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TOTAL NEGATION IN GENERAL TOPOLOGY

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To Professor Samuel Verblunsky, on the occasion
of his eighty-fifth birthday.

Introduction

A recurring theme in general topology is the pursuit of examples and characterizations, for each homeomorphic invariant P , of those spaces which are *hereditarily* P in the sense that all of their subspaces are P spaces. Implicit in this programme is the corresponding problem for hereditarily non- P spaces: indeed from a purely logical standpoint the two quests are co-extensive since the negation of an invariant is an invariant. There is however a practical difference between them because, with few exceptions, the invariants of principal interest are shared by all spaces of sufficiently small cardinality; for each such invariant P this simple observation serves both to guarantee a supply of (admittedly superficial) examples of hereditarily P spaces, and to disprove the existence of hereditarily non- P spaces *unless* we modify the question by choosing to disregard these small, "inevitably- P " subspaces. It is from this modification that the study of total negation, surveyed in the present article, arises.

The topic has three historical roots, of which perhaps the most obvious concerns connectedness. The one-point sets (and only they) will be connected irrespective of the choice of topology on the space surrounding them, and so the nearest we can approach to a 'hereditarily non-connected' space is one in which the only connected subspaces are the singletons: that is to say, a *totally disconnected* space. Secondly, the invariant 'perfectness' (absence of isolated points) gives rise to that of a hereditarily non-perfect (that is, *scattered*) space as one in which every subspace possesses a (relatively) isolated point: there being no 'small' subspaces to disregard this time since every non-null set can support a non-perfect topology. Thirdly, from the late 'sixties onwards there has been increasing interest in those spaces (then called pseudo-finite by Albert Wilansky [28] and cf by Norman Levine [16], latterly *anti-compact*) in which only the finite subsets are compact; as finite sets cannot carry non-compact topologies, these are likewise as close as one can get to 'hereditarily non-compact' spaces.

In 1979 Paul Bankston published the pivotal paper [4] of the study. Recognising the pattern in the previous examples, he united them in initiating the general theory of *anti- P* spaces (P here denoting an arbitrary invariant), meaning spaces within which the only P subspaces were those whose cardinalities alone guaranteed that they would be P . The same article presented major contributions to the exploration of anti-compactness and anti-sequential compactness which was already being actively pursued (in different terminology) by Ivan Reilly and M.K. Vamanamurthy, and of anti-lindelöf spaces and similar ideas. Reilly and Vamanamurthy and their co-workers have played a central role in subsequent developments on anti- P spaces, which have mostly focussed on the cases in which P is either a compactness condition or a separation/regularity axiom.

We shall now give a detailed exposition of the elements of the general theory of the "anti-"operation; this is derived from Bankston's article, together with presentational details of our own which have proved useful in explaining the ideas to our colleagues and to each other. This is followed by a survey of what has been established (and by whom) about anti- P spaces for specific in-

variants P , to be read in conjunction with the list of references which we have endeavoured to make fairly complete. Lastly we comment on our perceptions of possible further developments, including projects we have in hand and some open problems.

The "Anti-"operation in general

Bankston's operation is generally viewed as acting on classes of spaces (closed under homeomorphism) rather than on homeomorphic invariants (identified whenever co-extensive). This is unlikely to cause any confusion since one may identify each invariant with the class of all spaces possessing it. We adopt the convention of using the same name for the invariant and the class, but spelling the latter with a capital letter. Thus, for example, ' $\text{compact} \Rightarrow \text{lindelöf}$ ' and ' $\text{Compact} \subseteq \text{Lindelöf}$ ' are to be regarded as interchangeable. We denote by \aleph_0 the cardinality of a countably infinite set. Contrary to the practice of most writers on this topic we choose not to allow 0 as the cardinal number of a topological space, believing it more in keeping with general conventions to insist that topologies be defined only on non-empty sets.

Each invariant P partitions the positive cardinals into (at most) three subclasses $\text{spec}(P)$, $\text{proh}(P)$ and $\text{ind}(P)$: for if α is a cardinal then either *every* space of cardinality α is a P space, or *none* is, or *some* are and some are not; we shall accordingly describe α as *specific*, or *prohibitive*, or *indecisive* for P , and the three types of cardinal constitute the subclasses to which we refer. Clearly $1 \notin \text{ind}(P)$, but this is the only restriction on the resulting division:

Proposition. Let $\{S, P, I\}$ be a pairwise-disjoint covering of the class of positive cardinals. There is a homeomorphic invariant P for which $S = \text{spec}(P)$ and $P = \text{proh}(P)$ and $I = \text{ind}(P)$ if and only if $1 \notin I$.

Proof Given that $1 \notin I$, choose for each $i \in I$ a space X_i of cardinality i . If we call X a P space when *either* X is homeomorphic to one of the X_i chosen *or* the cardinality $|X|$ of X is in S , then P is as required.

It is usually an easy exercise to identify these subclasses for a given invariant. We illustrate the ideas by the following table of

simple but important examples. Note that $\text{spec}(P)$ is known as the *spectrum* of P .

P	$\text{spec}(P)$	$\text{ind}(P)$	$\text{proh}(P)$
Connected	$\{1\}$	$[2,)$	ϕ
Perfect	ϕ	$[2,)$	$\{1\}$
Compact	$[1, \aleph_0)$	$[\aleph_0,)$	ϕ
Lindelöf	$[1, \aleph_0]$	$(\aleph_0,)$	ϕ
Infinite	$[\aleph_0,)$	ϕ	$[1, \aleph_0)$
Normal	$\{1, 2\}$	$[3,)$	ϕ

A space X is called *anti- P* if, for every P subspace Y of X , we have $|Y| \in \text{spec}(P)$. By *Anti- P* we understand the class of all such spaces. One sees immediately that anti-connected \equiv totally disconnected, that anti-perfect \equiv scattered, and that anti-compact is as described earlier. Bankston's key observations on *Anti- P* in general are now summarised:

Theorem.

- (i) *Anti- $P \neq \phi$.*
- (ii) *Anti- P is hereditary; that is, membership of the class is a hereditary invariant.*
- (iii) *If P is hereditary then (a) $\text{spec}(P)^+ \subseteq \text{spec}(\text{Anti-}P)$, (b) $P \subseteq \text{Anti-Anti-}P$.*
- (iv) *Anti- $P \subseteq \text{Anti-Anti-Anti-}P$.*
- (v) *Suppose $\text{spec}(P) = \text{spec}(Q)$; then $P \subseteq Q$ implies $\text{Anti-}Q \subseteq \text{Anti-}P$.*

Proof (i) Bearing in mind that $1 \notin \text{ind}(P)$, the one-point spaces either are or are not P : and in either eventuality they are *anti- P* .

(ii) is immediate from the definition.

(iii)(a) Assuming P to be hereditary, consider a (P) subspace Y of a space X for which $|X| \in \text{spec}(P)$. Any space Z having the same cardinality as Y will be a subspace of some X_1 with $|X_1| = |X|$; thus Z will be P because X' is, and we deduce that $|Y| \in \text{spec}(P)$. This shows that X is *anti- P* , and so $|X| \in \text{spec}(\text{Anti-}P)$.

(b) Now let Y be an *anti- P* subspace of a P space X . With P being hereditary, Y must be a P subspace of itself, which implies $|Y| \in \text{spec}(P)$. In view of (a) this shows X to be *anti-anti- P* .

- (iv) is immediate from (ii) and (iii)(b).
- (v) again follows directly from the definitions.

Much use has been made of part (v) in particular, for it transpires that there are many sets of important invariants having the same spectrum and being implicationally related: for instance, compact and countably compact; lindelöf and σ -compact; T_0 , T_1 , T_2 , T_3 and T_4 .

The enunciation of the other major result in this area, due to Brian Scott, also appears in [4]. It asserts that every hereditary class, apart from one easily-recognised set of exceptions, is of the form *Anti- P* for some suitably chosen P . As far as we can determine, its demonstration has never been published, so we were obliged to prove it for ourselves. Subsequently Bankston sent us a lot of information on the origin of the topic, which we acknowledge with sincere gratitude, including details of Scott's proof. Since it differs substantially (albeit not radically) from our own, and in view of the utility and striking elegance of the conclusion, we outline below our method of verification.

Scott's Theorem. Let Q be a non-empty hereditary class, and consider the following condition:

there is a positive integer n such

that $n \in \text{spec}(Q)$ and $n+1 \in \text{proh}(Q) \dots (*)$

- (I) If $(*)$ holds, then Q is not of the form *Anti- P* .
- (II) If $(*)$ does not hold, then
 - (i) Q is of the form *Anti- P* , and
 - (ii) we can arrange that P shall have empty spectrum.
- (III) If, further, $\text{proh}(Q) = \phi$, then $Q = \text{Anti-}(\text{Not } Q)$.

Proof (I) Suppose that $(*)$ holds and that $Q = \text{Anti-}P$. Every space of cardinality less than n is a subspace of some n -element space, and is therefore Q ; thus $\{1, 2, \dots, n\} \subseteq \text{spec}(Q)$. Now each $(n+1)$ -element set X is not *anti- P* , and must contain a P subspace Y with $|Y| \notin \text{spec}(P)$; contradictions follow whether Y is proper or $Y = X$.

(III) follows directly from the definitions, since

$$\text{spec}(\text{Not } Q) = \text{proh}(Q)$$

in general. In proving (II) we may therefore assume that $\text{proh}(Q)$ has a least element β , and will be of the form $[\beta,)$ since Q is hereditary.

Case 1: $\text{ind}(Q) = \phi$. Here, β must be infinite. We propose calling a space X β -schizoid if $|X| = \beta$ and either it is T_0 or β is the least upper bound of the cardinalities of the point-closures $\{\bar{x}\}$, $x \in X$. It is easily verified that $\text{spec}(\beta\text{-Schizoid}) = \phi$ and that every space of cardinality not less than β possesses a β -schizoid subspace; it follows that $\text{Anti-}\beta\text{-Schizoid} = Q$.

Case 2: $\text{ind}(Q) \neq \phi$, and possesses a least element α . The hereditary character of Q forces $\text{spec}(Q) = [1, \alpha)$, $\text{ind}(Q) = [\alpha, \beta)$. If β is infinite we can compromise between the two previous choices, by taking P to comprise the non- Q spaces with cardinalities in $[\alpha, \beta)$ together with the β -schizoid spaces. If on the other hand β is finite, choose a $(\beta - 1)$ -element non- Q space Y , form a β -element space Y^* by adjoining a point to Y in any fashion, and take P to comprise the non- Q spaces with cardinalities in $[\alpha, \beta)$ together with all β -element spaces except those homeomorphic to Y^* . In both eventualities it is easily seen that $\text{spec}(P) = \phi$ and that $\text{Anti-}P = Q$.

Problem: under what conditions on Q will there exist a hereditary property P such that $Q = \text{Anti-}P$?

Specific Anti-Properties

We shall first summarise what is known about anti-properties derived from compactness and from related conditions, under four sub-headings: sources of examples, operations which preserve the conditions, characterizations, and implications between them.

Examples Let N denote the set of positive integers. For any p in $\beta N \setminus N$, $N \cup \{p\}$ is anticompact [28]. So are the discrete, the co-countable, the included-point topologies, and the topology of decreasing sets on N [16]. The MI spaces of [11], of which many connected T_2 examples can be constructed [1], are anti- λ compact for every cardinal λ , and therefore anti-compact and anti-lindelöf in particular [14, 26]. The T_1 ' P '-spaces (those in which each countable intersection of open sets is open) are anti-compact [4], and exist in abundance (see, e.g., [3], [20]). An example is known of a connected $T_{3\frac{1}{2}}$ anti-compact space [4].

Preservation Open bijections, finite products, and finite unions of open or of closed subsets preserve both anti-compactness [16] and anti-sequential compactness [12, 22]. Indeed, arbitrary unions of open subsets preserve anti-compactness, and so any locally anti-compact space is anti-compact [21]. "Compact covering" maps (continuous maps for which each compact set in the range is the image of a compact set) preserve anti-compactness, and likewise the anti-lindelöf property is preserved by "lindelöf covering" maps, as well as by open bijections and finite products [4]. Infinite products preserve these properties only when cofinitely many factors degenerate [4, 22], but stronger preservation behaviour obtains for various modified products (box products, topological ultraproducts, see [4] for details).

Characterizations

The condition:

for each point p and each infinite set A , p has an open neighbourhood G such that $A \setminus G$ is not compact

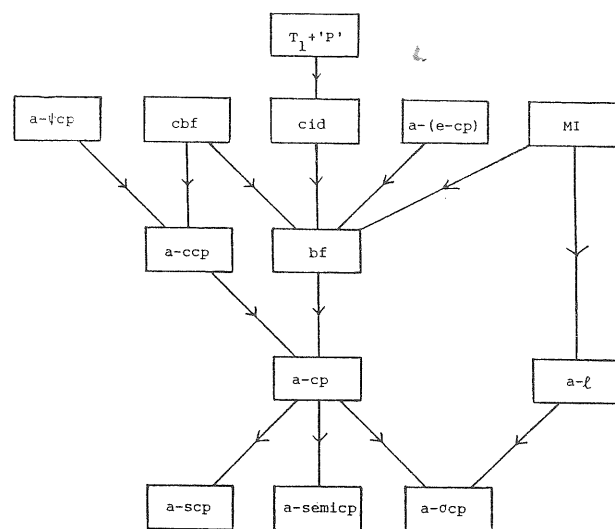
is equivalent to anti-compactness [22]. If we change *compact* to 'countably compact'/'finite' we obtain characterizations of anti-countable compactness [23]/anti-sequential compactness [22]. Alternatively, change *infinite* to 'uncountable' and *compact* to 'lindelöf' and a local characterization of anti-lindelöf spaces arises [23]. An analogous description [24] of anti-semicompactness is founded by replacing *open* by 'semi open' and *compact* by 'semi compact'.

The condition 'no sequence of distinct points has a convergent subsequence' is necessary and sufficient for anti-sequential compactness [22]. Among first-countable spaces, anti-compactness and anti-sequential compactness coincide, and are then identified by each point possessing a finite neighbourhood [12] or, equivalently, by no sequence of distinct points having a cluster-point [16]. Among T_2 spaces, the anti-compact ones are those whose co-compact reflections are cofinite: for further details on this and related matters see [8]. Anti-anti- λ compact and anti-anti-anti- λ compact spaces (for regular cardinals λ) have also been characterized in [23]: the most striking results being that anti-anti-compact/lindelöf is the same as hereditarily compact/lindelöf, and that every T_2 space is anti-anti-anti-compact. This is echoed by the observation that the anti-anti-semi compact spaces are the

hereditarily semi compact ones [24]. The Stone-Čech remainder $\beta X \setminus X$ of a T_4 space X is anti-compact if and only if Y is T_4 for all Y such that $X \subseteq Y \subseteq \beta X$ [5].

Implications

The 'mainstream' connections (see [23], also [12]) are: anti-pseudocompact ($a - \psi cp$) implies anti-countably compact ($a - ccp$) implies anti-compact ($a - cp$) implies both anti-sequentially compact ($a - scp$) and anti- σ compact ($a - \sigma cp$), anti-lindelöf ($a - \ell$) implies $a - \sigma cp$, $a - cp$ implies anti-semi-compact [24]. Many recent developments [15] centre on the ideas of a *bounded* subset of a topological space, i.e. one for which each open cover of the space has a finite subcover of the set, and of a *bf* space as one in which every bounded set is finite; the *cbf* spaces are defined similarly but considering only *countable* open covers; the *essentially-compact* ($e - cp$) spaces [10] are those possessing a bounded dense subset. Lastly, the *cid* spaces ([25]; also called *A-spaces* [21]) are those in which each countably infinite subset is discrete. The relationships between these concepts are summarised in the following diagram.



The converse implications are generally false, and counterexamples will be found in the literature cited. Known partial converses are: $a - scp$ implies $a - cp$ for first-countable spaces [12] and for spaces of small infinite cardinality ([22], see also [17]); $a - cp$ implies bf for regular spaces and for those in which every bounded set is closed [15]; bf implies anti- $(e - cp)$ for T_1 spaces [15].

There is also a small group of results indicating 'how close' certain types of space are to being discrete: we mention $a - cp + T_1 +$ first-countable implies discrete [16], $a - cp + T_2$ k -space implies discrete [4], $a - scp + T_1 +$ sequential implies discrete [22]. Miscellaneous implications include $a - cp +$ first-countable implies saturated [12], $a - \ell + a - scp + T_2$ implies $a - cp$, and $a - cp + T_1$ implies anti-pathconnected [4]; *cid* implies anti-separated, and anti-anti-compact implies anti-*cid* implies finite or not T_2 [25]. Also a T_2 space in which every set with dense interior is open must be $a - ccp$ [27], a space whose compact subsets have empty interiors and whose topology is maximal with respect to that property is $a - cp$ [18], and anti-anti-*bf* does not imply *bf* [15].

Lastly we examine the total negation of the axioms of separation and regularity, where in contrast to the preceding discussion the situation is almost disappointingly simple. In [23] it is shown that the anti- T_0 spaces are the trivial (indiscrete) ones, while for $P = T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4, T_5$, metrizable or discrete, anti- P coincides with nested (each two open sets are comparable). Also anti-trivial equals T_0 and anti-nested equals T_1 . For $P =$ complete regularity, regularity, or the weaker axioms R_1 or R_0 [6], the anti- P spaces are those which are T_0 and nested. The anti-normal spaces, identified in [9], comprise the one- and two-point spaces and the unique non-normal three-point space.

Future developments

The area is quite rich in unsolved (even unposed) problems and possible lines of further investigation. Many standard invariants as yet have no published characterizations of their total negations: we have for example encountered no references to those of local compactness, paracompactness, realcompactness, local connectedness, connectedness *in kleinen*, first countability or complete separability, nor (apart from the result mentioned

above) to that of separability. Our own preliminary investigations show that some at least of these can be dealt with fairly simply; a sample conclusion is that the anti-completely separable spaces are precisely the finite ones. Turning to separation axioms, nothing appears to be known of the 'anti's' of those lying in logical strength between T_1 and T_2 [28]. We have checked several between T_0 and T_1 [2], where the evidence is that the trivial spaces form the anti-class of most if not all of the known ones. An intriguing set of questions in the general theory concern whether it is possible to find a class P so that the classes in the sequence

P , Anti- P , Anti-Anti- P , Anti-Anti-Anti- P , ...

are all distinct, or include infinitely many distinct ones, or include arbitrarily long lists of distinct ones. We have been able to answer these in the negative by proving that, for any choice of P , the sequence contains at most four distinct classes, which recur in one of seven simple patterns [19]. Finally, it could be worth raising corresponding questions (if it has not already been done) in areas other than topology. Is there a significant body of knowledge concerning anti-abelian groups? anti-distributive lattices? anti-noetherian rings? anti-precompact uniformities?

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A SURVEY OF SUBNORMAL SUBGROUPS

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Introduction

Since the appearance of Helmut Wielandt's fundamental paper [27] over fifty years ago, much progress has been made in the theory of subnormal subgroups thanks to the contribution of many distinguished group theorists.

A comprehensive and masterly exposition of the theory of subnormal subgroups is due to Lennox and Stonehewer. The purpose of this article, based on a talk given at "Groups in Galway" is to present some of the remarkable results in the theory without encumbering the general reader with technical details or proofs. The selection of topics is not exhaustive and reflects a bias of the author, but it is hoped to whet the appetite of the reader, who is referred to Lennox and Stonehewer [12] in the first instance. Notation is standard and follows that of Lennox and Stonehewer [12] or Robinson [22].

Definition. If H is a subgroup of a group G such that

$$x^{-1}Hx = H \quad \forall x \in G$$

then H is normal in G , written $H \triangleleft G$.

If $L \triangleleft H$ and $H \triangleleft G$ it does not follow that $L \triangleleft G$, i.e. for subgroups of a group normality is not a transitive relation, as can be verified by examining the alternating group on 4 letters, \mathcal{A}_4 , for instance. This may serve as motivation for the following relation on subgroups which is transitive: