) *A has the* CPAP.

(c) *A^{**} is semi-discrete.*

(d) *A^{**} is injective.*

(e) *A is amenable.*

(f) $CB(A^{**}, A^{**}) = \operatorname{lin} CP(A^{**}, A^{**})$.

The various implications in this result are due to Connes [8], Choi and Effros [2], [3], Effros and Lance [12], and Haagerup [15], [17]. In order to explain the terminology we recall that a C^*-algebra A is said to be *nuclear* if for every C^*-algebra B all C^*-cross norms on $A \otimes B$ coincide, or equivalently, $A \otimes_{min} B = A \otimes_{max} B$ where

$$\|x\|_{min} = \sup\{\|\pi_1 \otimes \pi_2(x)\| \mid \pi_1,\ \pi_2 \text{ representations of } A,\ B\}$$

and

$$\|x\|_{max} = \sup\{\|\pi(x)\| \mid \pi \text{ a representation of } A \otimes B\}$$

are the minimal respectively the maximal C^*-cross norm. Among the class of nuclear C^*-algebras are all finite-dimensional and all commutative C^*-algebras, and inductive limits as well as tensor products of nuclear C^*-algebras are nuclear. The reduced group C^*-algebra $C_r^*(G)$ of a locally compact group G is nuclear if and only if G is amenable. For some more information, see e.g. [21] and [35]. A C^*-algebra A has the *completely positive approximation property* (CPAP) if the identity on the dual of A can be approximated by completely positive contractions of finite rank in the topology of simple convergence, while a W^*-algebra R is *semi-discrete* if the identity on R is approximated by normal completely positive contractions of finite rank in the topology of simple convergence on $(R, \sigma(R, R_*))$. Finally, A is *amenable* if every derivation $\delta: A \to E$, E a dual Banach A-bimodule, is inner. A recent discussion of Theorem 4.3 can be found in [28].

Injectivity also plays a role in the generalisation of the Jordan decomposition. The following result generally referred to as Wittstock's decomposition theorem was obtained independently in [14], [24], and [36].

Theorem 4.4. *Let A be a unital and B an injective C^*-algebra. Then $CB(A, B) = \operatorname{lin} CP(A, B)$. More precisely, if $\phi: A \to B$ is completely bounded, then there exists a completely positive map $\psi: A \to B$ with $\|\psi\|_{cb} \leq \|\phi\|_{cb}$ such that $\psi \pm \operatorname{Re}(\phi)$ and $\psi \pm \operatorname{Im}(\phi)$ are all completely positive.*

Here, the *real* and *imaginary parts* of a linear map ϕ are defined by $\operatorname{Re}(\phi)(x) = \frac{1}{2}(\phi(x) + \phi(x^*)^*)$ and $\operatorname{Im}(\phi)(x) = \frac{1}{2i}(\phi(x) - \phi(x^*)^*)$, respectively. The decomposition of a completely bounded linear map into a linear combination of completely positive maps is not always possible, e.g. if $A = B = C[0,1]$ [33]. If $A = B$ is a W^*-algebra, then the injectivity is also a necessary condition for the decomposition property as observed by Haagerup in [17].

It emerged that Theorem 4.4 is in fact an immediate consequence of the following representation theorem which was proved by Stinespring in 1955 for completely positive maps [34], and by Paulsen in 1984 for completely bounded maps [24].

Theorem 4.5. *Let A be a unital C^*-algebra and $\phi: A \to L(H)$ be a completely bounded (completely positive) linear map. Then there exist a representation (π, K) of A and $v_i \in L(H,K)$, $i = 1,2$ such that*

$$\phi = M_{v_1^*, v_2} \circ \pi \tag{2}$$

and $\|v_i\| = \|\phi\|_{cb}^{\frac{1}{2}}$, $i = 1,2$. If $\|\phi\|_{cb} = 1$, then v_1 and v_2 can be taken to be isometries, and if ϕ is completely positive, then v_1 and v_2 can be taken equal, equivalently, $M_{v_1^*, v_2}$ is completely positive.

Stinespring's paper from 1955 in which the notion of a completely positive map was introduced can be viewed as both the historical as well as the conceptual starting point of the whole theory. Originally intended as an extension of a dilation theorem due to Naimark, it also generalises the famous GNS-construction. In fact, if φ is a state of a C^*-algebra A, the GNS-construction yields a triple $(\pi_\varphi, H_\varphi, \xi_\varphi)$ consisting of a cyclic representation (π_φ, H_φ) with cyclic vector ξ_φ such that $\varphi(x) = (\pi_\varphi(x)\xi_\varphi \mid \xi_\varphi)$ for all $x \in A$, and by choosing $H = \mathbb{C}$, $K = H_\varphi$ and $v: H \to K$, $v1 = \xi_\varphi$ this translates into $\varphi = M_{v^*, v} \circ \pi_\varphi$. Generally, the triple (π, K, v) is called a *Stinespring representation* of the completely positive map ϕ, and it is easily seen that (π, K, v) is unique up to unitary equivalence if $\pi(A)vH$ is total in K. However, for the completely bounded case, no additional assumption is known making the above representation unique up to unitary equivalence. More information on this topic is contained in [26], [35], and also [13] where the Stinespring representation is derived from the Kolmogorov decomposition for positive-definite kernels.

From Theorem 4.5, Wittstock's decomposition theorem is quickly deduced (cf. [24] and [26]). To do this it suffices to take $B = L(H)$. If $\phi: A \to L(H)$ is completely bounded, then, by (1) and (2),

$$\phi = \tfrac{1}{4} \sum_{k=0}^{3} i^k \, M_{(v_2 + i^k v_1)^*, v_2 + i^k v_1} \circ \pi \tag{3}$$

whence ϕ can be linearly combined by four completely positive maps. A simple rearrangement of (3) shows that

$$\psi = \frac{1}{2}\,(M_{v_1^*, v_1} \circ \pi + M_{v_2^*, v_2} \circ \pi)$$

meets the conditions of Theorem 4.4.

The reader will have noticed the rather long time which passed after the representation and the extension theorems for completely positive maps until their counterparts for completely bounded maps were obtained. The reason for this was the lack of a method relating completely bounded maps to completely positive ones in a natural way. This was remedied by Paulsen's "off-diagonal technique" which concludes this section.

Lemma 4.6. (Paulsen 1982) *Let A and B be unital C^*-algebras, and let $M \subseteq A$ be an operator space. Define an operator system $S \subseteq M_2(A)$ by*

$$S = \left\{ \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} \;\middle|\; \lambda, \mu \in \mathbb{C}, a, b \in M \right\},$$

and for each linear map $\phi: M \to B$ a linear map $\Phi: S \to M_2(B)$ by

$$\Phi \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi(b)^* & \mu \end{pmatrix}.$$

Then ϕ is completely contractive if and only if Φ is completely positive.

In the surprisingly simple proof one uses first the canonical shuffle and a module property of ϕ_n to reduce to the case $n = 1$, i.e. to contractivity respectively positivity, and then an approximation as well as a factorisation argument to reduce further to consideration of $\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right)$ instead of arbitrary elements of S. Applying Lemma 1.2 twice accomplishes the proof.

This lemma is used for example in the proof of the extension theorem (Theorem 4.2) as follows. If $\phi: M \to L(H)$ is completely bounded with $||\phi||_{cb} = 1$, Lemma 4.6 yields a completely positive map $\Phi: S \to M_2(L(H)) = L(H^2)$ which can be extended to $\Psi \in CP(M_2(A), L(H^2))$ under preservation of the norm by Arveson's extension theorem. Letting w_1 respectively w_2 be the isometries from H onto $H \oplus 0$ respectively $0 \oplus H$ and $\iota: A \to M_2(A)$ the embedding into the upper left corner we obtain a complete contraction $\psi: A \to L(H)$ extending ϕ by

$$\psi = M_{w_1^*, w_2} \circ \Psi \circ M_{1, w_1 w_2^*} \circ \iota.$$

5. Completely bounded homomorphisms

This final section is devoted to a deduction of Haagerup's characterisation of those bounded unital homomorphisms which are similar to *-homomorphisms (Theorem 3.2) from the following result by Paulsen [25]. By an *operator algebra* we understand a unital subalgebra of some C^*-algebra.

Theorem 5.1. (Paulsen 1984) *For every completely bounded unital homomorphism $\rho: A \to L(H)$ on an operator algebra A there exists an invertible operator $S \in L(H)$ such that $||\rho||_{cb} = ||S^{-1}|| \, ||S||$ and $M_{S^{-1}, S} \circ \rho$ is a completely contractive homomorphism. Moreover,*

$$||\rho||_{cb} = \inf \{ ||R^{-1}|| \, ||R|| \mid M_{R^{-1}, R} \circ \rho \text{ is completely contractive} \}.$$

The main idea in the proof of this result is to use Theorem 4.2 to extend the homomorphism ρ to the C^*-algebra containing A and the representation theorem applied to the extended map in order to introduce a new norm on H which is equivalent to the original one such that ρ becomes completely contractive. Once this is done, Haagerup's theorem is immediate from Theorem 5.1 and Proposition 3.3.

At about the same time when Haagerup proved Theorem 3.2, Hadwin showed in [18] that a unital homomorphism from a C^*-algebra into $L(H)$ is similar to a *-homomorphism if and only if the homomorphism lies in the span of the completely positive maps. This together with Wittstock's decomposition theorem yields an alternative argument for Theorem 3.2, without giving the norm identity. Theorem 5.1 is also useful in other applications, for instance to Halmos' question whether every polynomially bounded operator is similar to a contraction, see [26].

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C*-DYNAMICAL SYSTEMS AND COVARIANCE ALGEBRAS

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The last two decades have seen extraordinary progress in the theory of operator algebras and an enormous increase in the range and power of its applications. In this paper we shall look at that part of the theory which deals with the interaction of C*-algebras and groups of their automorphisms. From the purely theoretical point of view, the motivation for studying this area is that it enables the construction of new interesting C*-algebras from old. Further motivation is provided by the sheer depth and elegance of the ideas of the theory, which involve a beautiful interplay of C*-algebras and harmonic analysis, and concern some of the deepest (and hardest) results of the theory of operator algebras. Historically, however, the main impetus to the development of the subject came from its applications in mathematical physics. For this reason we shall occasionally motivate a point by a brief reference to quantum physics.

§1. Simple and primitive C*-algebras.

We begin by reviewing some basic terminology. Let A be an algebra (all vector spaces and algebras are complex). An involution on A is a conjugate-linear map, $a \mapsto a^*$, such that $(ab)^* = b^*a^*$ and $a^{**} = a$ $(a, b \in A)$. A C*-algebra is an algebra endowed with an involution and a complete norm such that $\|ab\| \le \|a\|\,\|b\|$ and $\|a^*a\| = \|a\|^2$ $(a, b \in A)$. Obviously the complex field $\mathbb{C}$ is a C*-algebra. Less trivially, if Ω is a locally compact Hausdorff space, then the set $C_0(\Omega)$ of all complex-valued continuous functions on Ω vanishing at infinity is a C*-algebra (the operations are defined pointwise and the norm is given by $\|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)|$). By

the Gelfand representation, every commutative C*-algebra is of this form, up to isomorphism.

If H is a Hilbert space, let $B(H)$ denote the set of (bounded linear) operators on H. This is a C*-algebra with the operator norm and the involution defined by the usual adjoint. If A is a norm-closed subalgebra of $B(H)$ such that $T^* \in A$ whenever $T \in A$, then it is a C*-algebra. The Gelfand-Naimark theorem asserts that every C*-algebra is of this form (up to isomorphism).

A fundamental technique used in analysing a C*-algebra A is to represent it on various Hilbert spaces. A representation of A is defined to be a pair (H, φ), where H is a Hilbert space and $\varphi : A \to B(H)$ is a $*$-homomorphism, that is, a linear map preserving multiplication and involution. We say that (H, φ) is non-degenerate if H is the closed linear span of all elements $\varphi(a)\eta$ $(a \in A,\ \eta \in H)$; and we say that (H, φ) is irreducible if the only closed vector spaces K of H such that $\varphi(a)K \subseteq K$ $(a \in A)$ are $K = 0$ and $K = H$.

There are two classes of C*-algebras that play the role of "building blocks" in the theory—the simple and the primitive C*-algebras (their description as building blocks has to be taken *cum grano salis*). A primitive C*-algebra is one which admits an irreducible representation (H, φ) with φ injective. For example, $B(H)$ is primitive, but $C_0(\Omega)$ is not, unless Ω is a single point (in which case $C_0(\Omega) = \mathbb{C}$). A C*-algebra A is simple if its only closed ideals are the trivial ones, 0 and A. Simple C*-algebras are primitive, but not conversely. For instance, $B(H)$ is simple only when H is finite-dimensional. The ideal of compact operators is always simple.

In general it is a non-trivial task to exhibit examples of simple and primitive C*-algebras. The covariance algebras that we introduce in the next section play a vital role in the construction of many such examples.

§2. C*-dynamical systems and covariance algebras.

An automorphism of a C*-algebra A is a bijective $*$-homomorphism from A onto itself. We denote by Aut A the group of automorphisms of A.

A C^*-dynamical system is a triple (A, G, α), where A is a C*-algebra, G is a locally compact group, and the map $\alpha : G \to \operatorname{Aut} A$, $x \mapsto \alpha_x$, is a homomorphism that is continuous in the sense that the map $G \to A$, $x \mapsto \alpha_x(a)$, is continuous for each $a \in A$.

The terminology derives from the applications. In quantum physics the observables are non-commuting operators on a Hilbert space. In some models they "form" a C*-algebra A (more precisely, they form its self-adjoint part $A_{sa} = \{a \in A \mid a^* = a\}$). Time evolution and spatial translation of the observables are then described by a C^*-dynamical system.

If A is abelian, we can write $A = C_0(\Omega)$. In this case, the analysis of (A, G, α) relates to ergodic theory, since we get a corresponding action of G on Ω by a group of homeomorphisms α_x^t, where the map $\alpha_x^t : \Omega \to \Omega$ is determined by the equation

$$(\alpha_x f)(\omega) = f(\alpha_{x^{-1}}^t(\omega)) \qquad (x \in G,\ \omega \in \Omega, f \in A).$$

When $G = \mathbf{R}$, $\mathbf{Z}$ or $\mathbf{T}$ (the circle group), the study of $(C_0(\Omega), G, \alpha)$ is in essence classical topological dynamics. The motivation to work with A non-abelian came from the quantum physicists, who have to deal with non-commutating observables.

A unitary representation of G is a pair (H, U), where H is a Hilbert space, the map

$$U : G \to B(H), \quad x \mapsto U_x,$$

is a homomorphism into the group of unitary operators on H, and U is continuous in the sense that for arbitrary $\eta, \eta' \in H$ the function

$$G \to \mathbf{C}, \quad x \mapsto \langle U_x \eta, \eta' \rangle,$$

is continuous.

The analogous object to a representation of a C*-algebra A is a covariant representation of a C^*-dynamical system (A, G, α). This is a triple (H, φ, U), where (H, φ) is a representation of A, the pair (H, U) is a unitary representation of G, and φ, U interact via the condition

$$\varphi(\alpha_x(a)) = U_x \varphi(a) U_x^* \quad (a \in A,\ x \in G).$$

We can now introduce the covariance algebra of (A, G, α). The connection of this C*-algebra with (A, G, α) is that there is a natural one-one correspondence between its non-degenerate representations and the covariant representations of (A, G, α), so that, at least to some extent, the theory of covariant representations is reduced to that of ordinary representations.

Let m and Δ denote the left Haar measure and the modular function of G respectively. Denote by $K(G, A)$ the vector space of continuous maps from G to A having compact support. We endow $K(G, A)$ with a (convolution-type) multiplication and an involution defined by

$$(f * g)(y) = \int f(x)\alpha_x(g(x^{-1}y))\, dm(x)$$
$$f^*(x) = \Delta(x)^{-1}\alpha_x(f(x^{-1}))^*$$

for f, $g \in K(G, A)$ and $x, y \in G$.

By rather indirect means, one also equips $K(G, A)$ with a suitable norm making it almost a C*-algebra—the only requirement that is not satisfied is completeness. This defect is remedied simply by completing $K(G, A)$ and extending its operations by continuity to get a C*-algebra, denoted by $C^*(A, G, \alpha)$ or $A \times_\alpha G$ and called the covariance algebra of (A, G, α), or the crossed product of A with G (under the action α).

A primary motivation for this construction is that $C^*(A, G, \alpha)$ can be made simple or primitive by imposing suitable conditions on (A, G, α). Examples of simple and primitive C*-algebras are important not only for theoretical reasons, but also for applications. The algebras occurring in physics are often of this type—as D. Kastler remarks, nature does not have ideals. In physics the algebra of quantum observables is frequently obtained from the commutative algebra of the classical observables by taking something like the crossed product with the group generated by a set of "conjugate" variables of the classical variables.

A particular case of the crossed product construction is of great importance in the theory of unitary representations of locally compact groups. If G is one of these groups, we get a C^*-

dynamical system $(\mathbf{C}, G, \alpha)$ by letting G act trivially on $\mathbf{C}$. The covariance C*-algebra $\mathbf{C} \times_\alpha G$ is denoted by $C^*(G)$ and called the group C*-algebra of G. The theory of the unitary representations of G then becomes a part of the representation theory of C*-algebras, since they correspond to the representations of $C^*(G)$ (for details, see [2]). If G is abelian, then $C^*(G) = C_0(\hat{G})$, where $\hat{G}$ is the Pontryagin dual group of G, but in the non-abelian case the analysis of $C^*(G)$ can be very difficult.

Another class of C*-algebras that arise from the crossed product construction is the class of the irrational rotation algebras. These have been extensively studied. One reason for their importance is that they are motivating examples for the non-commutative differential geometry being developed by the Fields medalist Alain Connes.

Let $A = C(\mathbf{T})$ and let $u : \mathbf{T} \to \mathbf{C}$ be the inclusion function (u generates A). If we fix an irrational number θ in [0,1], then there is a unique automorphism α_1 of A such that $\alpha_1(u) = e^{i2\pi\theta}u$. Setting $\alpha_n = \alpha_1^n$, we get a C^*-dynamical system $(A, \mathbf{Z}, \alpha)$ whose covariance C*-algebra is denoted by A_θ and called an irrational rotation algebra. The action of $\mathbf{Z}$ on $\mathbf{T}$ corresponding to α on $C(\mathbf{T})$ is given by rotation through the irrational angle θ, hence the name. We shall return to these algebras in the next section.

§3. Ergodicty and simplicity.

Although the crossed product is the most powerful device for getting new C*-algebras, the process is very elusive and a great deal of effort has been required to give general conditions which imply it is simple or primitive. In this and the next section we discuss some of these conditions (there are others which are not suitable for inclusion here due to their complexity).

We shall make the following assumption:

In this and the next section, (A, G, α) is a C^*-dynamical system for which A is separable and G is countable, discrete and abelian.

Moreover, in this section only, we further assume that A is abelian.

Thus, we may write $A = C_0(\Omega)$. If $\omega \in \Omega$, its orbit is the set of all points $\alpha_x^t(\omega)$ $(x \in G)$. We say that α is ergodic if every orbit is dense in Ω. We write $f < g$ in A to mean $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$ and $f \neq g$. We define α to be free if for all non-zero elements $x \in G$ and all elements $f > 0$ in A, there exists an element $g > 0$ in A such that $g < f$ and $\alpha_x(g) \neq g$.
The following important result is due to E. Effros and F. Hahn.

Theorem [4]. *If (A, G, α) is as assumed above, and the action α is ergodic and free, then the crossed product $A \times_\alpha G$ is simple.*

Despite the considerable restrictions imposed, this is still a very useful result. We illustrate it by applying it to the C^*-dynamical system $(C(\mathbf{T}), \mathbf{Z}, \alpha)$ associated to the irrational rotation algebra A_θ: As is well known, the only closed subgroups of $\mathbf{T}$ are the finite ones and $\mathbf{T}$ itself. The irrationality of θ implies that the set $\{e^{i2\pi n\theta} \mid n \in \mathbf{Z}\}$ is infinite, and therefore the closed subgroup generated by $e^{i2\pi\theta}$ is equal to $\mathbf{T}$. It follows that every orbit is dense in $\mathbf{T}$, that is, α is ergodic. If f is an element of $C(\mathbf{T})$ such that $\alpha_n(f) = f$ for some non-zero integer n, then $\alpha_{mn}(f) = f$ for all $m \in \mathbf{Z}$. Hence, $f(e^{i2\pi mn\theta}) = f(1)$, and therefore, by density of the set $\{e^{i2\pi mn\theta} \mid m \in \mathbf{Z}\}$ in $\mathbf{T}$, the function f is constant. This easily implies that α is free. Since all the conditions of the Effros-Hahn theorem hold, we conclude that A_θ is simple.

§4. The Olesen-Pedersen spectral theory.

A subset S of A is said to be G-invariant if $\alpha_x(S) = S$ $(x \in G)$. If A is abelian, the ergodicity condition defined in the preceding section means that the only G-invariant closed ideals of A are the trivial ideals 0 and A. When A is not (necessarily) abelian, we use the term G-simple for this reformulated condition. We say that A is G-prime if every pair of non-zero G-invariant closed ideals of A have a non-zero intersection.

The Arveson spectrum $Sp(\alpha)$ of α is the set of all $\gamma \in \hat{G}$ such that there exists a sequence of unit vectors a_n in A for which

$$\lim_{n\to\infty} \|\alpha_x(a_n) - \gamma(x)a_n\| = 0 \quad (x \in G).$$

(The $\gamma(x)$ are "joint approximate eigenvalues" of α_x.) The appropriate spectral object for C^*-dynamical systems is not this spectrum, however, but rather another spectrum derived from it which we now describe. If B is a G-invariant C^*-subalgebra of A, we get a new C^*-dynamical system $(B, G, a|B)$ by restriction of α to B. The Connes spectrum of α is

$$\Gamma(\alpha) = \cap_B Sp(\alpha|B),$$

where B runs over all non-zero G-invariant hereditary C^*-subalgebras of A (B is hereditary if $BAB \subseteq B$). The computation of $\Gamma(\alpha)$ is helped by the fact that it is a closed subgroup of $\hat{G}$, but nevertheless its calculation is in general a non-trivial task.

The following result is due to Olesen and Pedersen.

Theorem [7] ,[8]. *If (A, G, α) satisfies the assumption in section 3 the following conditions are equivalent:*
(a) $A \times_\alpha G$ is primitive (respectively, simple);
(b) A is G-prime (respectively, G-simple) and $\Gamma(\alpha) = \hat{G}$.

This is a difficult result, involving a beautiful duality theory for C^*-dynamical systems due to Takesaki and Takai that is a sort of C^*-analogue of the Pontryagin duality theory for locally compact groups. We do not attempt a statement of what this duality involves, as it would require a disproportionate amount of detail.

§5. Crossed products by semigroups.

A question that is begged by the theory we have outlined above is what kind of results hold if we replace groups by semigroups. This situation has been analysed by a number of mathematicians in recent years. We shall briefly outline here some results of a theory developed by the author [5], [6]. Surprisingly (or perhaps not), the situation turns out to be radically different, but nevertheless we get new examples of primitive C*-algebras and, indirectly, of simple C*-algebras.

We redefine a C^*-dynamical system to be a triple (A, G, α), where A is a C*-algebra, G is a cancellative abelian semigroup

with zero, and the map $\alpha : G \to \operatorname{Aut} A$ is a homomorphism. To avoid trivialities we assume that A and G are non-zero. The previous construction of the crossed product using $K(G, A)$ does not work in this setting, but this difficulty is surmounted by constructing $A \times_\alpha G$ as something like the solution to a universal mapping problem (when G is a group, our crossed product is the same as before). The details are omitted as they are technical.

For G arbitrary, we can get a C^*-dynamical system $(\mathbf{C}, G, \alpha)$ by letting G act trivially on $\mathbf{C}$; we then denote $\mathbf{C} \times_\alpha G$ by $C^*(G)$. Observe that $C^*(\mathbf{Z}) = C(\mathbf{T})$, which is not something new. However, $C^*(\mathbf{N})$ is a much more complicated and interesting C*-algebra. It is called the Toeplitz C*-algebra, as it is (isomorphic to) the C*-algebra generated by all Toeplitz operators with continuous symbol on the unit circle $\mathbf{T}$. It plays an important role in K-theory, as indeed does the algebra $A \times_\alpha \mathbf{N}$, for any C^*-dynamical system $(A, \mathbf{Z}, \alpha)$ (this algebra is isomorphic to the generalised Toeplitz algebra of α as defined by Pimsner and Voiculescu in [10]). If G is an ordered group, that is, an abelian group endowed with a total order $\leq$ such that $x \leq y \Rightarrow x + z \leq y + z$, and if $G^+ = \{x \in G \mid 0 \leq x\}$, then $C^*(G^+)$ was shown to be primitive in [5]. A special case of these algebras was first studied by Douglas in [3], where he showed that for G a subgroup of $\mathbf{R}$ with the induced order, not only is $C^*(G^+)$ primitive, but in this case the commutator ideal (the closed ideal generated by all $ab - ba$) is simple.

Let (A, G, α) be a C^*-dynamical system and suppose that G is an ordered group. We get a new (non-classical) C^*-dynamical system (A, G^+, α) by restricting α to G^+. There is a canonical $*$-homomorphism from $A \times_\alpha G^+$ to $A \times_\alpha G$. We let $K(A, G, \alpha)$ denote its kernel.

The algebra $A \times_\alpha G^+$ is never simple, but we can still get new simple C*-algebras by indirect means from this construction, and it seems in some ways to be easier to get primitive C*-algebras using $A \times_\alpha G^+$.

The following theorem is the main result of [6].

Theorem. *If (A, G, α) is as above, then*
(a) If A is primitive, so is $A \times_\alpha G^+$;
(b) If A is simple and G is a subgroup of $\mathbf{R}$ with the induced order, then $K(A, G, \alpha)$ is simple.

A useful feature of this result is that one does not have to compute a Connes spectrum—this makes the hypothesis easy to verify.

Concluding remarks.
We have said nothing about the related theory of W*-dynamical systems. This involves the revolutionary Tomita-Takesaki theory and the deep results of Connes on factors. The reader wishing to learn about this vast subject can consult [9], or, for a quick survey of Tomita-Takesaki theory, Lance's preface to [1].

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PERFECT COMPACT T_1 SPACES

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Abstract: In a previous note in this bulletin, M. Ó Searcóid [6] proved several interesting results on perfect sets. In this article we prove some results on the existence of largish perfect sets (§1), use an Erdös-Rado partition relation to bound cardinalities (§2) and complete the cardinality picture in the final section.

§1. Perfect sets in perfect spaces. The existence of a perfect set in a topological space is a far from certain thing. Even compact Hausdorff spaces do not in general have perfect sets: the ordinal spaces in the order topology constitute the classical counterexample. For this reason one seeks representability conditions which imply the existence of perfect subsets. The condition studied in this section is the following: X is a *perfect* space if and only if every closed set is a G_δ. Examples are the reals, any metric space, any discrete space...; indeed for any topology T on X there is a smallest topology $T' \supset T$ in which X is a perfect space. The main result of §1 says that if X is an uncountable perfect Lindeloef T_1 space, then X contains a perfect subset P of cardinality $|X|$, and $X - P$ is countable. In other words, there is a Cantor-Bendixson theorem for perfect Lindeloef T_1 spaces too.

Definition. For $A \subset X$, $A' := \{x \in A : \text{for all } N \in N_x |A \cap N| > 1\}$ where N_x is the family of open neighbourhoods of x. For each ordinal α define $X^0 := X$, $X^{\alpha+1} := (X^\alpha)'$ and $X^\alpha := \cap_{\beta<\alpha} X^\beta$ if α is a limit ordinal. We use ω for the (cardinality of) the set of natural numbers.

Lemma 1. *For all $\alpha < \omega_1$, (the first uncountable cardinal) if X is uncountable perfect Lindeloef T_1, then (1) $X^\alpha - X^{\alpha+1}$ is countable; (2) $|X^\alpha| = |X|$; (3) $X - X^\alpha$ is countable.*

Proof. X is perfect Lindeldoef T_1, so X^α is also (since X^α is closed in X). Prove (1), (2) and (3) by induction on α. For $\alpha = 0 : X - X'$ is a discrete subset of X, X' is closed in X, so $X' = \cap_{n\in\omega} G_n$, G_n open, and so $X - X' = \cup_{n\in\omega} F_n$, F_n closed. If $|X - X'| > \omega$, then for some $n |F_n| > \omega$; F_n is Lindeloef and discrete—a contradiction. Thus (1), (2) and (3) hold. For $\alpha = \beta + 1$: from (1), (2) and (3) of the inductive hypothesis for β, $|X^\alpha| = |X^\beta| = |X|$, so X^α inherits the uncountability condition too, and (as for $\alpha = 0$) $|X^\alpha - X^{\alpha+1}| \le \omega$, $|X - X^\alpha| = |\cup_{\gamma\le\beta} X^\gamma - X^{\gamma+1}| \le |\beta| \cdot \omega \le \omega$. For α a limit ordinal: $X^\alpha := \cap_{\beta<\alpha} X^\beta$ so $|X - X^\alpha| = |\cup_{\beta<\alpha} X^\beta - X^{\beta+1}| \le |\alpha| \cdot \omega = \omega$; hence $|X^\alpha| = |X|$ and again X^α inherits all the conditions on X and so $|X^\alpha - X^{\alpha+1}| \le \omega$.

Lemma 2. *There exists $\alpha < \omega_1$ such that $X^\alpha = X^{\alpha+1}$.*

Proof. Suppose not. Then for each α and $x \in X^\alpha - X^{\alpha+1}$, there exists $V(x)$ an open neighbourhood of x with $V(x) \cap X^{\alpha+1} = \phi \cdot X^{\omega_1} = \cap_{\alpha<\omega} X^\alpha$ is closed so $X - X^{\omega_1} = \cup_{\alpha<\omega_1} X^\alpha - X^{\alpha+1} = \cup_{n\in\omega} F_n$, F_n closed.

By lemma 1, $X^\alpha - X^{\alpha+1}$ can be enumerated as $\langle x(n,\alpha) : n \in \omega\rangle$. Thus $\cup_{n\in\omega} F_n = \{x(n,\alpha) : \alpha < \omega_1, n < \omega\}$, so for some m, some $B \subseteq \omega_1$, B cofinal (unbounded) in ω_1, and C_α, $\alpha \in B$, $\phi \ne C_\alpha \subset \omega$ one has:

$$F_m = \{x(n,\alpha) : \alpha \in B, n \in C_\alpha\}.$$

Now $\{V(x(n,\alpha)) \cap F_m : \alpha \in B, n \in C_\alpha\}$ is an open cover of F_m (closed hence Lindeloef), so for some countable $A \subset B$, and $D_\alpha \subset C_\alpha (\alpha \in A)$

$$F_m \subseteq \bigcup \{V(x(n,\alpha)) : \alpha \in A, n \in D_\alpha\} \qquad (*)$$

But $\sup A < \omega_1$ since A is a countable set of countable ordinals. So one can find $\beta \in B - (\sup A + 1)$. Consider $x(r,\beta)$ for any $r \in C_\beta : x(r,\beta) \in \cup\{V(x(n,\alpha)) : \alpha \in A, n \in D_\alpha\}$ since $x(r\beta) \in X^{\alpha+1}$ and $V(x(n,\alpha)) \cap X^{\alpha+1} = \phi$ for all $\alpha \in A$ and $n \in D_\alpha$. Of course $x(r,\beta) \in F_m$—contradicting $(*)$.

Therefore there exists $\alpha < \omega_1$ with $X^\alpha = X^{\alpha+1}$

Theorem 3. *X contains a perfect subset P of cardinality $|X|$, and $X - P$ is countable.*

Proof. Choose the first $\alpha < \omega_1$ such that $X^\alpha = X^{\alpha+1}$. Then $P := X^\alpha$ is required.

§2 Partition relations and the power of perfect Lindeloef T_1 spaces.

Question: if from a palette of μ colours one assigns a colour to each n element subset of a set X, can one be sure of finding a large subset $H \subset X$ which is monochromatic: every n element subset of H receives the same colour? It depends. The study of this kind of problem by F. P. Ramsey and later by P. Erdös and co-workers initiated the partition calculus [2], whose many applications include a proof of a famous theorem of Arhangel'skiĭ that every first countable compact (or even Lindeloef) Hausdorff space has power at most continuum (also in [2]).

Some notation: $[X]^n := \{A \subset X : |A| = n\}$; for cardinals $\kappa, \lambda, \mu : \kappa \rightarrow (\lambda)^n_\mu$ read: "κ arrows λ super n sub μ") abbreviates the statement: for every set X of power κ, for every function $f : [X]^n \rightarrow \mu$, there exist $H \subset X$ and $\alpha < \mu$ such that (i) $|H| = \lambda$ (ii) for every $A \in [H]^n$, $f(A) = \alpha$. Intuitively speaking, the partition relation $\kappa \rightarrow (\lambda)^n_\mu$ holds if for every colouring f of $[X]^n$ by μ colours, there is a monochromatic (homogeneous) $H \subset X$ of power λ.

For orientation, here are some partition relations which are theorems of ZFC (Zermelo-Fraenkel set theory with the axiom of choice). (κ^+ is the next cardinal after κ.)

Theorem. *([2], [4], [1])*

(1) $\omega \rightarrow (\omega)^n_k$ *$n, k \in \omega$ (Ramsey)*

(2) $(\exp_n(\lambda))^+ \rightarrow (\lambda^+)^{n+1}_\lambda$ *(Erdös-Rado)*
where $\exp_0(\lambda : \lambda, \exp_{n+1}(\lambda) := \exp_n(2^\lambda)$;

(3) $2^\lambda \not\rightarrow (\lambda^+)^2_2$ *(Sierpiński)*
where $\not\rightarrow$ *means that the relation is false.*

We'll need only the special case $\lambda = \omega$, $n = 1$ of [2]:

$$(2^\omega)^+ \rightarrow (\omega_1)^2_\omega. \qquad (**)$$

Theorem 4. *[7]: If X is perfect Lindeloef T_1, then X has power at most 2^ω.*

The proof of Theorem 4 illustrates very well how partition relations bound cardinality:

Lemma 5. *If X is a topological space in which every singleton subset is a G_δ and if X has no uncountable discrete subsets, then X has power at most 2^ω.*

Proof: For $x \in X$, $\{x\} = \cap_{n\in\omega} G(n,x)$, $G(n,x)$ open. Set $U(n,x) := \cap_{m\leq n} G(m,x)$ so that $U(n,x)$ is open, $U(n_1,x) \supseteq U(n_2,x)$ for $n_1 \leq n_2$ and $\cap_{n\in\omega} U(n,x) = \{x\}$. $x \neq y \in X$ implies that there exists $k \in \omega$ such that:

$$(*)_k \begin{cases} x \in U(k,x) & y \notin U(k,x) \\ y \in U(k,y) & x \notin U(k,y) \end{cases}$$

Define $f : [X]^2 \rightarrow \omega$ by $f(\{x,y\}) :=$ the least k such that $(*)_k$ holds.

Suppose now that $|X| \geq (2^\omega)^+$. Then by $(**)$ there exist $H \subset X$ and $k_0 \in \omega$ such that

(i) $$|H| = \omega_1$$

and

(ii) for $$x \neq y \in H f(\{x,y\}) = k_0.$$

H is discrete, for if $y \neq x \in H$, then by (ii) $y \in U(k_0, x)$ so that $H \cap U(k_0,x) = \{x\}$.

To finish the proof of Theorem 4, one employs the simple

Lemma 6. *If X is perfect Lindeloef T_1, then every discrete subset of X is countable.*

Proof. Let $Y \subset X$ be discrete. Put $F := \{x \in X :$ for all $N \in N_x |Y \cap N| > 1\}$. It's easy to check that

(i) F is closed in X and

(ii) $Y \cap F = \emptyset$, $Y \subset X - F$ and Y is closed in $X - F$. From (i) and (ii) Y is an F_σ in X, say $Y = \cup_{n\in\omega} F_n$, F_n closed in X. Since Y is discrete and X is Lindeloef, F_n is discrete and Lindeloef, hence countable. So $|Y| \leq \sum_{n\in\omega} |F_n| \leq \omega \cdot \omega = \omega$.

Proof of Theorem 4. Every singleton subset is closed in a T_1 space; apply lemmas 5 and 6.

Remarks:

(1) Lemma 6 and Theorem 4 are from [7]; Lemma 5 comes from [3].

(2) Similarly it is easy to show: if X is a perfect κ^+-compact T_1 space, then X has power at most 2^κ.

(3) Recall that X has the Souslin property (the countable chain condition) if and only if there is no uncountable family of pairwise disjoint non-empty open subsets of X. Using (**) one can prove that if X is a first countable Hausdorff space with the Souslin property, then X has power at most 2^ω.

§3Uncountable perfect compact T_1 spaces have power 2^ω.

Theorem 4 says that uncountable perfect compact T_1 spaces have power at most 2^ω. In fact any such space has power exactly 2^ω.

Lemma 7. *If $A \subset X$ is a closed uncountable set, then it is possible to find disjoint closed sets B, C, $B \subset A$ with $A \cap B$, $A \cap C$ both uncountable.*

Proof. Choose $a \in A$ such that $A \cap G$ is uncountable whenever Ga is open. (a exists, since otherwise for all $a \in A$ there exists $G(a)$ open, $a \in G(a)$ and $G(a) \cap A$ is countable; A is compact and so $A \subset (G(a_1) \cup G(a_2) \cup \cdots \cup G(a_n)) \cap A$ giving $|A| \leq \omega$ contradiction.)

$\{a\} = \cap_{n\in\omega} G_n$, G_n open since X is perfect T_1.

$$\left| \{a\} \cup \bigcup_{n\in\omega} (A - G_n) \right| = |A|,$$

so for some n, $A - G_n$ is uncountable. $a \in G_n$ implies that $A \cap G_n$ is uncountable; also $G_n = \bigcup_{m\in\omega} F_m$, F_m closed, so for some m, $a \cap F_m$ is uncountable. Now $B := A \cap (X - G_n)$ and $C := A \cap F_m$ are as required.

Corollary 8. *If $P \subset X$ is perfect uncountable, then there exist P_1, P_2 disjoint uncountable perfect subsets of P.*

Proof Split P to find B, C as in Lemma 7; by Theorem 3 there are $P_1 \subset B$ and $P_2 \subset C$ perfect, $|P_1| = |B|$, $|P_2| = |C|$ and $P_1 \cap P_2 = \emptyset$.

Theorem 9. *Suppose that X is a perfect compact T_1 space. If X is uncountable then X has power 2^ω.*

Proof By Theorem 3, X contains a perfect subset P, $|P| = |X|$. It's enough to show $|P| = 2^\omega$.

Define by induction on ${}^{<\omega}2$ (finite sequences of 0's and 1's) a family of sets P_s for $s \in {}^{<\omega}2$ as follows:
$P_{<>} := P$ ($<>$ is the empty sequence in ${}^{<\omega}2$); if $s \in {}^{<\omega}2$ and P_s is defined so that P_s is uncountable and perfect, choose $P_{s\bar{0}}$, $P_{s\bar{1}}$, disjoint uncountable perfect subsets of P_s (by Corollary 8).

Now define for $f \in {}^\omega 2$ (functions form ω into $\{0,1\}$)

$$P_f := \bigcap_{n\in\omega} P_{f|n}$$

where $f|n$ is the restriction of f to n giving the finite sequences $\langle f(0), f(1), \ldots, f(n-1)\rangle$ in ${}^{<\omega}2$.

For $m \in \omega$, $\bigcap_{n\leq m} P_{f|n} \neq \emptyset$, so by compactness, $P_f \neq \emptyset$.

Thus $\{P_f : f \in {}^\omega 2\}$ si a family of pairwise disjoint non-empty subsets of P. So $2^\omega \leq |P| = |X| \leq 2^\omega$ (By Theorem 4).

Remarks

(1) Some representability condition is necessary, as evidenced by the space $\omega_1 + 1$ with $2^\omega > \omega_1$; similarly a discrete space of power ω_1 with $2^\omega > \omega_1$ indicates the necessity of some degree of compactness.

(2) Theorem 9 resembles the classical theorem that a first countable compact Hausdorff space is either countable or has power 2^ω.

(3) It turns out that Theorem 9 is true under the weaker assumption: if X is compact T_1 and every point of X is a G_δ, then

either $|X| \leq \omega$ or $|X| = 2^{\omega}$. Some of the proof can be found in [5].

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Book Review

DIFFERENTIAL EQUATIONS: A DYNAMICAL SYSTEMS APPROACH, PART I

Texts in Applied Mathematics 5

J. H. Hubbard and B. H. West

Springer-Verlag, 1991, 348 pages,
ISBN 0-387-97286-2.

Reviewed by Donal O'Regan

The book of Hubbard and West provides roughly about one third of a year's undergraduate course in ordinary differential equations for senior undergraduate mathematics students. The authors give a very nice up to date treatment of first order (one dimensional) ordinary differential equations in normal form, namely $x' = f(t, x)$; their own software package MacMath is used and referred to throughout the text to compliment the material.

The book consists of five chapters. Chapter 1 is devoted to qualitative description of solutions; Hubbard and West begin with a discussion of such standard topics as direction fields and computer graphics. However the major part of the chapter is devoted to the introduction of the terms fences, funnels and antifunnels. The authors motivate and illustrate very convincingly how these concepts can be used to examine the behaviour of solutions. Chapter 2 discusses standard methods for solving differential equations analytically; here Hubbard and West provide some lovely insights into some very well known problems. Numerical solutions of differential equations are examined in chapter 3. Here the standard one step methods are discussed and again a very nice treatment is given. Chapter 4 is devoted to the study of existence and uniqueness of solutions. In addition the error bounds stated in chapter 3 are deduced using the ideas (in particular a Dieudonné type inequality) of this section. The final chapter in this book examines iteration methods. This leads naturally to a discussion of intervals of stability for numerical solutions of ordinary differential equations. Hubbard and West finish with a very brief discussion of iteration in one complex dimension.

Overall this book provides an interesting and enlightening introduction to the theory of ordinary differential equations. However in teaching a course on this subject I feel that the book would be more suitable as a supplementary or reference text. The reason for this is that certain sections would have to be optional reading and therefore the main text would only cover about one third of a course. Hence another book would be required and this is far too costly to the student! The book is surprisingly free of typo's; the few I did find are hardly worth mentioning.

Hubbard and West's book will be of interest to those who at some stage were influenced, motivated, frustrated or even baffled by the theory of differential equations. Differential Equations: A Dynamical Systems Approach will provide some of the answers. Moreover it will motivate one to explore more; it is certainly a credit and does credit to the elegant theory of differential equations. If the reader receives even half the pleasure this reviewer obtained from this book then he or she will have purchased wisely. The book is that good.

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Book Review

FUNCTIONAL ANALYSIS AND LINEAR OPERATOR THEORY

Carl L. DeVito

Addison-Wesley, 1990,
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Reviewed by A. Christofides

This is a book with a strong personal flavour and a clear sense of purpose. The author explains in the preface that it is based on courses he gave over the years, which were designed not only for students of mathematics, but also for advanced engineering and science students. As one reads, one soon realises that there is a central theme. This theme is the spectral theory of linear operators—the whole book is designed to lead rapidly to the description of this theory and to the formulation and proof of its main theorems. The mathematical prerequisites are, a sound knowledge of basic analysis and linear algebra and familiarity with the elements of general topology. With such equipment, progress will be swift, and one will soon be immersed in the main topic. Some of the material that one might perhaps expect in a general introduction to functional analysis is left out, while other topics, such as fixed point theorems, are treated parenthetically, as illustrations, rather than for their own sake. Naturally, a lot of important basic material is necessary in order to understand spectral theory and this is discussed both carefully and concisely.

A long first chapter covers all this introductory material: We are introduced to normed vector spaces, Hilbert spaces and, in particular, to $L^2[a,b]$ and to l^2. Here there is no compromise with regard to precision, and the definition of L^2 is preceded by a brief section on Lebesgue outer measure and Lebesgue measurable sets.

Most of the results on Lebesgue measure are stated without proof, but one or two simple key results (such as the fact that a countable set is measurable and has measure zero) are proved. There is even an attempt to motivate Carathéodory's mysterious definition of a measurable set and though the section does not seek to provide a full introduction to measure theory, it succeeds, I think, in demystifying the subject sufficiently to allow an uninitiated student to proceed, without feeling an attack of panic whenever it is subsequently mentioned. This chapter also contains a concise treatment of Fourier series of functions in $L^2[a,b]$, Féjer's theorem and Weierstrass' approximation theorem.

The second chapter introduces bounded linear operators and their norms. Shortly afterwards, the spectrum of a linear operator is defined and two interesting examples are carefully described: We encounter the shift operators on l^2 and the multiplication operator on $L^2[a,b]$. The spectra of these operators are worked out in detail. By the end of the chapter the reader is aware that the spectrum of a bounded linear operator is a closed bounded set, has seen examples of spectra, and realises that a real number can be in the spectrum of a linear operator without being an eigenvalue. Here again, precision has not been sacrificed in any way, yet the flow of the narrative has only barely been interrupted—once in order to prove a necessary point about closed subsets of normed vector spaces and, another time, to state the open mapping theorem—without proof.

Subsequent chapters follow one another with perfect logic: We encounter first the Riesz theory of the spectrum of a compact operator on a Banach space, then the spectral theory of a compact Hermitian operator and that of a compact normal operator. Then we have the spectral theory of general bounded Hermitian operators and a chapter on unbounded operators. A final chapter, on $L^2[\mathbb{R}]$ and the related problems of Fourier analysis on the real line, illustrates some of the results of previous chapters and provides interesting examples of linear operators.

Many interesting topics are encountered on the way. The more "geometric" chapters, which deal with the spectral theory of compact Hermitian operators on a Hilbert space, include a comprehensive review of the spectral theory in the finite-dimensional case. There is a good, clear account of the adjoint of a bounded linear operator on a Hilbert space, with a complete proof that the adjoint of a compact operator on a Hilbert space is compact. This, naturally, involves the weak topology on a Hilbert space, which, like everything in this book, is treated carefully, but with minimum fuss. The same can also be said of the complicated spectral theory of non-compact Hermitian operators. As has already been indicated, new concepts are often introduced with the help of a key example. One such example, the Hilbert-Schmidt operator on $L^2[a,b]$, first makes it appearance in the second chapter, and recurs throughout the book. The related topic of integral equations is repeatedly used to illustrate the theory. The lists of exercises, at the end of each section, are well chosen to complement the text and the occasional comments, concerning more advanced topics of the theory and some of the famous problems that have been preoccupying the experts, help to bring the subject to life.

Maybe a list of topics that are not covered might be of interest. None of the so called "main theorems", such as the Hahn-Banach theorem or the open mapping theorem, are proved, but those that are needed in the text are carefully stated and references are given, indicating where one can find a proof. There is no treatment of L^p-spaces, or l^p-spaces, for $1 < p < \infty$ and $p \neq 2$. Also, the exercises and examples are deliberately and strictly mathematical. The author explains his attitude on this matter in the preface: "My students", he writes, "told me that they want to know that what they are learning has applications but they don't want to see the details. To do so would mean learning the concepts and terminology of the application's subject." He feels that students are not sufficiently interested in each others' disciplines to justify the inclusion of examples from non-mathematical areas. Be that as it may in the case of experimental scientists, I must confess that, as a pure mathematician, I find the application of subtle mathematical concepts to the physical sciences very exciting. However, here, as in other matters, the author had to be selective, and his choice of exercises is a perfectly valid one.

The introductory remarks at the beginning of each chapter are useful and illuminating. The proofs are very complete and none of the details necessary to understand a proof are omitted. There is however a certain clumsiness of presentation, which I shall come to in a moment. Another feature, that I have already mentioned, is the practice of introducing concepts early on and returning to them repeatedly. In the preface, the author refers to this as the "spiral approach". The following extract from the preface gives an interesting insight in to the author's pedagogical method and to the objectives that he has set himself:

> "The style in writing mathematics for mathematics students is to say something once and only once and we train our students to be aware of this. This book, however, is written for students who are primarily interested in using mathematics. As important as mathematics is to their course of study, they forget material that hasn't been discussed for awhile and appreciate a brief review. So I do repeat myself and go over some topics more than once."

On the whole, this method seems to work quite well. Combined with the single-minded pursuit of a central theme, it adds to the cohesion of the book and provides a link between new material and old, so that, when a new topic is introduced, this often throws further light on already familiar concepts. The effect is also reassuring, like seeing a familiar face in strange surroundings. There is, however, a negative aspect to this approach. The mathematical practice of saying something "once and only once", for all its drawbacks, does have the effect of making the reader particularly alert to the introduction of new concepts. There is a tendency, in this book, for concepts to drift in quietly unannounced and, as with familiar faces, one sometimes finds it difficult to recall the circumstances of one's first encounter. The problem is heightened by the rather indifferent system of cross referencing, and this takes me to the book's most serious defect, which has to do with typesetting, presentation and typographical accuracy.

The first half of the book is literally peppered with typographical errors of all kinds. I have counted as many as thirty-four in the first five chapters. Most of them are perfectly harmless, but some, which occur in displayed formulae and in mathematical symbols, result in statements that are either untrue or meaningless as they stand. In an exercise on page 19, the reader is asked to prove that, in an inner product space, elements u and v satisfy $< u, v >= ||u||\,||v||$ if and only if u and v are linearly independent. Further examples of confusing mistakes will be found on pages 10, 19, 21, 22, 23, 41, 69, 90, 105, 108, 138, 184. The number of misprints falls off in the second half of the book, but some still occur. There is also a certain clumsiness and lack of consistency in notation. We find vectors being denoted by the symbols u and v and also by $\vec{u}$ and $\vec{v}$ within the same section of a chapter, (Section 1.6). On page 42, a trigonometric polynomial is denoted by $\varepsilon(x)$, and this gives rise to the following rather strange formula:

$$||f - \varepsilon||_\infty < \varepsilon \, .$$

Again, expressions such as $|||Tx - Tx_0||| < \varepsilon$, or $|\sigma(x,y)| \leq ||A||||x||||y||$ are not easy on the eye, while some readers might find it a little unnerving to be asked to note, in the closing sentence of a proof, on page 77, that the notation "has changed slightly" and that "μ is now $\frac{1}{\mu}$."

I mentioned that the author's "spiral method" can give rise to difficulties. Here is an example of what I mean. The following operator on $L^2[a,b]$ is defined on page 92:

$$P(f) = tf(t).$$

Leaving aside the rather pedantic objection that the argument t appears only on one side of this equation, one reads on, to find a clear and useful discussion which, three pages later, concludes with the the theorem that the spectrum of the multiplication operator on $L^2[a,b]$ is $[a,b]$ and that this operator has no eigenvalues. The fact that this multiplication operator is our friend P, is not made clear when P is first introduced. Instead, half-way through the discussion, the author begins to refer to P by name. The index, which on the whole is good, is of no use in this instance, since it lists eight references to the multiplication operator, but not the one where the term is first encountered.

Life is also made harder than necessary by the absence of Q.E.D. signs, or some equivalent indication of where proofs end. Chapters are referred to as sections, making it difficult to distinguish between chapters and sections within a chapter, numbered equations and results are sometimes referred to by the wrong number, unnumbered equations by number, and various typographical styles of numbering are used.

Throughout the book one is struck by the contrast between the content, treatment, and organisation of material, which are excellent, and these shortcomings in presentation. One sometimes has the impression that, what we have here is an excellent set of notes, which were rather hastily brought out in book form. In spite of its drawbacks, this is a very good introduction to the spectral theory of linear operators and a new, more careful, edition is bound to be popular.

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Book Review

AN INTRODUCTION TO ALGEBRAIC TOPOLOGY

(Graduate Texts in Mathematics 119)

Joseph J. Rotman

Springer-Verlag, 1988, 433pp.
ISBN 0-387-96678-1

Reviewed by Graham Ellis

This is a well written, often chatty, introduction to algebraic topology which "goes beyond the definition of the Klein bottle, and yet is not a personal communication to J. H. C. Whitehead." Having read this book, a student would be well able to use J.F.Adams' *Algebraic Topology: A Student's Guide* to find direction for further study. The book begins with a sketch proof of the Brouwer fixed point theorem: if $f\colon D^n \to D^n$ is continuous, then there is an $x \in D^n$ such that $f(x) = x$. *Functorial* properties of *homology groups* imply that the sphere S^n is not a retract of the disc D^n, and then a simple argument by contradiction shows that f must have a fixed point. This illustrates the basic idea of studying topological spaces by assigning algebraic entities to them in a functorial way. There follows a rigorous account of the singular homology of a space which assumes only a modest knowledge of point-set topology and a familiarity with groups and rings. The account includes the Hurewicz map from the fundamental group to the first homology group, and ends with a proof of the Mayer-Vietoris sequence. By p. 110 a complete proof of Brouwer's theorem has been given. Singular homology is good for obtaining theoretical results, but not so good for computations. So simplicial homology is introduced in Chapter 7, and used to compute the homology groups of some simple spaces such as the torus and the real projective plane. A proof of the Seifert-Van Kampen theorem for polyhedra is given at the end of the chapter. Continuing the search for effective means of computing homology groups, Chapter 8 introduces CW complexes and their cellular homology. Chapter 9 begins with a statement (without proof) of the axiomatic characterization of homology theories due to Eilenberg and Steenrod, and then introduces enough homological algebra to prove the Eilenberg-Zilber theorem and Künneth formula for the homology of a product of spaces. Chapter 10 deals with covering spaces. The higher homotopy groups are studied in Chapter 11 using the suspension and loop functors. Results obtained include the exact homotopy sequence of a fibration, and its application to the fibration $S^3 \to S^2$ to show that the group $\pi_3(S^2)$ is non-trivial. The isomorphism $\pi_3(S^2) = \mathbf{Z}$ is beyond the scope of the book. In the final chapter a short discussion on de Rahm cohomology is used to motivate the study of the cohomology ring of a space.

The book is nicely structured, with explanations of where the theory is heading given at frequent intervals. Important definitions are often accompanied by a discussion on their origins. Many exercises are given at the end of sections. Proofs are usually given in full detail. Even though probably every result in the book (and many more besides) can be found in E.H. Spanier's classic text *Algebraic Topology*, J.J. Rotman's style of exposition makes the book a useful reference. However a lecture course based on this book may turn out to be a bit slow and dry. (Unfortunately the book corresponds to the syllabus of a one year course given at the University of Illinois, Urbana.) For example the homology of a space is defined on p. 66 but we have to wait until p. 157 until the homology of the torus is calculated, and until p. 226 for the homology of a lens space. The fundamental group is introduced on p. 44 but isn't calculated for a wedge of two circles until p. 171. Maybe too much rigour and generality in a first course on any topic is not a good thing!

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Book Review

INTRODUCTORY MATHEMATICS THROUGH SCIENCE APPLICATIONS

J. Berry, A. Norcliffe & S. Humble

Cambridge University Press, 1989,
stg£45 (hardback) ISBN 0 521 24119 7,
stg£15 (paperback) ISBN 0 521 28446 5.

Reviewed by Martin Stynes

For most of this century, pure (as opposed to applied) mathematics has held the centre of the mathematics stage. The last twenty years have seen a significant change in emphasis; today, applied mathematics is at least an equal partner. This trend has been reflected at the teaching level by the introduction of "new" topics such as discrete mathematics and dynamical systems, but it has not yet had much effect on the teaching of traditional courses such as calculus and linear algebra (except that sometimes these traditional courses disappear to make room for new courses). Textbooks for traditional courses now tend to use more applied material than heretofore, but the ratio of " applied" to " non-applied" examples is still low in the vast majority of cases. In this respect the book by Berry, Norcliffe & Humble is to be welcomed. Most of its examples are applied; they come from biology, chemistry and especially physics. As the authors state: " There is a growing awareness that we must not teach mathematics in isolation from its applications".

The book is intended for first-year service courses in science or engineering. It devotes approximately 150p. to pre-calculus material, 80p. to differentiation, 70p. to integration, 60p. to ordinary differential equations, 60p. to partial differentiation, and 100p. to probability and statistics. The topics and techniques covered are quite standard.

Each chapter begins with a section entitled " scientific context", which seeks by example to motivate the material in that chapter. This motivation is an excellent idea, and overall it works well, but in some cases it becomes so involved as to discourage the learner. For instance, in Chapter 13 (Second-order ordinary differential equations), a cooling fin on a motor-cycle engine is modelled. This is quite an interesting example, but the explanation assumes a model for convection and Fourier's law for conductive heat flow, in order to derive a second-order ordinary differential equation.

Th order of the material in the book is sometimes surprising. For example, on pp. 37-41, we meet the function e^x and learn that

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Then on p.44, the book explains why $a^m/a^n = a^{m-n}$ for $a > 0$ and m, n positive integers!

Returning to pp. 37-41, the authors are guilty here of presenting too many things at once. While learning about e^x, the novice reader encounters for the first time the binomial expansion, the limit of a sequence, the sigma notation, and the sum of an infinite series. It's all too much! Surely it would be better to discuss these other ideas before analyzing e^x?

This reflects my main criticism of the text: its explanations are often not as clear as they could be. Sometimes they are misleading, as on p. 139, where the idea of $\lim_{x\to a} f(x)$ is being introduced: "...the value of $f = (x^2-1)/(x-1)$ is not so obvious when $a = 1$ ...dividing top and bottom by $x - 1$ gives $f(x) = x + 1$. Now setting $x = 1$ we have $f(1) = 2$."

The discussion of points of inflection on pp. 194-6 is puzzling insofar as only points where the first derivative vanishes seem to be considered. This suspicion is confirmed on p. 467 where we read: "At a point of inflexion we know that both the first and second derivatives are zero". In fact on p. 467, it happens that at

the point in question (the critical point where the liquid, vapour and gas phases meet on the surface $p = f(v,t)$ given by van der Waal's equation of state) one has both derivatives vanishing, so the example is not in error; the harm is that the innocent reader will carry away a nonstandard definition of a point of inflexion.

While the book in its preface says that it does not claim to give a rigorous treatment, arguments are presented later as apparent proofs without any disclaimer. Thus the chain rule is justified by

$$\begin{aligned}\Big(g(f(x))\Big)' &= \lim_{h\to 0}\frac{g(f(x+h)) - g(f(x))}{h}\\ &= \lim_{h\to 0}\frac{g(f(x+h) - g(f(x))}{f(x+h)-f(x)}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= g'(f(x))\cdot f'(x).\end{aligned}$$

I do think that this calculation has definite heuristic value, but the reader deserves a little warning!

To summarize, let me divide the book into examples, exercises and exposition. The material of the examples and exercises is very good, with many applications that were unfamiliar to me; a real effort is made to show how mathematics is used to solve problems in science and engineering. However, the exposition is only fair. The book is thus a useful source for lecture and examination material, but I would be reluctant to use it as a textbook.

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Problem Page

Editor: Phil Rippon

My first problem this time is a remarkable result about spherical triangles, which was apparently first proved by a computer!

26.1 *Prove that if the area of a spherical triangle is one quarter of the area of the sphere, then the midpoints of its sides form an equilateral spherical triangle with angles of* $90°$.

A discussion of the algebraic verification of theorems in geometry and a BASIC program to prove this result can be found in the article *A new method of automated theorem proving* by Yang Lu ('The mathematical revolution inspired by computing' edited by J. H. Johnson and M. J. Loomes, Oxford University Press, 1991). It might be argued that a computer program cannot tell you why the result holds, in the way that a conventional proof should do.

Next is a problem that I heard recently from my school mathematics teacher, Mr Harold Taylor. It was inspired, he says, by a discussion of the relative sizes of bifurcating blood vessels, given on a television science programme.

26.2 *A pipe from A is split into two smaller pipes at P to supply B and C. Given that the pipe AP costs k times as much per unit length as do PB and PC, determine the position of P so that the total cost is a minimum.*

Now, here is some recent news about one of my older problems. Problem 11.2 asked you to prove that the sequence

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $a_0,\ a_1 \in \mathbf{R}$, is always periodic with period 9. Just before last Christmas, Alan Beardon noticed a connection between this problem and the theory of Hecke groups (certain discrete groups of Möbius transformations). This insight has led to a number of extensions and related results, now being written up by Alan, Shaun Bullett and myself; for example, the sequence

$$a_{n+2} = 2\cos(\pi/p)|a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots,$$

where $p \in \{2, 3, \ldots\}$ and $a_0,\ a_1 \in \mathbf{R}$, is always periodic with period p^2. For $p = 3$, we obtain the sequence (1).

Finally, here is a solution to problem 23.2 which appeared in issue 23.

23.2 *Let $s(n)$ denote the number of triples (a, b, c), where a, b, c are positive integers with*

$$a + b + c = n, \quad a \le b \le c \text{ and } a + b > c.$$

Determine a simple formula for $s(n)$.

The motivation behind this counting problem is that each such triple (a, b, c) determines an integer-sided triangle, which is unique up to congruence. We denote the set of such triples by

$$S_n = \{(a, b, c) : a, b, c \in \mathbf{N},\ a + b + c = n,\ a \le b \le c, a + b > c\},$$

and record below the elements of S_n, for $0 \le n \le 10$.

n	S_n	$s(n)$
0		0
1		0
2		0
3	$(1,1,1)$	1
4		0
5	$(1,2,2)$	1
6	$(2,2,2)$	1
7	$(2,2,3),(1,3,3)$	2
8	$(2,3,3)$	1
9	$(3,3,3),(2,3,4),(1,4,4)$	3
10	$(3,3,4),(2,4,4)$	2

On the basis of this table, it is clear that $s(n)$ is somewhat irregular, but it appears that $s(n+3) = s(n)$ if n is odd. Indeed, it is clear that if $(a,b,c) \in S_n$, then $(a+1,b+1,c+1) \in S_{n+3}$ and the reverse implication holds also if n is *odd* (because if $a+b+c$ is odd, then $a+b-c$ is odd, so that

$$\begin{aligned}(a+1)+(b+1) > c+1 &\Longrightarrow a+b > c-1 \\ &\Longrightarrow a+b > c).\end{aligned}$$

Thus

$$s(2m+1) = s(2m+4), \quad m = 0,1,2,\ldots, \tag{2}$$

and so the problem reduces to the evaluation of $s(2m)$, $m = 0,1,2,\ldots$ To do this, we first prove that

$$s(2m+3) = s(2m) + \left[\frac{1}{2}(m+2)\right], \tag{3}$$

where $[x]$ denotes the integer part of x. For, if $(a+1,b+1,c+1) \in S_{2m+3}$ but $(a,b,c) \notin S_{2m}$, then

$$a+1+b+1 > c+1 \quad \text{and} \quad a+b \le c,$$

so that $a+b=c$. Hence

$$2m = a+b+c \Longleftrightarrow a+b = m \Longleftrightarrow (a+1)+(b+1) = m+2.$$

Now, there are $[\frac{1}{2}(m+2)]$ pairs $(a+1, b+1)$ with $a+1 \le b+1$ and $(a+1)+(b+1) = m+2$, so that (3) follows.

Combining (2) and (3) gives, for $m = 0,1,2,\ldots$,

$$s(2m+6) = s(2m) + \left[\frac{1}{2}(m+2)\right]$$

and hence

$$\begin{aligned}s(2m+12) &= s(2m) + \left[\frac{1}{2}(m+2)\right] + \left[\frac{1}{2}(m+5)\right] \\ &= s(2m) + m + 3.\end{aligned}$$

Applying this recurrence relation repeatedly, we find that if $2m = 12k + 2i$, where $i = 0,1,2,3,4,5$ and $k = 0,1,2,\ldots$, then

$$\begin{aligned}s(2m) &= s(2i) + (i+3) + (i+9) + \cdots + (i + 6(k-1) + 3) \\ &= s(2i) + 6(k-1)k/2 + k(i+3) \\ &= s(2i) + k(3k+i) \\ &= s(2i) + (m^2 - i^2)/12,\end{aligned}$$

since $k = (m-i)/6$. Thus, in this case,

$$s(2m) - m^2/12 = s(2i) - i^2/12.$$

On examining the table above, we find that, for $i = 0,1,2,3,4,5$,

$$s(2i) \text{ is the nearest integer to } i^2/12.$$

Hence, for $m = 0,1,2,\ldots$,

$$s(2m) \text{ is the nearest integer to } m^2/12,$$

so that, by (2),

$$s(2m+1) \text{ is the nearest integer to } (m+2)^2/12.$$

To get some feeling for this formula, it is a nice exercise to find the first value of n for which $s(n) > n$.

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