A CONTEXT FOR ADDITION FORMULAE

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§1. Introduction

For a function w of a complex variable, an *addition formula* is an explicit expression for $w(\zeta + \alpha)$. This is less precise than an (algebraic) *addition theorem* in which $w(\zeta + \alpha)$, $w(\zeta)$ and $w(\alpha)$ are to be related algebraically [4, p. 440, 519 and 595].

The context that we have in mind for w is that of satisfying a homogeneous ordinary linear differential equation. For some differential equation of order 1, we find an addition formula pretty much as we might expect, similar to that for the exponential function. However for a differential equation of order 2, we find that there is a pair of addition formulae shared by two linearly independent solutions, similar to those for the cosine and sine functions, although this pattern is obscured when there is a constant solution as then the addition formula appears to involve only one function. More generally for a differential equation of order $n \geq 2$, the method shows that there are n additions formulae shared by n linearly independent solutions, although when there is a constant solution only $n - 1$ seem to be involved.

§2 Basic Theory

We present our material largely in terms of second order equations and consider

$$p_2(z)w''(z) + p_1(z)w'(z) + p_0(z)w(z) = 0 \tag{2.1}$$

where p_2, p_1 and p_0 are functions analytic (holomorphic) in some neighbourhood of $z = 0$ and p_2 is not identically 0. If $p_2(0) \neq$

0, then by continuity $p_2(z)$ is zero-free in some disc $N(0,\delta)$; if $p_2(0) = 0$, then since the zeros of a non-constant analytic function are isolated, there is some deleted disc $N^*(0,\delta) = \{z : 0 < |z| < \delta\}$ in which $p_2(z)$ is zero-free. Thus we can assume that p_2, p_1 and p_0 are analytic in $N(0,\delta)$ and either

$$p_2(z) \neq 0 \quad \text{for all } z \in N(0,\delta) \tag{2.2}$$

or

$$p_2(z) \neq 0 \quad \text{for all } z \in N^*(0,\delta). \tag{2.3}$$

Then [3, p. 34, 47–49], (2.1) will have a solution w on D_δ where in case (2.2) $D_\delta = N(0,\delta)$ and in case (2.3) D_δ is the slit disc

$$\overline{N}(0,\delta) = N(0,\delta) \setminus \{z : z = x, -\delta \leq x \leq 0\}.$$

For $0 < \eta < \delta/2$, let us denote by $D_{\delta,\eta}$ the set of points in D_δ such that z is at a distance exceeding η from the complement of D_δ. Then $D_{\delta,\eta}$ is a domain and for $\zeta \in N(0,\eta)$ and $\alpha \in D_{\delta,\eta}$ we have $\zeta + \alpha \in D_\delta$.

Now let $W(\zeta) = w(\zeta + \alpha)$. Then corresponding to (2.1) we have

$$p_2(\zeta+\alpha)W''(\zeta) + p_1(\zeta+\alpha)W'(\zeta) + p_0(\zeta+\alpha)W(\zeta) = 0. \tag{2.4}$$

For $\alpha \in D_{\delta,\eta}$, clearly $p_2(\alpha) \neq 0$ and so $\zeta = 0$ is an ordinary point of (2.4).

Consequently there is a fundamental set of solutions $W_1(\zeta,\alpha)$, $W_2(\zeta,\alpha)$ of (2.4) on $N(0,\eta)$ satisfying

$$\begin{aligned} W_1(0,\alpha) &= 1, \quad W_1'(0,\alpha) = 0 \\ W_2(0,\alpha) &= 0, \quad W_2'(0,\alpha) = 1 \end{aligned} \tag{2.5}$$

and any solution of (2.4) on $N(0,\eta)$ can be expressed as a linear combination of these [2, p. 16]. Then for any solution w of (2.1) on D_δ, we have that

$$w(\zeta+\alpha) = c_1W_1(\zeta,\alpha) + c_2W_2(\zeta,\alpha)$$

where c_1 and c_2 are independent of ζ. On putting $\zeta = 0$ we see that $c_1 = w(\alpha)$; on differentiating once with respect to ζ and then putting $\zeta = 0$ we have $c_2 = w'(\alpha)$. Thus

$$w(\zeta + \alpha) = w(\alpha)W_1(\zeta, \alpha) + w'(\alpha)W_2(\zeta, \alpha) \tag{2.6}$$

for $\alpha \in D_{\delta,\eta}$ and $\zeta \in N(0, \eta)$. This shows the structure of an addition formula for w.

We note moreover that (2.1) will have linearly independent solutions w_1 and w_2 on D_δ, and then by (2.6) we get the following result.

Let w_1, w_2 be linearly independent solutions of the differential equation (2.1) on D_δ, and for $\alpha \in D_{\delta,\eta}$, let W_1, W_2 be a fundamental set of solutions of the differential equation (2.4) on $N(0, \eta)$ satisfying (2.5). Then we have the addition formulae

$$\begin{aligned} w_1(\zeta + \alpha) &= w_1(\alpha)W_1(\zeta, \alpha) + w_1'(\alpha)W_2(\zeta, \alpha) \\ w_2(\zeta + \alpha) &= w_2(\alpha)W_1(\zeta, \alpha) + w_2'(\alpha)W_2(\zeta, \alpha). \end{aligned} \tag{2.7}$$

In the particular case when p_0 is identically 0, i.e.

$$p_2(z)w''(z) + p_1(z)w'(z) = 0 \tag{2.8}$$

we take $w_1(z) = 1$ and $W_1(\zeta, \alpha) = 1$ identically. Then with the same notation as before we have the following.

Let w_2 be any non-constant solution of the differential equation (2.8) on D_δ, and for $\alpha \in D_{\delta,\eta}$, let W_2 be the solution of the differential equation corresponding to (2.4), on $N(0, \eta)$, satisfying

$$W_2(0, \alpha) = 0, \quad W_2'(0, \alpha) = 1.$$

Then we have the addition formula

$$w_2(\zeta + \alpha) = w_2(\alpha) + w_2'(\alpha)W_2(\zeta, \alpha). \tag{2.9}$$

§3. Examples of Full Type

There is a difficulty in providing examples that make a ready impact, a difficulty which stems from our incomplete knowledge of differential equations. Although we may start with a familiar equation in (2.1), it is only too common that in the corresponding (2.4) we cannot find a solution in any explicit form, and have not any individual distinctive notation for functions which satisfy such an equation.

As a well known example we could derive the addition formulae for the cosine and sine functions from the differential equation

$$w''(z) + w(z) = 0$$

as is done in [2, p. 54]. We obtain a more substantial example, containing this, as follows.

Example 1 Consider the equation

$$(a_2 + b_2 z)w''(z) + (a_1 + b_1 z)w'(z) + (a_0 + b_0 z)w(z) = 0 \tag{3.1}$$

with $a_2 \neq 0$. Let us denote by

$$\begin{aligned} w_1(z) &= F_1(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ z) \\ w_2(z) &= F_2(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ z) \end{aligned}$$

the solutions of this which satisfy respectively

$$\begin{aligned} w_1(0) = 1, &\quad w_1'(0) = 0, \\ w_2(0) = 0, &\quad w_2'(0) = 1. \end{aligned}$$

Then by (2.7),

$$\begin{aligned} &F_1(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \zeta + \alpha) = \\ &F_1(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha) \\ &\quad F_1(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta) \\ &+ F_1'(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha) \\ &\quad F_2(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta) \end{aligned}$$

$$F_2(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \zeta + \alpha) =$$
$$F_2(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha)$$
$$F_1(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta)$$
$$+ F_2'(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha)$$
$$F_2(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta)$$

We could perform a similar analysis if in (3.1) we replaced the coefficients by quadratic or cubic polynomials, or polynomials of a fixed higher degree, or polynomials in $\exp z$, or trigonometric polynomials.

Example 2 We can construct an example by taking

$$w_1(z) = (1-z)^{-b}, \quad w_2(z) = (1+z)^{-b}$$

where $b \neq 0$, these being independent solutions of

$$(1-z^2)w''(z) - 2(b+1)zw'(z) - b(b+1)w(z) = 0.$$

This is a particular case of the ultraspherical equation. The corresponding equation (2.4) does not seem to have a name, but we can calculate

$$W_1(\zeta, \alpha) = \frac{1}{2}\left[(1-\alpha)(1-\frac{\zeta}{1-\alpha})^{-b} + (1+\alpha)(1+\frac{\zeta}{1+\alpha})^{-b}\right]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}\left[(1-\alpha)^{1-n} + (-1)^n(1+\alpha)^{1-n}\right]\zeta^n$$
$$W_2(\zeta, \alpha) = \frac{1-\alpha^2}{2b}\left[(1-\frac{\zeta}{1-\alpha})^{-b} - (1+\frac{\zeta}{1+\alpha})^{-b}\right]$$
$$= \frac{1-\alpha^2}{2b}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}\left[(1-\alpha)^{-n} - (-1)^n(1+\alpha)^{-n}\right]\zeta^n$$

where $(b)_0 = 1$, $(b)_n = b(b+1)\ldots(b+n-1)$, for $n \geq 1$. Then (2.7) applies.

§4. Examples of Restricted Type

In this section we deal with some examples of the type (2.8).

Example 3 Consider the Euler homogeneous equation

$$z^2w''(z) + (1+b)zw'(z) = 0 \tag{4.1}$$

For it we take $w_1(z) = 1$, $W_1(\zeta, \alpha) = 1$ for all z and ζ, respectively, and

$$w_2(z) = \begin{cases} \frac{z^{-b}-1}{-b} = \int_1^z s^{-b-1}\,ds & \text{,when } b \neq 0, \\ \ln z & \text{,when } b = 0. \end{cases}$$

The corresponding equation (2.4) can be written as

$$-\frac{\zeta}{\alpha}\left(1 - \frac{-\zeta}{\alpha}\right)\frac{d^2w}{d(-\zeta/\alpha)^2} + (1+b)\frac{\zeta}{a}\frac{dw}{d(-\zeta/\alpha)} = 0$$

which is of the hypergeometric type. In fact we note that when $b \neq 0$, we have $w_2'(\alpha) = \alpha^{-b-1}$ and

$$w_2(\zeta+\alpha) - w_2(\alpha) = \int_\alpha^{\zeta+\alpha} s^{-b-1}\,ds$$
$$= \alpha^{-b-1}\int_0^\zeta (1+t/\alpha)^{-b-1}\,dt$$
$$= \alpha^{-b-1}\zeta\sum_{n=0}^{\infty}\frac{(b+1)_n(1)_n}{n!(2)_n}(-\zeta/\alpha)^n$$
$$= \alpha^{-b-1}\zeta\,{}_2F_1(b+1, 1;\ 2-\zeta/\alpha)$$

where ${}_2F_1$ is the hypergeometric function [4]. Thus in this case we have

$$W_2(\zeta, \alpha) = \zeta\,{}_2F_1(b+1, 1;\ 2;\ -\zeta/\alpha)$$

for (2.9).

Now when $b = 0$ we can write

$$\ln(\zeta+\alpha) = \ln\alpha + \frac{1}{\alpha}\alpha\ln(1+\zeta/\alpha) \tag{4.2}$$

which identifies $W_2(\zeta, \alpha) = \alpha w_2(1+\zeta/\alpha)$ directly in terms of w_2. When $b \neq 0$ we can express

$$w_2(\zeta+\alpha) = w_2(\alpha) + w_2\left\{\left[1 + (\zeta+\alpha)^{-b} - \alpha^{-b}\right]\right\}$$

in a similar fashion, but this more self-contained form seems additional to what our general theory provides.

We can make a similar example from the differential equation

$$(1-z^2)w''(z) - 2bzw'(z) = 0$$

for which we take

$$w_2(z) = \int_0^z (1-s^2)^{-b}\, ds.$$

The cases

$$w_2(z) = \frac{1}{2} \ln \frac{1+z}{1-z}$$

when $b = 1$ [including the arctan function on replacing z by iz], and

$$w_2(z) = \arcsin z$$

when $b = \frac{1}{2}$ are instances of this.

Example 4. Consider the equation

$$(a_2 \cos z + b_2 \sin z)w''(z) + (a_1 \cos z + b_1 \sin z)w'(z) = 0$$

with $a_2 \neq 0$, and denote by

$$G(a_2, b_2;\ a_1, b_1;\ z)$$

the solution w of this which satisfies $w(0) = 0$, $w'(0) = 1$. Then on using the addition formulae for cosine and sine to expand the coefficients in the differential equation, we find that

$$\begin{aligned} &G(a_2, b_2;\ a_1, b_1;\ \zeta + \alpha) \\ &\quad = G'(a_2, b_2;\ a_1, b_1;\ \alpha) \\ &\qquad * G(a_2 \cos\alpha + b_2 \sin\alpha, -a_2 \sin\alpha + b_2 \cos\alpha; \\ &\qquad\quad a_1 \cos\alpha + b_1 \sin\alpha, -a_1 \sin\alpha + b_1 \cos\alpha;\ \zeta). \end{aligned}$$

Here $G(1, 0;\ 0, -2;\ z) = \tan z$.

§5. A Recognition.

We recall that for a non-homogeneous equation

$$p_2(z)w''(z) + p_1(z)w'(z) + p_0(z)w(z) = \phi(z)$$

a particular solution, obtainable by the method of variation of parameters, is

$$w_p(z) \int_0^z \frac{w_1(t)w_2(z) - w_2(t)w_1(z)}{W(w_1, w_2;\ t)} \frac{\phi(t)}{p_2(t)}\, dt, \tag{5.1}$$

where $W(w_1, w_2;\ t)$ is the Wronskian. In the case where the coefficients p_2, p_1 and p_0 are all constant, this solution can be put in the form of a convolution as we can express

$$\frac{w_1(t)w_2(z) - w_2(t)w_1(z)}{W(w_1, w_2;\ t)} \tag{5.2}$$

in the form $K(z-t)$ for an appropriate function K [1, p249]. In the general case we ask if z enters in the form $z - t$, that is if (5.2) has the form $\psi(z-t, t)$. On putting $z - t = \zeta$ we see that (5.2) equals

$$\frac{w_1(t)w_2(\zeta + t) - w_2(t)w_1(\zeta + t)}{W(w_1, w_2;\ t)}$$

and on solving the equations (2.7) for $W_2(\zeta, \alpha)$, we recognise this as $W_2(\zeta, t)$. Thus (5.1) can be written as

$$w_p(z) = \int_0^z W_2(z-t, t) \frac{\phi(t)}{p_2(t)}\, dt.$$

References

[1.] R. P. Agnew, *Differential Equations*, McGraw-Hill, New York, 1960.

[2.] G. Birkhoff and G-C. Rota, *Ordinary Differential Equations*, Ginn, Boston etc., 1962.

[3.] J. C. Burkill, *The Theory of Ordinary Differential Equations*, Oliver and Boyd, Edinburgh, 1962.

[4.] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge, 4th edition, 1927.

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TOTAL NEGATION IN GENERAL TOPOLOGY

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To Professor Samuel Verblunsky, on the occasion of his eighty-fifth birthday.

Introduction

A recurring theme in general topology is the pursuit of examples and characterizations, for each homeomorphic invariant P, of those spaces which are *hereditarily* P in the sense that all of their subspaces are P spaces. Implicit in this programme is the corresponding problem for hereditarily non-P spaces: indeed from a purely logical standpoint the two quests are co-extensive since the negation of an invariant is an invariant. There is however a practical difference between them because, with few exceptions, the invariants of principal interest are shared by all spaces of sufficiently small cardinality; for each such invariant P this simple observation serves both to guarantee a supply of (admittedly superficial) examples of hereditarily P spaces, and to disprove the existence of hereditarily non-P spaces *unless* we modify the question by choosing to disregard these small, "inevitably-P" subspaces. It is from this modification that the study of total negation, surveyed in the present article, arises.