PROBLEM PAGE

Editor: Phil Rippon

The first problem this time is elegant, simple to state and yet rather surprising. I heard it first from Tom Laffey, who indicated that it may have an application to checking the accuracy of computer calculations!

25.1 *Let*

$$m * n = mn + [\varphi m][\varphi n],$$

where m, n are positive integers, φ is the golden ratio $\frac{1}{2}(1+\sqrt{5})$ and $[x]$ denotes the integer part of x. Prove that $*$ is associative.

The next problem came from John Toland at the University of Bath. By checking special cases one can 'guess' the solution, but producing a proof is a different matter!

25.2 *What are the eigenvalues of the matrix*

$$\begin{pmatrix} 0 & -(n-1) & & & \\ 1 & 0 & -(n-2) & & 0 \\ & 2 & 0 & & \\ & & & \ddots & \\ & 0 & & 0 & -1 \\ & & & n-1 & 0 \end{pmatrix} ?$$

Finally, here is a tantalising 'find the next term in the sequence' problem, which I heard first from Derek Goldrei here at the OU.

25.3 *Find the next term in the sequence*

$$2, 4, 16, 37, 58, 89, \ldots$$

How do such sequences behave in general?

Now here are the solutions to the problems in Issue 22. Problem 22.1 was concerned with a relative of the Mandelbrot set, which was found by my colleagues David Crowe, Robert Hasson and Peter Strain-Clark. To define this set, consider the recurrence relation

$$z_{n+1}(c) = \overline{z_n(c)}^2 + c, \quad n = 0, 1, 2, \ldots,$$

where c is complex and $z_0(c) = 0$. Without the complex conjugate, such sequences are used to define the Mandelbrot set and so it makes sense to give the name Mandelbar set to

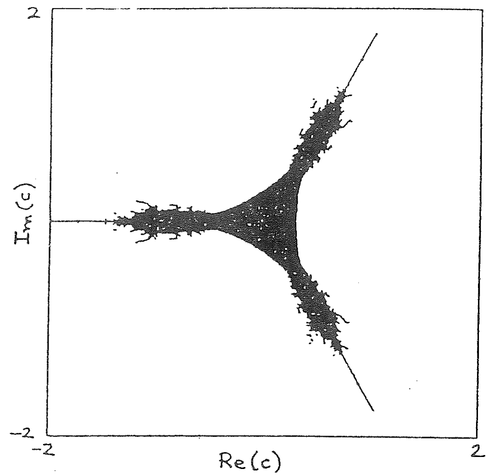
$$M_{\mathrm{BAR}} = \{c : z_n(c) \to \infty \text{ as } n \to \infty\}.$$

By a simple argument, this is equivalent to

$$M_{\mathrm{BAR}} = \{c : |z_n(c)| \le 2, \quad \text{for } n = 1, 2, \ldots\}.$$

22.1 *Prove that M_{BAR} has a rotational symmetry.*

When M_{BAR} is plotted using (1) the following picture appears.



The picture suggests that M_{BAR} is symmetric under a rotation through $2\pi/3$ about the origin, and this can be verified as follows. Let

$$f_c(z) = \overline{z}^2 + c,$$

so that

$$z_n(c) = f_c^n(0),$$

where f_c^n denotes n applications of the function f_c. Now put $\omega = e^{2\pi i/3}$, so that $\overline{\omega}^2 = \omega$, and hence

$$f_{\omega c}(z) = \overline{z}^2 + \omega c = \omega((\overline{z/\omega}^2 + c)) = \omega f_c(z/\omega).$$

By induction, therefore,

$$f_{\omega c}^n(z) = \omega f_c^n(z/\omega),$$

and so, on putting $z = 0$, we obtain

$$z_n(\omega c) = \omega z_n(c).$$

Thus $z_n(\omega c)$ tends to ∞ if and only if $z_n(c)$ tends to ∞, as required.

In fact it turns out that M_{BAR} has been studied for some time by John Milnor, who uses the more descriptive name of **tricorn**, T, for it. The set arose first in connection with the Mandelbrot set for cubics (a subset of $\mathbf{C}^2$), which has been studied in great depth by Milnor, John Hubbard and Bodil Branner. More details about $M_{BAR} = T$ and further references can be found in:

D. Crowe, R. Hasson, P.E.D. Strain-Clark and P.J. Rippon, 'On the structure of the Mandelbar set', Non-linearity **2**(1989), 541-553.

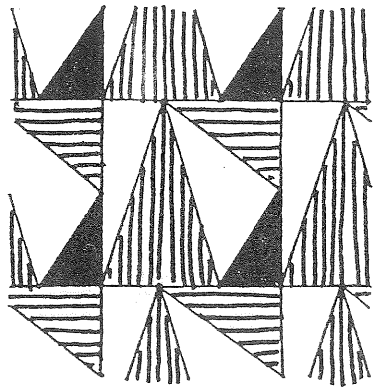
J. Milnor, 'Remarks on iterated cubic maps', Preprint, Stonybrook Institute for Mathematical Sciences.

R. Winters, 'Bifurcations in families of antiholomorphic functions in biquadratic maps', Ph.D Thesis, Boston University, 1990.

22.2 *Prove that it is impossible to tile the plane with triangles in such a way that at most 5 triangles meet at each vertex.*

I came across this problem many years ago while living for a time in West Africa. The local library had just a few maths books, including a Hungarian problem book, which included 22.2. Unfortunately, my wording of 22.2 was not quite precise in view of

examples like the following, which I found when writing up these solutions.



In this example, vertices of some of the triangles are allowed to lie on sides of others. If such a configuration is forbidden, then the problem is correctly posed and the solution goes as follows.

Suppose, if possible, that there does exist a tiling of the plane in which at most 5 triangles meet at each vertex. Consider a large circle and form the union of those triangles in the tiling all of whose vertices lie in or on the circle. This union forms a polygon and we note that at most 3 of the polygon's constituent triangles can meet at one of its boundary vertices. Indeed if 4 of its constituent triangles were to meet at one of its boundary vertices, then there would be exactly one triangle from the tiling outside the polygon at this vertex and this triangle would have to lie in the polygon (because its 3 vertices would be in or on the circle).

Suppose now that the polygon consists of t triangles from the tiling, and that it has e edges and v interior vertices. Summing all the angles of the t triangles and using the fact that the interior angles around the boundary of the polygon sum to $(e-2)\pi$, we obtain the equation

$$t\pi = (e-2)\pi + 2v\pi,$$

so that

$$t = e - 2 + 2v. \tag{1}$$

Counting all the angles of the t triangles, we obtain the inequality

$$3t \leq 3e + 5v, \tag{2}$$

and, from (1) and (2), it follows that $v \leq 6$. This is a contradiction, however, if the circle is large enough, since we also have $t \leq 5v$ because each triangle of the polygon has at least one interior vertex.

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