IRISH MATHEMATICAL SOCIETY

BULLETIN

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, in March and December. The Bulletin is supplied free of charge to to members by Local Representatives, or by surface mail abroad. Libraries may subscribe to the Bulletin for IR£20 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files. Articles submitted in the form of typed manuscripts will be given the same consideration as articles in TEX.

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IMS MEMBERSHIP

Ordinary Membership
Ordinary of the Irish Mathematical Society is open to all persons interested in the activities of the Society. Application forms are available from the Treasurer and from Local Representatives. Special reciprocity rates apply to members of the Irish Mathematics Teachers Association and of the American Mathematical Society.

Institutional Membership
Institutional Membership is a valuable support to the Society. Institutional members receive two copies of each issue of the Bulletin and may nominate up to five students for free membership.

Subscriptions rates The rates are listed below. The membership year runs from 1st October to 30th September. Members should make payments by the end of January either direct to the Treasurer or through Local Representatives. Members whose subscriptions are more than eighteen months in arrears are deemed to have resigned from the Society.

Ordinary Members	IR£5
IMS–IMTA Combined	IR£6.50
Reciprocity Members from IMTA	IR£1.50
Reciprocity Members from AMS	US$6
Institutional Members	IR£35

Note: Equivalent amounts in foreign currency will also be accepted.

THE 1990 SEPTEMBER MEETING

Donal O'Regan

The 1990 September meeting was hosted by Dublin City University. This very successful meeting was organised by Alastair Wood.

Professor J. Mawhin (Louvain) opened the conference on Thursday 6 September with a talk entitled "Topological Results for Periodic Solutions of ODEs". He described via continuation methods how the existence question for period solutions of ODEs could be approached.

Dr Donal O'Regan followed with "Some Old and Some New Results for Certain Classes of BVPs for ODEs" in which a brief description of recent results for systems of BVPs was presented.

The morning session concluded with Professor N. Everitt (Birmingham) who spoke on "Recent Developments in Computer Programmes for eigenvalues of the Sturm-Liouville problem". Regular and singular (limit circle type only) Sturm-Liouville problems were discussed and also a brief discussion was presented of the computer programme SLEIGN 1 for computing eigenvalues.

The afternoon session began with Mr N. Steele's (Coventry) talk "An Introduction to Neural Networks". A description of the learning algorithm was presented and computer displays of his ideas were also given.

Next Dr T Murphy (TCD) spoke on "Cubic Art". He described how cubic splines could be presented via affine geometry (with Besier curves) to secondary school teachers.

The final lecture of the afternoon session "Semi-homomorphisms of Groups/Rings" was given by Dr M Leeney (UCC). He gave a brief account of how semi-homomorphisms of division rings occur in projective geometry. Recent results on semi-homomorphisms of rings were presented.

Professor F. Holland (UCC) began the morning session on Friday 7 September with a talk entitled "Hankel Operators between Weighted Sequence Spaces". He introduced his subject with a description of Hankel operators between sequence spaces and then recent results for weighted sequence spaces were presented.

Next, Professor T. J. Laffey (UCD) spoke on "A review of Unitarity Similarity of Matrices". He gave a short description of the general problem, i.e. given two $n \times n$ matrices, how can one decide "in a nice way", if there exists a unitary matrix U transforming one of the given matrices into the other via similarity? Results for the cases $n = 2$ and 3 were discussed.

After coffee Dr C Budd (Bristol) spoke on "Coronas in Nonlinear Electrostatics". He began with a description of the electric fields in the vicinity of a high voltage "coronating" electrode. Mathematical models were given and then a brief account of the steady state problem was presented.

The conference concluded with Professor M. Berry's talk "The Stokes Phenomenon in Wave Asymptotics." He gave a historical introduction on how "Stokes phenomena" came about. Recent developments in the subject were also included.

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NOTICE OF MOTION TO CHANGE THE RULES

D. A. Tipple

At the Ordinary Meeting of the Irish Mathematical Society to be held in December of 1991 a motion will be proposed to change **four** of the Rules of the Society. The Constitution requires that written notice of such a motion must be given to all members. Accordingly the motion to be proposed is given below. Significant changes to the existing rules are written in italics.

Motion

The following changes shall be made to the Rules of the Irish Mathematical Society.

Rule **3.** shall be amended to read as follows.

3. The election of the Office-Bearers and the additional members of the Committee shall take place at the *last* Ordinary Meeting of each session.

Rule **4.** shall be amended to read as follows.

4. The term of office of the Office-Bearers and the Committee shall be two *sessions starting on the first day of January that follows the Ordinary Meeting at which they were elected* . The President and the Vice-President may not continue in office for more than two consecutive terms.

Rule **6.** shall be amended to read as follows.

6. Each session shall commence on the 1st day of *January* and last until the following *31st day* of *December* .

Rule **9.** shall be amended to read as follows.

9. A Financial Statement for each session *shall be written by the Treasurer holding office in that session* , shall be duly audited by two persons appointed by the Committee, and shall be submitted to the First Ordinary Meeting *that follows that session* .

The Committee wishes to make these changes to the Rules for the following reasons.

(i) The Society's financial year and the terms of office of its officers and committee members should be brought into line.

(ii) The annual financial statement written by the treasurer should be based on the accounts maintained by that treasurer. Since no rules are laid down for the keeping of the Society's accounts, different treasurers may well use different systems of book-keeping and this could cause problems.

A copy of the current Constitution and Rules of the Society is given below.

Irish Mathematical Society

Constitution

(*as amended by the Ordinary Meeting held on 21 December 1984*)

1. The Irish Mathematical Society shall consist of Ordinary and Honorary Members.

2. Any person may apply to the Treasurer for membership by paying one year's membership fee. His admission to membership must then be confirmed by the Committee of the Society.

Candidates for honorary membership may be nominated by the Committee only, following a proposal of at least three members of the Society. Nominations for honorary membership must be made at one Ordinary Meeting of the Society and voted upon at the next, a simple majority of the members present being necessary for election.

3. Every Ordinary member shall pay subscription to the funds of the Society at the times and of the amounts specified in the Rules.

4. The Office-Bearers shall consist of a President, a Vice-President, a Secretary, a Treasurer. The Office of President or Vice-President may be held in conjunction with any of the other offices.

5. The Committee shall consist of the President, the Vice-President, the Secretary, the Treasurer, and eight additional members. No person shall serve as an additional member for more than three terms consecutively.

6. There shall normally be at least two Ordinary Meetings in a session.

7. Notice of a motion to repeal or alter part of the Constitution shall be given at one Ordinary Meeting. Written notice of one month shall be given to all members before the next Ordinary Meeting at which the motion shall be voted upon, being carried if it receives the consent of two-thirds of the members present.

8. One month's written notice of a motion to repeal or alter a Rule, or to enact a new Rule, shall be given to all members before the Meeting at which it is to be voted upon, the motion being carried if it receives the consent of a simple majority of the members present.

9. All questions not otherwise provided for in the Constitution and Rules shall be decided by a simple majority of members

present at a Meeting. Eleven Ordinary members shall form a quorum for such business.

Rules

(*as amended by the Ordinary Meeting* held on 21 December 1984)

These rules shall be subject to the over-riding authority of the Constitution.

Subscriptions:

1. Every Ordinary Member shall pay, on election to membership and during January in each succeeding session, an annual subscription to be determined by the Committee. A change in the annual subscription shall be ratified by a Meeting of the Society.

2. Ordinary Members whose subscriptions are more than eighteen months in arrears shall be deemed to have resigned from the Society.

Officers and Committee

3. The election of the Office-Bearers and the additional members of the Committee shall take place at the first Ordinary Meeting of each session.

4. The term of office of the Office-Bearers and the Committee shall be two years. The President and the Vice-President may not continue in office for more than two consecutive terms.

5. On alternate years elections for the following positions will take place :

   (a) President, Vice-President, and half of the additional members of the Committee.

   (b) Secretary, Treasurer, and one half of the additional members of the Committee.

6. Each session shall commence on the 1st day of October and last until the following 30th of September.

7. The Committee shall meet at least twice during each session, the President to be convener. Five shall form a quorum.

8. The Secretary shall keep minutes of the Meetings of the Society and of the Committee and shall issue notice of meetings to members resident in Ireland.

9. At the first Ordinary Meeting of each session the Treasurer shall submit a Financial Statement for the previous session, duly audited by two persons appointed by the Committee.

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Change of ISSN

When this **Bulletin** changed its name from Irish Mathematical Society Newsletter to its present name, the ISSN was inadvertently retained for several issues.

The ISSN has been changed with this issue.

GRADUATES SURVEY
Preliminary report

Donal P. O'Donovan

The background

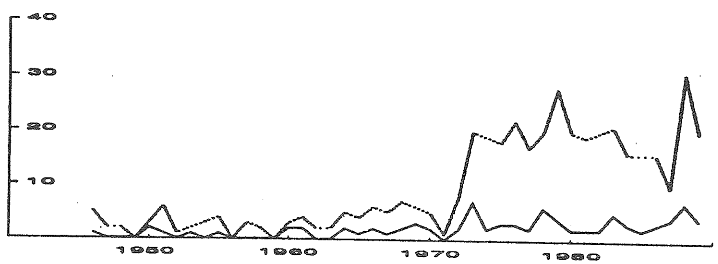
The David Report on Mathematics in the U.S.A. which appeared in 1984, was successful in raising the awareness and funding of mathematics within that country. Having studied this and an earlier Canadian report, I began advocating the desirability of a broad based study of the mathematical sciences in Ireland. This was several years ago. In a document requested by the Irish Mathematical Society, I outlined the reasoning behind such a report, suggested a form that it could take, and hazarded a guess at the likely requirements in terms of time and money. At the September 1989 meeting of the organisation, at Maynooth, I chaired a discussion about my proposals, and there was broad agreement that something along the lines suggested should be undertaken. It was accepted that somebody from outside mathematics should be in charge, and some possibilities were put forward. Shortly thereafter the I.M.S. decided to recommend that the first step in the proposal, a graduate survey, should be carried out. A form for Trinity graduates was prepared by Richard Timoney and myself, and submitted to the I.M.S. as a specimen. Suggestions were taken on board, and the questionnaire was mailed in April 1990 to all Trinity mathematics graduates since 1945.

There has been a reasonable response. From a total of 401 questionnaires mailed, 86 were returned completed, and 19 arrived back marked not known at this address. Mailed with the questionnaire was a letter which described the survey, and included a financial appeal for the Trinity Quatercentenary Fund. This could have reduced the number responding, or biased the sample that did respond, if people chose to respond to both or neither. Also since the early portion of the questionnaire dealt with career data, perhaps those satisfied with there careers were most likely to respond. The numbers responding were just too small to deal with these questions.

To obtain more complete information on recent years we have undertaken a repeat mailing to those from the last fifteen years who had failed to reply. This was done in March 1991. To date no other College has to my knowledge even started the first part of such a survey, which is acquiring/compiling their graduate data base, so in the hopes that a little further prodding might help, I would like to summarise some of the survey results. I say further prodding as David Simms has already compiled and presented much of this information at the 1990 September meeting of the I.M.S. at Dublin City University. Also the ubiquitous Richard Timoney must be credited with much of the work.

The results

First here is a profile of the respondents by year of graduation, compared with the number to whom the survey was mailed.



Number surveyed . . . and number responding — (1946–1989)

Here are some of the questions, and the percentage responses.

Employment status.

(a) What general category would you put your job in?

First Employment		Present Employment
21	Academic, third level	22
12	Academic, primary or secondary	07
20	Actuarial	11
00	Statistical	01
10	Financial services	09
15	Computer services	20
04	Business	01
02	Management	16
02	Civil Service	00
04	Engineering	00
04	Industrial R&D	05
05	Other	04
05	No Answer	05

(b) What rôle did your mathematics background play in attaining these positions?

02	None	05
15	Only in so far as I had a degree in something	10
13	A small rôle	30
28	A large rôle	23
28	It was an absolute requirement	23

(c) Would you recommend a mathematics degree for people interested in such a position?

15	No	21
68	Yes	66
17	No Answer	12

Further Study.
What, if any, further study did you undertake?

	All years	1970-75
Diploma	25	38
Masters	33	42
Doctorate	25	08
Professional qualification	33	46
Other	17	20
None	14	17

These numbers are also percentages. To see how things were changing, I have included a column for seventies graduates. This relates to 24 students. The number of doctorates in the All Years column is strikingly high, but less in the seventies column, by which time numbers of students were increasing. Separate data that David Simms has suggests that this seventies figure for doctorates is too low, which highlights the need for more complete data. The numbers taking a professional qualification are quite large. The bulk of these are an accountancy, actuarial, or insurance qualification. As the response to a later question, there were suggestions that some courses in business and economics should be available to mathematics students. But only some courses, as most people stressed that the most important thing that they had carried away from their mathematics studies was the ability to treat problems in a disciplined and logical way.

Use of Mathematics
During your career, how often have you made use of your mathematical training?

never	01
rarely	11
sometimes	28
frequently	22
constantly	37

The remaining questions were specific to the Trinity degree, so I will not bother to comment on them here. The exercise is still

incomplete, but already we have learned much from it. In many cases it has simply confirmed what most of us would have imagined. But imagined information will not be sufficient for dealing with deans, or mandarins. I hope that all mathematics departments will proceed with a similar study, and that in the meantime the I.M.S. will be putting in place the structures needed to complete the picture of all aspects of Mathematical Sciences in Ireland.

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ENGINEERING MATHEMATICS FOR THE 1990's REPORT OF IMS/SEFI MWG JOINT MEETING, University College, Dublin September 3rd and 4th 1990

This meeting was organised by the Irish Mathematical Society and the Mathematics Working Group of S.E.F.I. (Société Européan pour la Formation de Ingénieurs) to coincide with and complement the S.E.F.I. Annual Conference, hosted this year by the Engineering Faculty of University College, Dublin.

The meeting was held in UCD's new Engineering Building and the delegates were welcomed by the Dean of Engineering, Professor V. McCabe. Then the chairman of the S.E.F.I. Mathematics Working Group (S.M.W.G.), Professor L. Rade (Gothenburg) formally opened the meeting. The papers can be divided into three (related) themes:

- curriculum and teaching developments
- current research in mathematics related to engineering
- use of computers in engineering mathematics

Curriculum and Teaching

A major reason for holding the meeting was to air the (nearly completed) S.M.W.G. report on a core curriculum in engineering mathematics.

P. Nuesch (Lausanne) (S.E.F.I.'s next president) spoke about the need for mathematicians to adapt to the newer types of engineering, for co-operation between universities in Western Europe and in Eastern Europe, for S.E.F.I. and S.M.W.G. to address the issue of mutual recognition of qualifications, as required by the E.C. from 1993.

E. Murphy (Limerick) addressed the issue of the role of statistics in engineering education but with particular reference to qual-

ity and reliability. He quoted a 1989 report by American Management Association which said that engineering students must study more statistics to enable them to cope with these issues. Attempts to reduce statistics to a black box package should be resisted, rather we should ensure that our graduates have a good scientific base, to enable them to benefit fully from the rest of their 'formation'.

G. James (Coventry), former chairman of the S.M.W.G. presented the report on the curriculum. Among the factors listed were:

- need to balance theory and applications, analytic and numerical methods
- need to provide coverage of mathematical ideas and techniques of current applicability
- need to provide coverage of mathematical ideas that will provide a foundation for future study

The document caused much discussion, and although generally welcomed, there were reservations about some its recommendations. For example D. McHale (Cork) agreed that one role of mathematics was to improve an engineer's creativity, whereas F. Hodnett (Limerick) was disappointed to see that special functions had been squeezed out.

Addressing pedagogical issues, M. Attenborough (London) argued that the concept of a system is fundamental to engineering, as it is to applied mathematics. And J. Kennedy (Dublin) discussed the integration of numerical methods and computing into mathematics teaching, and he illustrated some of the problems that packages implementing standard numerical methods can give rise to if used without understanding.

Mathematics Related to Engineering

The papers presented on this theme were designed to illustrate the type of mathematics that is currently used by mathematicians when addressing engineering problems, and so may reflect back on the engineering mathematics curriculum

P. Fitzpatrick (Cork) gave an illuminating talk showing how aspects of abstract algebra and number theory are now finding application in computer engineering. For example in coding theory, cyclic codes draw on the rich algebraic structure of ideals, and convolution codes use an extended version of the familiar Euclidean algorithm. Similarly in the study of IIR filters algebraists have introduced a "signed binary redundant number system" to ensure that the most significant digits in a calculation are those that are calculated first.

A. Wood (Dublin) described a problem which originated in the real engineering requirement of estimating leakage at bends in optical fibres. His approach to this modern problem showed how some very classical ideas in Applied Mathematics (Sturm-Liouville problems, Airy functions, contour integration, Stoke's phenomenon) retain their relevance to engineering.

M. Newman and A. Roberts (Belfast) jointly described a stochastic feedback control problem where the approach to the solution depends on a spectral decomposition of polynomial matrices.

P. Boland (Dublin) gave a brief review of reliability—a topic that uses probability, statistics and calculus.

Also on the topic of reliability, L. Rade (Gothenburg) described his use of the symbolic algebra package MATHEMATICA for doing calculations that arise in reliability theory.

Rade was followed by C. Wolfram from Wolfram Research, the designers of MATHEMATICA. He gave an overview of the package.

Computing in Mathematics Teaching

A number of case studies relating to this theme were presented. In Eindhoven, computing is fully integrated into the early calculus and linear algebra courses. J. Smits described how PC-MATLAB is used as the vehicle to teach a first linear algebra course which has matrix decomposition as its core. At present the course is still examined in the traditional way, but they are moving to a situation in which computing will be involved in that process too.

D. Sprevak (Belfast) described two programs (FLIP and NAN-OPT) that he uses to teach numerical optimisation to engineering students.

In separate talks P. Boieri (Turin) and C. Mate (Madrid) described their approaches to the introduction of computing into engineering mathematics courses.

The concluding paper of the meeting was a well illustrated talk by W. Schauffelberger (Zürich) who described the Project-

Zentrum IDA in the ETH in Zürich. This government financed centre is concerned with integrating computers, especially workstations, into engineering and scientific education. Its aim is to equip ETH with one workstation per five students by 1991 so that students can spend roughly one day per week on computer driven tasks.

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A CONTEXT FOR ADDITION FORMULAE

P. D. Barry and D. J. Hurley

§1. Introduction

For a function w of a complex variable, an *addition formula* is an explicit expression for $w(\zeta+\alpha)$. This is less precise than an (algebraic) *addition theorem* in which $w(\zeta+\alpha)$, $w(\zeta)$ and $w(\alpha)$ are to be related algebraically [4, p. 440, 519 and 595].

The context that we have in mind for w is that of satisfying a homogeneous ordinary linear differential equation. For some differential equation of order 1, we find an addition formula pretty much as we might expect, similar to that for the exponential function. However for a differential equation of order 2, we find that there is a pair of addition formulae shared by two linearly independent solutions, similar to those for the cosine and sine functions, although this pattern is obscured when there is a constant solution as then the addition formula appears to involve only one function. More generally for a differential equation of order $n \geq 2$, the method shows that there are n additions formulae shared by n linearly independent solutions, although when there is a constant solution only $n-1$ seem to be involved.

§2 Basic Theory

We present our material largely in terms of second order equations and consider

$$p_2(z)w''(z)+p_1(z)w'(z)+p_0(z)w(z)=0 \tag{2.1}$$

where p_2, p_1 and p_0 are functions analytic (holomorphic) in some neighbourhood of $z=0$ and p_2 is not identically 0. If $p_2(0) \neq$

0, then by continuity $p_2(z)$ is zero-free in some disc $N(0,\delta)$; if $p_2(0)=0$, then since the zeros of a non-constant analytic function are isolated, there is some deleted disc $N^*(0,\delta)=\{z: 0<|z|<\delta\}$ in which $p_2(z)$ is zero-free. Thus we can assume that p_2, p_1 and p_0 are analytic in $N(0,\delta)$ and either

$$p_2(z)\neq 0 \quad \text{for all } z\in N(0,\delta) \tag{2.2}$$

or

$$p_2(z)\neq 0 \quad \text{for all } z\in N^*(0,\delta). \tag{2.3}$$

Then [3, p. 34, 47–49], (2.1) will have a solution w on D_δ where in case (2.2) $D_\delta = N(0,\delta)$ and in case (2.3) D_δ is the slit disc

$$\overline{N}(0,\delta)=N(0,\delta)\setminus\{z: z=x, -\delta\leq x\leq 0\}.$$

For $0<\eta<\delta/2$, let us denote by $D_{\delta,\eta}$ the set of points in D_δ such that z is at a distance exceeding η from the complement of D_δ. Then $D_{\delta,\eta}$ is a domain and for $\zeta\in N(0,\eta)$ and $\alpha\in D_{\delta,\eta}$ we have $\zeta+\alpha\in D_\delta$.

Now let $W(\zeta)=w(\zeta+\alpha)$. Then corresponding to (2.1) we have

$$p_2(\zeta+\alpha)W''(\zeta)+p_1(\zeta+\alpha)W'(\zeta)+p_0(\zeta+\alpha)W(\zeta)=0. \tag{2.4}$$

For $\alpha\in D_{\delta,\eta}$, clearly $p_2(\alpha)\neq 0$ and so $\zeta=0$ is an ordinary point of (2.4).

Consequently there is a fundamental set of solutions $W_1(\zeta,\alpha)$, $W_2(\zeta,\alpha)$ of (2.4) on $N(0,\eta)$ satisfying

$$\begin{aligned} W_1(0,\alpha)&=1, \quad W_1'(0,\alpha)=0\\ W_2(0,\alpha)&=0, \quad W_2'(0,\alpha)=1 \end{aligned} \tag{2.5}$$

and any solution of (2.4) on $N(0,\eta)$ can be expressed as a linear combination of these [2, p. 16]. Then for any solution w of (2.1) on D_δ, we have that

$$w(\zeta+\alpha)=c_1W_1(\zeta,\alpha)+c_2W_2(\zeta,\alpha)$$

where c_1 and c_2 are independent of ζ. On putting $\zeta = 0$ we see that $c_1 = w(\alpha)$; on differentiating once with respect to ζ and then putting $\zeta = 0$ we have $c_2 = w'(\alpha)$. Thus

$$w(\zeta + \alpha) = w(\alpha)W_1(\zeta, \alpha) + w'(\alpha)W_2(\zeta, \alpha) \tag{2.6}$$

for $\alpha \in D_{\delta,\eta}$ and $\zeta \in N(0, \eta)$. This shows the structure of an addition formula for w.

We note moreover that (2.1) will have linearly independent solutions w_1 and w_2 on D_δ, and then by (2.6) we get the following result.

Let w_1, w_2 be linearly independent solutions of the differential equation (2.1) on D_δ, and for $\alpha \in D_{\delta,\eta}$, let W_1, W_2 be a fundamental set of solutions of the differential equation (2.4) on $N(0, \eta)$ satisfying (2.5). Then we have the addition formulae

$$\begin{aligned} w_1(\zeta + \alpha) &= w_1(\alpha)W_1(\zeta, \alpha) + w_1'(\alpha)W_2(\zeta, \alpha) \\ w_2(\zeta + \alpha) &= w_2(\alpha)W_1(\zeta, \alpha) + w_2'(\alpha)W_2(\zeta, \alpha). \end{aligned} \tag{2.7}$$

In the particular case when p_0 is identically 0, i.e.

$$p_2(z)w''(z) + p_1(z)w'(z) = 0 \tag{2.8}$$

we take $w_1(z) = 1$ and $W_1(\zeta, \alpha) = 1$ identically. Then with the same notation as before we have the following.

Let w_2 be any non-constant solution of the differential equation (2.8) on D_δ, and for $\alpha \in D_{\delta,\eta}$, let W_2 be the solution of the differential equation corresponding to (2.4), on $N(0, \eta)$, satisfying

$$W_2(0, \alpha) = 0, \quad W_2'(0, \alpha) = 1.$$

Then we have the addition formula

$$w_2(\zeta + \alpha) = w_2(\alpha) + w_2'(\alpha)W_2(\zeta, \alpha). \tag{2.9}$$

§3. Examples of Full Type

There is a difficulty in providing examples that make a ready impact, a difficulty which stems from our incomplete knowledge of differential equations. Although we may start with a familiar equation in (2.1), it is only too common that in the corresponding (2.4) we cannot find a solution in any explicit form, and have not any individual distinctive notation for functions which satisfy such an equation.

As a well known example we could derive the addition formulae for the cosine and sine functions from the differential equation

$$w''(z) + w(z) = 0$$

as is done in [2, p. 54]. We obtain a more substantial example, containing this, as follows.

Example 1 Consider the equation

$$(a_2 + b_2 z)w''(z) + (a_1 + b_1 z)w'(z) + (a_0 + b_0 z)w(z) = 0 \tag{3.1}$$

with $a_2 \neq 0$. Let us denote by

$$\begin{aligned} w_1(z) &= F_1(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ z) \\ w_2(z) &= F_2(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ z) \end{aligned}$$

the solutions of this which satisfy respectively

$$\begin{aligned} w_1(0) = 1, \quad w_1'(0) = 0, \\ w_2(0) = 0, \quad w_2'(0) = 1. \end{aligned}$$

Then by (2.7),

$$\begin{aligned} &F_1(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \zeta + \alpha) = \\ &F_1(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha) \\ &\quad F_1(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta) \\ &+ F_1'(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha) \\ &\quad F_2(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta) \end{aligned}$$

$$F_2(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \zeta + \alpha) =$$
$$F_2(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha)$$
$$F_1(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta)$$
$$+ F_2'(a_2, b_2;\ a_1, b_1;\ a_0, b_0;\ \alpha)$$
$$F_2(a_2 + b_2\alpha, b_2;\ a_1 + b_1\alpha, b_1;\ a_0 + b_0\alpha, b_0;\ \zeta)$$

We could perform a similar analysis if in (3.1) we replaced the coefficients by quadratic or cubic polynomials, or polynomials of a fixed higher degree, or polynomials in $\exp z$, or trigonometric polynomials.

Example 2 We can construct an example by taking

$$w_1(z) = (1 - z)^{-b}, \quad w_2(z) = (1 + z)^{-b}$$

where $b \neq 0$, these being independent solutions of

$$(1 - z^2)w''(z) - 2(b + 1)zw'(z) - b(b + 1)w(z) = 0.$$

This is a particular case of the ultraspherical equation. The corresponding equation (2.4) does not seem to have a name, but we can calculate

$$W_1(\zeta, \alpha) = \frac{1}{2}\left[(1 - \alpha)(1 - \frac{\zeta}{1 - \alpha})^{-b} + (1 + \alpha)(1 + \frac{\zeta}{1 + \alpha})^{-b}\right]$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}\left[(1 - \alpha)^{1-n} + (-1)^n(1 + \alpha)^{1-n}\right]\zeta^n$$
$$W_2(\zeta, \alpha) = \frac{1 - \alpha^2}{2b}\left[(1 - \frac{\zeta}{1 - \alpha})^{-b} - (1 + \frac{\zeta}{1 + \alpha})^{-b}\right]$$
$$= \frac{1 - \alpha^2}{2b}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}\left[(1 - \alpha)^{-n} - (-1)^n(1 + \alpha)^{-n}\right]\zeta^n$$

where $(b)_0 = 1$, $(b)_n = b(b + 1)\dots(b + n - 1)$, for $n \geq 1$. Then (2.7) applies.

§4. Examples of Restricted Type

In this section we deal with some examples of the type (2.8).

Example 3 Consider the Euler homogeneous equation

$$z^2w''(z) + (1 + b)zw'(z) = 0 \tag{4.1}$$

For it we take $w_1(z) = 1$, $W_1(\zeta, \alpha) = 1$ for all z and ζ, respectively, and

$$w_2(z) = \begin{cases} \frac{z^{-b} - 1}{-b} = \int_1^z s^{-b-1}\,ds & \text{,when } b \neq 0, \\ \ln z & \text{,when } b = 0. \end{cases}$$

The corresponding equation (2.4) can be written as

$$-\frac{\zeta}{\alpha}\left(1 - \frac{-\zeta}{\alpha}\right)\frac{d^2w}{d(-\zeta/\alpha)^2} + (1 + b)\frac{\zeta}{a}\frac{dw}{d(-\zeta/\alpha)} = 0$$

which is of the hypergeometric type. In fact we note that when $b \neq 0$, we have $w_2'(\alpha) = \alpha^{-b-1}$ and

$$w_2(\zeta + \alpha) - w_2(\alpha) = \int_\alpha^{\zeta+\alpha} s^{-b-1}\,ds$$
$$= \alpha^{-b-1}\int_0^\zeta (1 + t/\alpha)^{-b-1}\,dt$$
$$= \alpha^{-b-1}\zeta\sum_{n=0}^{\infty}\frac{(b + 1)_n(1)_n}{n!(2)_n}(-\zeta/\alpha)^n$$
$$= \alpha^{-b-1}\zeta\,{}_2F_1(b + 1, 1;\ 2 - \zeta/\alpha)$$

where ${}_2F_1$ is the hypergeometric function [4]. Thus in this case we have

$$W_2(\zeta, \alpha) = \zeta\,{}_2F_1(b + 1, 1;\ 2;\ -\zeta/\alpha)$$

for (2.9).

Now when $b = 0$ we can write

$$\ln(\zeta + \alpha) = \ln\alpha + \frac{1}{\alpha}\alpha\ln(1 + \zeta/\alpha) \tag{4.2}$$

which identifies $W_2(\zeta, \alpha) = \alpha w_2(1 + \zeta/\alpha)$ directly in terms of w_2. When $b \neq 0$ we can express

$$w_2(\zeta + \alpha) = w_2(\alpha) + w_2\left\{\left[1 + (\zeta + \alpha)^{-b} - \alpha^{-b}\right]\right\}$$

in a similar fashion, but this more self-contained form seems additional to what our general theory provides.

We can make a similar example from the differential equation

$$(1-z^2)w''(z) - 2bzw'(z) = 0$$

for which we take

$$w_2(z) = \int_0^z (1-s^2)^{-b}\, ds.$$

The cases

$$w_2(z) = \frac{1}{2}\ln\frac{1+z}{1-z}$$

when $b=1$ [including the arctan function on replacing z by iz], and

$$w_2(z) = \arcsin z$$

when $b=\frac{1}{2}$ are instances of this.

Example 4. Consider the equation

$$(a_2\cos z + b_2 \sin z)w''(z) + (a_1 \cos z + b_1 \sin z)w'(z) = 0$$

with $a_2 \neq 0$, and denote by

$$G(a_2, b_2;\ a_1, b_1;\ z)$$

the solution w of this which satisfies $w(0)=0$, $w'(0)=1$. Then on using the addition formulae for cosine and sine to expand the coefficients in the differential equation, we find that

$$\begin{aligned} &G(a_2, b_2;\ a_1, b_1;\ \zeta+\alpha) \\ &\quad = G'(a_2, b_2;\ a_1, b_1;\ \alpha) \\ &\qquad * G(a_2\cos\alpha + b_2\sin\alpha, -a_2\sin\alpha + b_2\cos\alpha; \\ &\qquad\quad a_1\cos\alpha + b_1\sin\alpha, -a_1\sin\alpha + b_1\cos\alpha;\ \zeta). \end{aligned}$$

Here $G(1,0;\ 0,-2;\ z) = \tan z$.

§5. A Recognition.

We recall that for a non-homogeneous equation

$$p_2(z)w''(z) + p_1(z)w'(z) + p_0(z)w(z) = \phi(z)$$

a particular solution, obtainable by the method of variation of parameters, is

$$w_p(z) \int_0^z \frac{w_1(t)w_2(z) - w_2(t)w_1(z)}{W(w_1, w_2;\ t)} \frac{\phi(t)}{p_2(t)}\, dt, \tag{5.1}$$

where $W(w_1, w_2;\ t)$ is the Wronskian. In the case where the coefficients p_2, p_1 and p_0 are all constant, this solution can be put in the form of a convolution as we can express

$$\frac{w_1(t)w_2(z) - w_2(t)w_1(z)}{W(w_1, w_2;\ t)} \tag{5.2}$$

in the form $K(z-t)$ for an appropriate function K [1, p249]. In the general case we ask if z enters in the form $z-t$, that is if (5.2) has the form $\psi(z-t, t)$. On putting $z - t = \zeta$ we see that (5.2) equals

$$\frac{w_1(t)w_2(\zeta+t) - w_2(t)w_1(\zeta+t)}{W(w_1, w_2;\ t)}$$

and on solving the equations (2.7) for $W_2(\zeta,\alpha)$, we recognise this as $W_2(\zeta, t)$. Thus (5.1) can be written as

$$w_p(z) = \int_0^z W_2(z-t, t)\frac{\phi(t)}{p_2(t)}\, dt.$$

References

[1.] R. P. Agnew, *Differential Equations*, McGraw-Hill, New York, 1960.

[2.] G. Birkhoff and G-C. Rota, *Ordinary Differential Equations*, Ginn, Boston etc., 1962.

[3.] J. C. Burkill, *The Theory of Ordinary Differential Equations*, Oliver and Boyd, Edinburgh, 1962.

[4.] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge, 4th edition, 1927.

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TOTAL NEGATION IN GENERAL TOPOLOGY

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To Professor Samuel Verblunsky, on the occasion of his eighty-fifth birthday.

Introduction

A recurring theme in general topology is the pursuit of examples and characterizations, for each homeomorphic invariant P, of those spaces which are *hereditarily* P in the sense that all of their subspaces are P spaces. Implicit in this programme is the corresponding problem for hereditarily non-P spaces: indeed from a purely logical standpoint the two quests are co-extensive since the negation of an invariant is an invariant. There is however a practical difference between them because, with few exceptions, the invariants of principal interest are shared by all spaces of sufficiently small cardinality; for each such invariant P this simple observation serves both to guarantee a supply of (admittedly superficial) examples of hereditarily P spaces, and to disprove the existence of hereditarily non-P spaces *unless* we modify the question by choosing to disregard these small, "inevitably-P" subspaces. It is from this modification that the study of total negation, surveyed in the present article, arises.

The topic has three historical roots, of which perhaps the most obvious concerns connectedness. The one-point sets (and only they) will be connected irrespective of the choice of topology on the space surrounding them, and so the nearest we can approach to a 'hereditarily non-connected'space is one in which the only connected subspaces are the singletons: that is to say, a *totally disconnected* space. Secondly, the invariant 'perfectness'(absence of isolated points) gives rise to that of a hereditarily non-perfect (that is, *scattered*) space as one in which every subspace possesses a (relatively) isolated point: there being no 'small'subspaces to disregard this time since every non-null set can support a non-perfect topology. Thirdly, from the late 'sixties onwards there has been increasing interest in those spaces (then called pseudo-finite by Albert Wilansky [28] and cf by Norman Levine [16], latterly *anti-compact*) in which only the finite subsets are compact; as finite sets cannot carry non-compact topologies, these are likewise as close as one can get to 'hereditarily non-compact'spaces.

In 1979 Paul Bankston published the pivotal paper [4] of the study. Recognising the pattern in the previous examples, he united them in initiating the general theory of *anti-P* spaces (P here denoting an arbitrary invariant), meaning spaces within which the only P subspaces were those whose cardinalities alone guaranteed that they would be P. The same article presented major contributions to the exploration of anti-compactness and anti-sequential compactness which was already being actively pursued (in different terminology) by Ivan Reilly and M.K. Vamanamurthy, and of anti-lindelöf spaces and similar ideas. Reilly and Vamanamurthy and their co-workers have played a central role in subsequent developments on anti-P spaces, which have mostly focussed on the cases in which P is either a compactness condition or a separation/regularity axiom.

We shall now give a detailed exposition of the elements of the general theory of the "anti-"operation; this is derived from Bankston's article, together with presentational details of our own which have proved useful in explaining the ideas to our colleagues and to each other. This is followed by a survey of what has been established (and by whom) about anti-P spaces for specific invariants P, to be read in conjunction with the list of references which we have endeavoured to make fairly complete. Lastly we comment on our perceptions of possible further developments, including projects we have in hand and some open problems.

The "Anti-"operation in general

Bankston's operation is generally viewed as acting on classes of spaces (closed under homeomorphism) rather than on homeomorphic invariants (identified whenever co-extensive). This is unlikely to cause any confusion since one may identify each invariant with the class of all spaces possessing it. We adopt the convention of using the same name for the invariant and the class, but spelling the latter with a capital letter. Thus, for example, 'compact $\Rightarrow$ lindelöf'and 'Compact $\subseteq$ Lindelöf'are to be regarded as interchangeable. We denote by $\aleph_0$ the cardinality of a countably infinite set. Contrary to the practice of most writers on this topic we choose not to allow 0 as the cardinal number of a topological space, believing it more in keeping with general conventions to insist that topologies be defined only on non-empty sets.

Each invariant P partitions the positive cardinals into (at most) three subclasses spec(P), proh(P) and ind(P): for if α is a cardinal then either *every* space of cardinality α is a P space, or *none* is, or *some* are and some are not; we shall accordingly describe α as *specific*, or *prohibitive*, or *indecisive* for P, and the three types of cardinal constitute the subclasses to which we refer. Clearly $1 \notin$ ind(P), but this is the only restriction on the resulting division:

Proposition. *Let $\{\mathcal{S}, \mathcal{P}, \mathcal{I}\}$ be a pairwise-disjoint covering of the class of positive cardinals. There is a homeomorphic invariant P for which $\mathcal{S}$=spec(P) and $\mathcal{P}$ = proh(P) and $\mathcal{I}$ = ind(P) if and only if $1 \notin \mathcal{I}$.*

Proof Given that $1 \notin \mathcal{I}$, choose for each $i \in \mathcal{I}$ a space X_i of cardinality i. If we call X a P space when *either* X is homeomorphic to one of the X_i chosen *or* the cardinality $|X|$ of X is in $\mathcal{S}$, then P is as required.

It is usually an easy exercise to identify these subclasses for a given invariant. We illustrate the ideas by the following table of simple but important examples. Note that spec(P) is known as the *spectrum* of P.

P	spec(P)	ind(P)	proh(P)
Connected	$\{1\}$	$[2, \ )$	ϕ
Perfect	ϕ	$[2, \ )$	$\{1\}$
Compact	$[1, \aleph_0)$	$[\aleph_0, \ )$	ϕ
Lindelöf	$[1, \aleph_0]$	$(\aleph_0, \ )$	ϕ
Infinite	$[\aleph_0, \ )$	ϕ	$[1, \aleph_0)$
Normal	$\{1, 2\}$	$[3, \ )$	ϕ

A space X is called *anti-P* if, for every P subspace Y of X, we have $|Y| \in$ spec(P). By Anti-P we understand the class of all such spaces. One sees immediately that anti-connected $\equiv$ totally disconnected, that anti-perfect $\equiv$ scattered, and that anti-compact is as described earlier. Bankston's key observations on Anti-P in general are now summarised:

Theorem.
(i) Anti-P $\neq \phi$.
(ii) Anti-P is hereditary; that is, membership of the class is a hereditary invariant.
(iii) If P is hereditary then (a) spec(P) $\subseteq$ *spec(Anti-P), (b) P* $\subseteq$ *Anti-Anti-P.*
(iv) Anti-P $\subseteq$ *Anti-Anti-Anti-P.*
(v) Suppose spec(P) = spec(Q); then P $\subseteq$ *Q implies Anti-Q* $\subseteq$ *Anti-P.*

Proof (i) Bearing in mind that $1 \notin$ ind(P), the one-point spaces either are or are not P: and in either eventuality they are anti-P.

(ii) is immediate from the definition.

(iii)(a) Assuming P to be hereditary, consider a (P) subspace Y of a space X for which $|X| \in$ spec(P). Any space Z having the same cardinality as Y will be a subspace of some X_1 with $|X_1| = |X|$; thus Z will be P because X' is, and we deduce that $|Y| \in$ spec(P). This shows that X is anti-P, and so $|X| \in$ spec(Anti-P).

(b) Now let Y be an anti-P subspace of a P space X. With P being hereditary, Y must be a P subspace of itself, which implies $|Y| \in$ spec(P). In view of (a) this shows X to be anti-anti-P.

(iv) is immediate from (ii) and (iii)(b).

(v) again follows directly from the definitions.

Much use has been made of part (v) in particular, for it transpires that there are many sets of important invariants having the same spectrum and being implicationally related: for instance, compact and countably compact; lindelöf and σ-compact; T_0, T_1, T_2, T_3 and T_4.

The enunciation of the other major result in this area, due to Brian Scott, also appears in [4]. It asserts that every hereditary class, apart from one easily-recognised set of exceptions, is of the form Anti-P for some suitably chosen P. As far as we can determine, its demonstration has never been published, so we were obliged to prove it for ourselves. Subsequently Bankston sent us a lot of information on the origin of the topic, which we acknowledge with sincere gratitude, including details of Scott's proof. Since it differs substantially (albeit not radically) from our own, and in view of the utility and striking elegance of the conclusion, we outline below our method of verification.

Scott's Theorem. *Let Q be a non-empty hereditary class, and consider the following condition:*

there is a positive integer n such that n $\in$ *spec(Q) and n + 1* $\in$ *proh(Q) . . . (*)*

(I) If (*) holds, then Q is not of the form Anti-P.
(II) If (*) does not hold, then
(i) Q is of the form Anti-P, and
(ii) we can arrange that P shall have empty spectrum.
(III) If, further, proh(Q) = ϕ*, then Q = Anti-(Not Q).*

Proof (I) Suppose that (*) holds and that Q = Anti-P. Every space of cardinality less than n is a subspace of some n-element space, and is therefore Q; thus $\{1, 2, \ldots n\} \subseteq$ spec(Q). Now each (n+1)-element set X is not anti-P, and must contain a P subspace Y with $|Y| \notin$ spec(P); contradictions follow whether Y is proper or $Y = X$.

(III) follows directly from the definitions, since

$$\text{spec(Not } Q) = \text{proh}(Q)$$

in general. In proving (II) we may therefore assume that proh(Q) has a least element β, and will be of the form $[\beta, \ )$ since Q is hereditary.

Case 1: ind(Q) $= \phi$. Here, β must be infinite. We propose calling a space X *β-schizoid* if $|X| = \beta$ and *either* it is T_0 *or* β is the least upper bound of the cardinalities of the point-closures $\{\overline{x}\}$, $x \in X$. It is easily verified that spec(β-Schizoid)$=\phi$ and that every space of cardinality not less than β possesses a β-schizoid subspace; it follows that Anti-β-Schizoid$=Q$.

Case 2: ind(Q) $\neq \phi$, and possesses a least element α. The hereditary character of Q forces spec(Q) $= [1, \alpha)$, ind(Q) $= [\alpha, \beta)$. If β is infinite we can compromise between the two previous choices, by taking P to comprise the non-Q spaces with cardinalities in $[\alpha, \beta)$ together with the β-schizoid spaces. If on the other hand β is finite, choose a $(\beta - 1)$-element non-Q space Y, form a β-element space Y^* by adjoining a point to Y in any fashion, and take P to comprise the non-Q spaces with cardinalities in $[\alpha, \beta)$ together with all β-element spaces except those homeomorphic to Y^*. In both eventualities it is easily seen that spec(P) $= \phi$ and that Anti-$P = Q$.

Problem: under what conditions on Q will there exist a *hereditary* property P such that $Q =$ Anti-P?

Specific Anti-Properties

We shall first summarise what is known about anti-properties derived from compactness and from related conditions, under four sub-headings: sources of examples, operations which preserve the conditions, characterizations, and implications between them.

Examples Let N denote the set of positive integers. For any p in $\beta N \backslash N, N \cup \{p\}$ is anticompact [28]. So are the discrete, the co-countable, the included-point topologies, and the topology of decreasing sets on N [16]. The *MI* spaces of [11], of which many connected T_2 examples can be constructed [1], are anti-λ compact for every cardinal λ, and therefore anti-compact and anti-lindelöf in particular [14,26]. The T_1 'P'-spaces (those in which each countable intersection of open sets is open) are anti-compact [4], and exist in abundance (see, e.g., [3], [20]). An example is known of a connected $T_{3\frac{1}{2}}$ anti-compact space [4].

Preservation Open bijections, finite products, and finite unions of open *or* of closed subsets preserve both anti-compactness [16] and anti-sequential compactness [12,22]. Indeed, arbitrary unions of open subsets preserve anti-compactness, and so any locally anti-compact space is anti-compact [21]. "Compact covering" maps (continuous maps for which each compact set in the range is the image of a compact set) preserve anti-compactness, and likewise the anti-lindelöf property is preserved by "lindelöf covering" maps, as well as by open bijections and finite products [4]. Infinite products preserve these properties only when cofinitely many factors degenerate [4,22], but stronger preservation behaviour obtains for various modified products (box products, topological ultraproducts, see [4] for details).

Characterizations The condition:

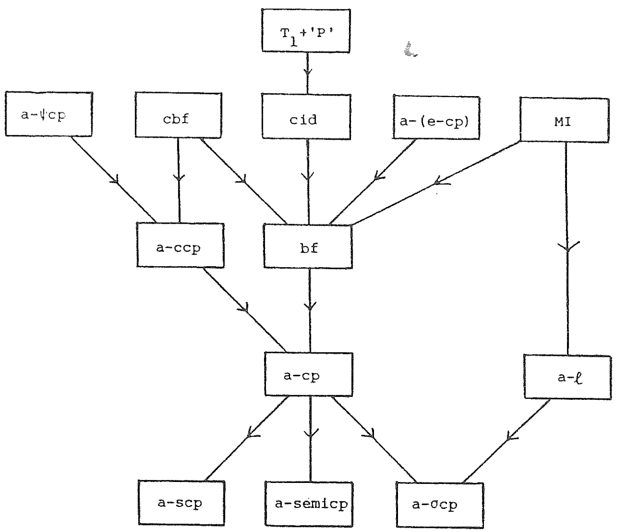
for each point p and each **infinite** *set A, p has an* **open** *neighbourhood G such that A\G is not* **compact**

is equivalent to anti-compactness [22]. If we change *compact* to 'countably compact'/'finite'we obtain characterizations of anti-countable compactness [23]/anti-sequential compactness [22]. Alternatively, change *infinite* to 'uncountable'and *compact* to 'lindelöf'and a local characterization of anti-lindelöf spaces arises [23]. An analogous description [24] of anti-semicompactness is founded by replacing *open* by 'semi open'and *compact* by 'semi compact'.

The condition 'no sequence of distinct points has a convergent subsequence'is necessary and sufficient for anti-sequential compactness [22]. Among first-countable spaces, anti-compactness and anti-sequential compactness coincide, and are then identified by each point possessing a finite neighbourhood [12] or, equivalently, by no sequence of distinct points having a cluster-point[16]. Among T_2 spaces, the anti-compact ones are those whose co-compact reflections are cofinite: for further details on this and related matters see [8]. Anti-anti-λ compact and anti-anti-anti-λ compact spaces (for regular cardinals λ) have also been characterized in [23]: the most striking results being that anti-anti-compact/lindelöf is the same as hereditarily compact/lindelöf, and that every T_2 space is anti-anti-anti-compact. This is echoed by the observation that the anti-anti-semi compact spaces are the hereditarily semi compact ones [24]. The Stone-Čech remainder $\beta X \backslash X$ of a T_4 space X is anti-compact if and only if Y is T_4 for all Y such that $X \subseteq Y \subseteq \beta X$ [5].

Implications

The 'mainstream'connections (see [23], also [12]) are: anti-pseudocompact ($a - \psi cp$) implies anti-countably compact ($a - ccp$) implies anti-compact ($a - cp$) implies both anti-sequentially compact ($a - scp$) and anti-σ compact ($a - \sigma cp$), anti-lindelöf ($a - \ell$) implies $a - \sigma cp$, $a - cp$ implies anti-semicompact [24]. Many recent developments [15] centre on the ideas of a *bounded* subset of a topological space, i.e. one for which each open cover of the space has a finite subcover of the set, and of a *bf* space as one in which every bounded set is finite; the *cbf* spaces are defined similarly but considering only *countable* open covers; the *essentially-compact* ($e - cp$) spaces [10] are those possessing a bounded dense subset. Lastly, the *cid* spaces ([25]; also called A-spaces [21]) are those in which each countably infinite subset is discrete. The relationships between these concepts are summarised in the following diagram.



The converse implications are generally false, and counterexamples will be found in the literature cited. Known partial converses are: $a - scp$ implies $a - cp$ for first-countable spaces [12] and for spaces of small infinite cardinality ([22], see also [17]); $a - cp$ implies bf for regular spaces and for those in which every bounded set is closed [15]: bf implies anti-($e - cp$) for T_1 spaces [15].

There is also a small group of results indicating 'how close'certain types of space are to being discrete: we mention $a - cp + T_1 +$ first-countable implies discrete [16], $a - cp + T_2$ k-space implies discrete [4], $a - scp + T_1 +$ sequential implies discrete [22]. Miscellaneous implications include $a - cp +$ first-countable implies saturated [12], $a - \ell + a - scp + T_2$ implies $a - cp$, and $a - cp + T_1$ implies anti-pathconnected [4]; *cid* implies anti-separated, and anti-anti-compact implies anti-*cid* implies finite *or* not T_2 [25]. Also a T_2 space in which every set with dense interior is open must be $a - ccp$ [27], a space whose compact subsets have empty interiors and whose topology is maximal with respect to that property is $a - cp$ [18], and anti-anti-bf does not imply bf [15].

Lastly we examine the total negation of the axioms of separation and regularity, where in contrast to the preceding discussion the situation is almost disappointingly simple. In [23] it is shown that the anti-T_0 spaces are the trivial (indiscrete) ones,while for $P = T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4, T_5$, metrizable or discrete, anti-P coincides with nested (each two open sets are comparable). Also anti-trivial equals T_0 and anti-nested equals T_1. For $P =$ complete regularity, regularity, or the weaker axioms R_1 or R_0 [6], the anti-P spaces are those which are T_0 and nested. The anti-normal spaces, identified in [9], comprise the one- and two-point spaces and the unique non-normal three-point space.

Future developments

The area is quite rich in unsolved (even unposed) problems and possible lines of further investigation. Many standard invariants as yet have no published characterizations of their total negations: we have for example encountered no references to those of local compactness, paracompactness, realcompactness, local connectedness, connectedness *im kleinen*, first countability or complete separability, nor (apart from the result mentioned above) to that of separability. Our own preliminary investigations show that some at least of these can be dealt with fairly simply; a sample conclusion is that the anti-completely separable spaces are precisely the finite ones. Turning to separation axioms, nothing appears to be known of the 'anti's'of those lying in logical strength between T_1 and T_2 [28]. We have checked several between T_0 and T_1 [2], where the evidence is that the trivial spaces form the anti-class of most if not all of the known ones. An intriguing set of questions in the general theory concern whether it is possible to find a class P so that the classes in the sequence

P, Anti-P, Anti-Anti-P, Anti-Anti-Anti-P, ...

are all distinct, or include infinitely many distinct ones, or include arbitrarily long lists of distinct ones. We have been able to answer these in the negative by proving that, for any choice of P, the sequence contains at most four distinct classes, which recur in one of seven simple patterns [19]. Finally, it could be worth raising corresponding questions (if it has not already been done) in areas other than topology. Is there a significant body of knowledge concerning anti-abelian groups? anti-distributive lattices? anti-noetherian rings? anti-precompact uniformities?

References

[1] D. R. Anderson, *On connected irresolvable Hausdorff spaces* Proc. Amer. Math. Soc. **16** 1965 463-466

[2] A. E. Aull, W. J. Thron *Separation axioms between T_0 and T_1* Indag. Math. **24** 1962 26-37

[3] P. Bankston *Ultraproducts in topology* General Topology and Appl. **7** 1977 283-308

[4] P. Bankston *The total negation of a topological property* Illinois J. Math. **23** 1979 241-252

[5] D. Barr, D. Hajek *Normality inducing spaces* General Topology and Appl. **7]** 1977 89-97

[6] A. S. Davis *Indexed systems of neighbourhoods for general topological spaces* Amer. Math. Monthly **68** 1961 886-893

[7] M. Ganster, I. L. Reilly, M. K. Vamanamurthy *On spaces whose denumerable subspaces are discrete* Mat. Vesnik **39** 1987 283-292

[8] D. B. Gauld, M. Mršević, I. L. Reilly,M.K. Vamanamurthy

Colindelöf topologies and ℓ-continuous functions Glasnik Mat. **19** 1984 297-308

[9] D. B. Gauld, I. L. Reilly, M. K. Vamanamurthy *On normal subspaces* Bull. Austral. Math. Soc. **23** 1981 1-4

[10] S. H. Hechler *On a notion of weak compactness in non-regular spaces* Studies in Topology, Proc. Conf. Charlotte;eds. N. Stravrakas, K. Allen, Academic Press 1975

[11] E. Hewitt *A problem of set-theoretic topology* Duke Math. J. **10** 1943 309-333

[12] B. Hutton, I. L. Reilly *On compactness and finiteness in topological spaces* Mat. Vesnik **13** 1976 59-61

[13] D. S. Janković, I. L. Reilly, M. K. Vamanamurthy *On strongly compact topological spaces* Questions Answers Gen. Topology **6** 1988 29-40

[14] M.R. Kirch *A class of spaces in which compact sets are finite* Amer. Math. Monthly **76** 1969 42.

[15] P. T. Lambrinos, I. L. Reilly, M. K. Vamanamurthy *Boundedness and finiteness* Boll. Un. Mat. Ital. (6) **5-A** 1986 103-110

[16] N. Levine*On the equivalence of compactness and finiteness in topology* Amer. Math. Monthly **75** 1968 178-180

[17] N. Levine *On compactness and sequential compactness* Proc. Amer. Math. Soc. **54** 1976 401-402

[18] N. Levine *On spaces nowhere locally compact* Kyungpook Math. J. **22** 1982 167-174

[19] J. Matier,T.B.M. McMaster *Iteration of Bankston's 'Anti'operation"*. (submitted).

[20] A. K. Misra *A topological view of P-spaces* General Topology and Appl. **2** 1972 349-362

[21] H. B. Potoczny*A class of spaces in which compactness and finiteness are equivalent* Mat. Vesnik **36** 1984 159-160

[22] I. L. Reilly, M. K. Vamanamurthy*Compactness and finiteness* Colloq. Math. Soc. János Bolyai **23** 1978 1033-1041

[23] I. L. Reilly, M. K. Vamanamurthy*Some topological anti-properties* Illinois J. Math. **24** 1980 382-389

[24] I. L. Reilly, M. K. Vamanamurthy*On semi compact spaces* Bull. Malaysian Math. Soc. **(7)2** 1984 61-67

[25] I.L. Reilly,M.K. Vamanamurthy *On spaces in which every denumerable subspace is discrete* Mat. Vesnik **38** 1986 97-102
[26] P. L. Sharma *The Lindelöf property in MI-spaces* Illinois J. Math. **25** 1981 644-648
[27] P. L. Sharma *A class of spaces in which compact sets are finite* Canad. Math. Bull. **24** 1981 373-375
[28] A. Wilansky *Between T_1 and T_2* Amer. Math. Monthly **74** 1967 261-266

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A SURVEY OF SUBNORMAL SUBGROUPS

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Introduction

Since the appearance of Helmut Wielandt's fundamental paper [27] over fifty years ago, much progress has been made in the theory of subnormal subgroups thanks to the contribution of many distinguished group theorists.

A comprehensive and masterly exposition of the theory of subnormal subgroups is due to Lennox and Stonehewer. The purpose of this article, based on a talk given at "Groups in Galway" is to present some of the remarkable results in the theory without encumbering the general reader with technical details or proofs. The selection of topics is not exhaustive and reflects a bias of the author, but it is hoped to whet the appetite of the reader, who is referred to Lennox and Stonehewer [12] in the first instance. Notation is standard and follows that of Lennox and Stonehewer [12] or Robinson [22].

Definition. *If H is a subgroup of a group G such that*

$$x^{-1}Hx = H \quad \forall x \in G$$

then H is normal in G, written $H \lhd G$.

If $L \lhd H$ and $H \lhd G$ it does not follow that $L \lhd G$, i.e. for subgroups of a group normality is not a transitive relation, as can be verified by examining the alternating group on 4 letters, $\mathcal{A}_4$, for instance. This may serve as motivation for the following relation on subgroups which *is* transitive:

Definition. *A subgroup H is subnormal in a group G if H occurs as a term in a finite normal series*

$$H = H_m \lhd H_{m-1} \lhd \cdots \lhd H_0 = G \qquad (*)$$

from H to G, where $H_i \lhd H_{i-1}$ for each i.

Notation: $H \lhd\lhd G$ or also H *sn* G will mean H is subnormal in G.

Definition. *The length of the shortest normal series from H to G is called the defect of H in G, written $def(G, H)$.*

Definition. *The normal closure of H in G, H^G, is $\langle H^g | g \in G\rangle$, the group generated by all the conjugates of the subgroup $H (H^g := g^{-1}Hg)$, and this is of course a normal subgroup of G. It is the smallest normal subgroup of G which contains H.*

Thus if H is subnormal of defect m in G, we see that

$$H^G \leq H_1 \lhd H_0 = G.$$

Replacing G by H_1 we get

$$H^{H_1} \leq H_2 \lhd H_1$$

Iterating this process, we see that by taking successive normal closures, in at most m steps the process terminates with arrival at $H_m = H$.

Notation: Put $H_{(0)} = G$, set $H_{(i+1)} = H^{H_{(i)}}$ then $H_{(1)} = H^G \leq H_1$ (as in (*) above) $H_{(2)} = H^{H_{(1)}} \leq H^{H_1} \leq H_2$ and in general $H_{(i)} \leq H_i$, the *ith* term in the normal series (*).

The series $H_{(i)}$ is the most rapidly descending series from G to H, and $H \lhd\lhd G$ if and only if $H = H_{(i)}$ for some $i \geq 0$.

Examples:

(i) All normal subgroups of a group are subnormal of defect 1.

(ii) Subgroups of order 2 in $\mathcal{A}_4$ have defect 2 in $\mathcal{A}_4$.

(iii) In $D_{2^m} \simeq \langle a, b | a^{2^{m-1}} = b^2 = 1, b^{-1}ab = a^{-1}\rangle, m \geq 3$, the non-central subgroups of order 2 have defect precisely $m - 1$.

Some elementary facts concerning subnormal subgroups are:
(i) Transitivity: If $H \lhd\lhd K$ and $K \lhd\lhd G$ then $H \lhd\lhd G$.
(ii) If $H \lhd\lhd G$ and θ is a homomorphism of G onto G^θ, then $H^\theta \lhd\lhd G^\theta$, moreover $def(G^\theta, H^\theta) \leq def(G, H)$.
(iii) If $H \lhd\lhd G$ and K is a subgroup of G then

$$H \cap K \lhd\lhd K \quad \text{and} \quad def(K, H \cap K) \leq def(G, H)$$

(iv) If $H \lhd\lhd G$ then $N_G(H)$, the normalizer of H in G, $\{g | g^{-1}Hg = H\}$ is strictly larger than H i.e. $N_G(H) > H$ (the converse of this fact is false in general). Of course $N_G(H)$ is the largest subgroup of G which contains H as a *normal* subgroup.
(v) If each member of a (finite) collection of subgroups H_j is subnormal of defect at most k in G, then

$$\bigcap_j H_j$$

is also subnormal of defect at most k in G. (If the defects of the H_j are not bounded then an example on p.373 of Robinson [22] shows that (v) is false without this condition.)

Some of these facts (ii), (iii), (iv), (v) are valid also for normal subgroups.

On the other hand, whereas normal subgroups have the property of permuting with elements of G ($N \lhd G$, then $Ng = gN$) and thus with subgroups of G ($N \lhd G, H \leq G$ then $NH = HN$), this is not usually the case for subnormal subgroups. Another fact concerning two normal subgroups N_1, N_2 of a group G is that N_1N_2 is also normal in G. This leads to the earliest but most famous question concerning subnormal subgroups.

The Join Problem

Let H, K be subnormal in G. Under what circumstances will $\langle H, K\rangle$—the join of H and K—be subnormal in G? This problem has been the subject of intense research, starting with Wielandt's fundamental paper in 1939 [27] and culminating with a remarkable result of J.P. Williams [32].

In 1939 Wielandt proved that if G is finite and H, K are subnormal in G then $\langle H, K\rangle$ is also subnormal in G. Thus the set of subnormal subgroups of a finite group G forms a sublattice of the lattice of all subgroups of a finite group G. In fact Wielandt showed the result was true provided G satisfied the maximal condition for subnormal subgroups *max-sn* whereby every strictly ascending chain of subnormal subgroups of G has finite length, (so every non–empty set of subnormal subgroups of G contains a maximal member). In the case of G finite there is an elegant proof due to Kegel, using induction on the order of G, see [12] p.8.

A criterion to guarantee subnormality of $\langle H, K\rangle$ when $H, K \lhd \lhd G$ is that K should normalize H i.e. $K \leq N_G(H)$ (**) One shows that K normalizes $H_{(i)}$ and since $K \lhd \lhd H_{(i-1)}K$ and $H_i \lhd H_{(i-1)}K$ one obtains $H_{(i)}K \lhd \lhd H_{(i-1)}K$ with defect $\leq def(G, K)$. In fact $def(G, J) \leq def(G, H) \cdot def(G, K)$. The next result is useful also.

Lemma. *Let $H, K \lhd \lhd G$. Put $J = \langle H, K\rangle$ then the following are equivalent:*
(i) $J \lhd \lhd G$,
(ii) $H^K (= \langle H^k | k \in K\rangle) \lhd \lhd G$ and
(iii) $[H, K](= \langle [h, k](= h^{-1}k^{-1}hk), h \in H, k \in K\rangle) \lhd \lhd G$.

Since $[H, K] \lhd H^K \lhd J$ one only has to show (iii) $\Rightarrow$ (i). (See [12] p.4.)

If G is a nilpotent group of class c, that is, if

$$[\underbrace{G, G, \ldots, G}_{c+1 G's}] = \langle 1\rangle$$

then every subgroup of G has defect at most c. In particular if $G' = [G, G]$ is nilpotent then from part (iii) of the Lemma above it follows that for any subnormal subgroups H, K of G their join is also subnormal. A generalization of Wielandt's result is due to Robinson [19].

Theorem (Robinson). *Let $H, K \lhd \lhd G$ and suppose G' satisfies $max - sn$. Then $J := \langle H, K\rangle \lhd \lhd G$.*

Another easy criterion—a companion to (**)—is the following:

Lemma. *Let $H, K \lhd \lhd G$ and put $J = \langle H, K\rangle$. If $HK = KH$ then $J \lhd \lhd G$.*

(Of course, $KH = HK$ does not imply that $K \leq N_G(H)$.)

So it is clearly of interest to the join problem to determine conditions under which H and K permute. (If J equals HKH then in fact J equals HK, but a counterexample due to R.S. Dark [12] p.20 shows that one can have $H, K \lhd \lhd G, J = HKHK$ but J not subnormal in G!) A famous permutability criterion is due to Roseblade [23].

Theorem (Roseblade). *If H and K are subnormal subgroups of a group G such that the tensor product of the abelian groups (regarded as $\mathbf{Z}$-modules)*

$$H/H' \bigotimes K/K'$$

is trivial (one says H is orthogonal to K, written $H \perp K$) then $HK = KH$, and thus $\langle H, K\rangle \lhd \lhd G$. Moreover if H and K are not orthogonal then there exists a group G_0 such that $H \simeq H_0 \lhd \lhd G_0, K \simeq K_0 \lhd \lhd G_0$ and $H_0K_0 \neq K_0H_0$.

In 1958, Zassenhaus [33] published an example [Exercise 23, Appendix D in his book "Theory of Groups"] showing that the join of two subnormal subgroups could fail to be subnormal. The group G constructed by Zassenhaus consisted of a module with a specially defined basis, over $\mathbf{Z}$, extended by suitably chosen automorphisms. This group was countable and abelian by nilpotent of class 2 i.e. G/A was nilpotent of class 2, with A abelian. Two subnormal subgroups H, K each had defect 3 in G and their join $\langle H, K\rangle$ was nilpotent of class 2 but not subnormal in G since one shows $J^G = G$. (See also Robinson [22] p.375.) It is worth remarking that in an example H cannot have defect 2 since then $H \lhd H^G \lhd G$ and so $H^K \lhd H^G \lhd G$, K normalizes H^K and one would get $J = \langle H, K\rangle \lhd \lhd G$. So in an example the defect of $H(K)$ must be at least three which is the case in the Zassenhaus example. Also G' is not abelian and neither is J. Thus in some respects Zassenhaus' example is the minimum one can get away with!

To conclude with a necessary *and* sufficient criterion to ensure that the join of a pair of subnormal subgroups is subnormal, the following is a result of J.P. Williams [32]:

Theorem (Williams). *Let H, K be groups:*
(i) If $H/H' \otimes K/K'$ (as an abelian group) is the (direct) sum of a group U of finite rank and a periodic divisible group V (c.f. Robinson [22] pp. 94–97) then $\langle H, K\rangle$ is subnormal in all groups in which H and K can be subnormally embedded.
(ii) Conversely if $H/H' \otimes K/K'$ does not have the structure in (i) as an abelian group, then there is a group G containing H, K as subnormal subgroups such that $\langle H, K\rangle$ is not subnormal in G.

The proof of this theorem (which is the subject of chapter 5 in [12]) involves extensive development of ring-theoretic machinery first introduced by Philip Hall in the 1950's.

The Wielandt Subgroup

In [1] Baer defined the "Kern" of a group as the intersection

$$\bigcap_{H \leq G} N_G(H)$$

of the normalizers of all the subgroups of G. In 1958, Wielandt [28] considered an analogous intersection

$$w(G) = \bigcap_{S \lhd\lhd G} N_G(S)$$

i.e. the intersection of the normalizers of all the subnormal subgroups of G.

Whereas $w(G)$ may equal $\langle 1\rangle$ as in the case of $G = D_\infty$ the infinite dihedral group, Wielandt proved the following rather surprising results [28].

Theorem (Wielandt). *If $|G|$ is finite then $w(G) \neq \langle 1\rangle$.*

Theorem (Wielandt). *Let G be an arbitrary group. Then $w(G)$ contains*
(i) every simple non–abelian subnormal subgroup of G and

(ii) every minimal normal subgroup M of G where M satisfies the minimal condition for subnormal subgroups. $(min - sn)$.
[Indeed, if G satisfies $min - sn$ then Robinson [20] has shown that $|G : w(G)|$ is finite.]

If G is a finite group, then $w(G) \neq \langle 1\rangle$. Consequently

$$w(G/w(G)) \neq \langle 1\rangle.$$

Setting $w_0(G)) = \langle 1\rangle$ and $w_{i+1}(G)/w_i(G) := w(G/w_i(G))$, for some finite i one will have $w_i(G) = G$. The smallest such i is called the *Wielandt length* of the group G.

The Wielandt subgroup has been the subject of a paper by Camina [4] in which he investigates relations between the Wielandt length, derived length and Fitting length for a finite soluble group G. This work has been improved by Bryce and Cossey [3] who obtain best possible bounds for both the derived and Fitting length of a finite soluble group in terms of its Wielandt length. Casolo [8] has extended these results to infinite soluble groups of finite Wielandt length. Another result concerning the Wielandt subgroup due to Brandl, Franciosi and de Giovanni [2] is the following

Theorem. *Let G be a polycyclic group (G has a normal series with each factor cyclic) which is either*
(a) metanilpotent (an extension of a nilpotent group by a nilpotent group) or
(b) abelian by finite. Then $w(G)/Z(G)$ is finite.

The Wielandt subgroup also has the property that since $w(G) \lhd G$, a subnormal subgroup K of $w(G)$ is subnormal in G, hence $N_G(K) \geq w(G)$ thus $K \lhd w(G)$. In other words, $w(G)$ is a group in which normality *is* a transitive relation. Such groups are called T-groups and for groups in this class $G = w(G)$, and all subnormal subgroups of G have defect 1. Finite soluble T–groups have been classified by Gaschütz [9], and Robinson [18] has shown that in fact every soluble T–group is metabelian.

Groups with every subgroup subnormal

Of course, if G is an abelian group, then every subgroup of G is subnormal. More interesting is the case of non–abelian groups with every subgroup subnormal.

Suppose first G is a non–abelian group with every subgroup **normal**, then G is a non–abelian Dedekind group and the structure of G is described in [Robinson, [22] p.139]; for instance Q_8 is an example of a non–abelian Dedekind group.

In view of the fact that if G is a nilpotent group of class c then every subgroup of G is subnormal with defect at most c, Roseblade [24] was able to show that if G is a group in which every subgroup is subnormal of defect at most d then there is a function $f(d)$ such that G is nilpotent of class at most $f(d)$. A specific result of Heineken [10] and Mahdahvianary [14] in this area is the following: If all cyclic subgroups of G have defect at most 2, then G is nilpotent of class 3.

However, a celebrated example due to Heineken & Mohamed [11] shows that there are groups G in which every subgroup is subnormal (but there are no bounds on the defects) and $Z(G) = 1$, so G is not even hypercentral. Moreover, for $H < G$ one has $N_G(H) > H$, i.e. G satisfies the so-called *normalizer condition*, whereby every proper subgroup of G is a proper subgroup of its normalizer.

Casolo [6] has shown that if G is a group with every subgroup subnormal, then for some n, $G^{(n)} = G^{(n+1)}$, i.e. the derived series breaks off after finitely many terms. Recently Möhres [16] proved that a group G with every subgroup subnormal is in fact soluble. It would appear that groups in the class of groups with every subgroup subnormal, are in fact metanilpotent.

The class of B_n groups G in which subnormal subgroups have defect at most n has been investigated. B_1 groups are the aforementioned T–groups, in which normality is a transitive relation. Since simple groups trivially are B_1–groups, one restricts attention to soluble B_n–groups. An interesting problem is to try and bound (if possible) the derived lengths of soluble B_n–groups in terms of n. Examples due to Robinson [21], show that even in the class of B_2–groups all derived lengths can occur (using a construction based on iterated wreath products) but these groups are torsion free. In the case of periodic soluble B_2–groups Casolo [5] has shown that they have derived length at most 10. In the case of finite soluble B_2–groups Casolo has shown that they have derived length at most 5 and Fitting length at most 4. An example of McCaughan and Stonehewer [13] shows that this last result of Casolo is best possible. Thus there is much scope for investigating the interrelationship between the derived length and Fitting length of soluble periodic B_n–groups.

Criteria for Subnormality

Let H be a subgroup of G. Suppose that $HK = KH$ for any subgroup K of G, we say H is a *permutable* subgroup of G or H is *quasinormal* in G, written H *per* G. Clearly normal subgroups are permutable. Not every subnormal subgroup is permutable (one can verify this by examining A_4). Not every permutable subgroup is normal, if $G = \langle a, b | a^8 = b^2 = 1, b^{-1}ab = a^5 \rangle$ then $\langle b \rangle$ *per* G, but $\langle b \rangle$ is not normal in G. Ore [17] showed that a maximal permutable subgroup of a group G is normal in G and as a corollary one obtains that a permutable subgroup H of a finite group G is subnormal in G. This corollary has been extended by Stonehewer [25] to finitely generated groups.

A more restrictive form of permutability is that of permuting with conjugates i.e. $VV^g = V^gV$ for all $g \in G$. A result of Ore [17] and Szép [26], is that in a finite group G such a subgroup V which permutes with its conjugates is subnormal in G. Wielandt [29] has considered similar criteria and the following Theorem is due to him.

Theorem (Wielandt). *Let G be a finite group and A, B subgroups of G such that*

$$AB^x = B^xA \quad \text{for all} \quad x \in G.$$

Then
(i) If $G = AB^G = BA^G$ then $G = AB$.
(ii) $A^B \cap B^A \lhd\lhd G$.
(iii) If $AB \leq H \leq G$ then $A^H \cap B^H \lhd\lhd G$.
(iv) If X, Y are subsets of G then $[A^X, B^Y] \lhd\lhd G$.

Another result of Wielandt's follows: First note that if $N \lhd G$, then (i) $N \lhd \langle N, g\rangle \forall g \in G$ and (ii) $[n, g] \in N \forall g \in G$. Moreover the converse of (i) or (ii) implies $N \lhd G$. If $H \lhd\lhd G$ then clearly (i)$'$ $H \lhd\lhd \langle H, g\rangle \forall g \in G$ and (ii)$'$ for any $g \in G$ and some positive integer n and $h \in H$, $[g, \underbrace{h, \dots, h}_{n\ h's}] \in H$. Wielandt [30] has shown that for finite groups (i)$'$ and (ii)$'$ are each sufficient for H to be subnormal in G.

Theorem (Wielandt). *Suppose $H \leq G$ and G is finite. Then the following are equivalent to $H \lhd\lhd G$:*
(i) $H \lhd\lhd \langle H, g\rangle \quad \forall g \in G$
(ii) $H \lhd\lhd \langle H, H^g\rangle \quad \forall g \in G$.
(iii) $H \lhd\lhd \langle H, H^{h^g}\rangle \quad \forall h \in H, g \in G$.

To conclude our survey we mention the **subnormalizer** of a subgroup. The normalizer of a subgroup H in a group G, $N_G(H) = \{g \in G | H \lhd \langle H, g\rangle\}$, and $H \lhd G \Leftrightarrow N_G(H) = G$. Consider the following:

Definition: Denote by $S_G(H) = \{g \in G | H \lhd\lhd \langle H, g\rangle\}$ the *subnormalizer* (in G) of the subgroup H.

One must note that in general $S_G(H)$ is not a subgroup! Take $H = \langle (12)(34), (13)(24), (14)(23), (12)\rangle$ and $K = \langle (23)(45), (24)(35), (25)(34), (25)\rangle$. Then $|H| = |K| = 8$ and they are Sylow 2–subgroups of $\mathcal{S}_5$, the symmetric group of degree 5. $|H \cap K| = 2$ and $H \cap K = \langle (34)\rangle$ is not subnormal in $\mathcal{S}_5$. Of course $H \cap K \lhd\lhd H, K$. If the subnormalizer of $H \cap K$ were a subgroup, it would contain H and K and hence would contain $\langle H, K\rangle$ which is $\mathcal{S}_5$, a contradiction, because $H \cap K$ is not subnormal in $\mathcal{S}_5$. In this example $HK \neq KH$ as subsets. The following result shows when the subnormalizer *is* a subgroup.

Theorem (Maier [15], Wielandt [31]). *Suppose G is a finite group and $G = AB$ with $A, B \leq G$. If $H \lhd\lhd A$ and $H \lhd\lhd B$ then $H \lhd\lhd G$.*

Thus in a finite group G whenever $H \lhd\lhd U, H \lhd\lhd V (H, U, V$ subgroups of $G)$ implies $H \lhd\lhd \langle U, V\rangle$ then $S_G(H)$ *is* a subgroup. Wielandt [31] has formulated a number of conjectures regarding criteria for subnormality of a subgroup H in the finite group G where $G = AB$ for subgroups A and B.

The Class of s_n–groups

Call G an s_n–group if the subnormalizer of every subgroup of G is itself a subgroup. For any element x in a group G, denote by $E_G(x) = \{g \in G | [g, \underbrace{x, \dots, x}_{n\ x's}] = 1$ for some $n \in \mathbb{N}\}$. If we denote by E–groups the class of groups in which $E_G(x)$ is a subgroup for every x in G then a recent result due to Casolo [7] is the following:

Theorem (Casolo). *Let G be a finite group. Then G is an s_n–group if and only if G is an E–group.*

In addition, Casolo has proved that a finite group G is an s_n–group if and only if the intersection of any two Sylow subgroups of G is pronormal in G, whereby a subgroup H is pronormal in G if H is conjugate to H^g in $\langle H, H^g\rangle$. Thus as the reader can see, there are new areas of investigation in the theory of subnormal subgroups which yield surprising connexions with other topics in group theory such as Engel elements or pronormality.

References

[1.] R. Baer, *Der Kern, eine charakteristische Autogruppe*, Compositio Math. 1 (1934) 254–283.

[2.] R. Brandl, S. Franciosi and F. De Giovanni, *On the Wielandt subgroup of infinite soluble groups*, Glasgow Math. J. 32 (1990) 121–125.

[3.] R. A. Bryce and J. Cossey, *The Wielandt subgroup of a finite soluble group*, J. London Math. Soc. (2) 40 (1989) 244–256.

[4.] A. Camina, *The Wielandt length for finite groups*, J. Algebra, 15 (1970) 8–142.

[5.] C. Casolo, *Gruppi finite risolubili in cui tutti i sottogruppi subnormale hanno difetto al più 2*, Rend. Sem. Mat. Univ. Padova 71, (1984) 257–271.

[6.] C. Casolo, *Groups in which all subgroups are subnormal*, Rend. Acad. Nax. Sci. 10 (1986) 247–249.
[7.] C. Casolo, *Finite groups in which subnormalizers are subgroups*, Rend. Sem. Mat. Univ. Padova 82 (1989) 25–53.
[8.] C. Casolo, *Soluble groups with finite Wielandt length*, Glasgow Math. J. (1989) 329–334.
[9.] W. Gaschütz, *Gruppen, in denen das Normalteilersein transitivist*, J. Reine Angew. Math. 198 (1957) 87–92.
[10.] H. Heineken, *A class of three-Engel groups*, J. Algebra 17 (1971) 341–5.
[11.] H. Heineken and I. J. Mohamed, *A group with trivial centre satisfying the normalizer condition*, J. Algebra 10 (1968) 368–76.
[12.] J.C. Lennox and S.E. Stonehewer, *Subnormal Subgroups of Groups*, Clarendon Press, Oxford, 1987.
[13.] D.J. McCaughan and S.E. Stonehewer, *Finite soluble groups whose subnormal subgroups have defect at most two*, Arch. Math. (Basel) 35 (1980) 56–60.
[14.] S.K. Mahdavianary, *A special class of three-Engel groups*, Arch. Math. (Basel) 40 (1983) 193–9.
[15.] R. Maier, *Um problema da teoria dos subgrupos subnormais*, Bol. Soc. Brasil Mat. 8 (1977) 127–30.
[16.] W. Möhres, *Torsionsfreie Gruppen deren Untergruppen alle subnormal sind*, Math–Ann. 284 (1989) 245–249.
[17.] O. Ore, *On the application of structure theory to groups*, Bull. Amer. Math. Soc. 44 (1938) 801–6.
[18.] D. J. S. Robinson, *Groups in which normality is a transitive relation*, Proc. Cambridge Philos. Soc. 60 (1964) 21–38.
[19.] D. J. S. Robinson, *Joins of subnormal subgroups*, Illinois J. Math. 9 (1965) 144–68.
[20.] D. J. S. Robinson, *On the theory of subnormal subgroups*, Math. Z. 89 (1965) 30–51.
[21.] D. J. S. Robinson, *Wreath products and indices of subnormality*, London Math. Soc. (3) 17 (1967) 257–70.
[22.] D. J. S. Robinson, *A course in the theory of groups*, New York–Heidelberg–Berlin: Springer 1982.
[23.] J. E. Roseblade, *The permutability of orthogonal subnormal subgroups*, Math. Z. 90 (1965) 365–72.

[24.] J. E. Roseblade, *On groups in which every subgroup is subnormal*, J. Algebra 2 (1965) 402–12.
[25.] S. E. Stonehewer, *Permutable subgroups of infinite groups*, Math Z. 125 (1972) 1–16.
[26.] J. Szép, *Bemerkung zu einem Satz von O. Ore*, Publ. Math. Debrecen 3 (1953) 81–2.
[27.] H. Wielandt, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Z. 45 (1939) 209–44.
[28.] H. Wielandt, *Über den Normalisator der subnormalen Untergruppen*, Math Z. 69 (1958) 463–5.
[29.] H. Wielandt, *Vertauschbarkeit von Untergruppen und Subnormalität*, Math. Z. 133 (1973) 275–6.
[30.] H. Wielandt, *Kriterien für Subnormalität in endlichen Gruppen*, Math. Z. 138 (1974) 199–203.
[31.] H. Wielandt, *Subnormalität in faktorisierten endlichen Gruppen*, J. Algebra 69 (1981) 305–11.
[32.] J.P. Williams, *Conditions for subnormality of a join of subnormal subgroups*, Proc. Cambridge Philos. Soc. 92 (1982) 401–17.
[33.] H. Zassenhaus, *The theory of groups*, 2nd ed., New York: Chelsea 1958.

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EXACTNESS IN ELEMENTARY DIFFERENTIAL EQUATIONS

Robin Harte

ABSTRACT *A simple pattern from linear algebra is present in linear differential equations, recurrence relations and matrix theory.*

If $T : X \to Y$ and $S : Y \to Z$ are abelian group homomorphisms we shall call the pair (S,T) *(left,right) one-one* ([3] Ch 10) if there is inclusion

0.1 $$S^{-1}(0) \subseteq T(X),$$

and *exact* if in addition

0.2 $$ST = 0.$$

Sufficient for (0.1) is that there are homomorphisms $T' : Y \to X$ and $S' : Z \to Y$ for which

0.3 $$S'S + TT' = I;$$

when $T : X \to Y$ and $S : Y \to Z$ are continuous homomorphisms of topological groups, or linear between vector spaces, we shall require that S' and T' are also continuous, or linear. If in particular

0.4 $$T = \begin{pmatrix} A \\ B \end{pmatrix} : X \to \begin{pmatrix} X \\ X \end{pmatrix}\ ,\ S = (-B \quad A) : \begin{pmatrix} X \\ X \end{pmatrix} \to X$$

then condition (0.1) takes the form

0.5 $$Ax = By \Longrightarrow x = Bz\ ,\ y = Az\ ,$$

while condition (0.2) reduces to commutivity

0.6 $$BA = AB.$$

From (0.5) it follows in particular that

0.7 $$B^{-1}(0) \subseteq A\, B^{-1}(0)\ ,$$

and hence also that

0.8 $$(BA)^{-1}(0) \subseteq B^{-1}(0) + A^{-1}(0)\ .$$

Already this captures a familiar observation [2],[4] about linear equations with constant coefficients : with $D : X \to X$ the operation of differentiation on the space $X = C^\infty(\Omega)$ of infinitely differentiable real, or complex, functions on an open interval $\Omega \subseteq \mathbf{R}$, we have

Theorem 1. *If $p = qr$ is the product of polynomials q and r without nontrivial common factors then*

1.1 $$p(D)^{-1}(0) = q(D)^{-1}(0) + r(D)^{-1}(0).$$

Proof. The Euclidean algorithm gives polynomials q',r' for which

1.2 $$q'q + r'r = \mathrm{hcf}(q,r) = 1\ ;$$

since everything commutes we can now argue, with $A = q(D)$, $B = r(D)$, $A' = q'(D)$ and $B' = r'(D)$,

$$By = 0 \Longrightarrow y = AA'y \text{ with } BA'y = A'By = 0$$

and hence

$$BAx = 0 \Longrightarrow x = (I - A'A)x + A'Ax \in A^{-1}(0) + B^{-1}(0).$$

This is inclusion one way in (1.1), and the reverse is clear •

Something very similar to Theorem 1 is relevant to elementary matrix theory: if $p(A) = 0$ and $p = qr$ with $\mathrm{hcf}(q, r) = 1$ then [4]

1.3 $$q(A)^{-1}(0) = r(A)(X).$$

With $q = (z - \lambda)^k$ and $r(\lambda) \neq 0$ this shows that the *eigenvectors* of A lie in the column spaces of related polynomials $r(A)$. The conditions (0.2) and (0.3) say something about the solution of equations with coefficients S or T:

Theorem 2. *If*

2.1 $$S'S + TT' = I \text{ and } ST = 0$$

and

2.2 $$T'T + WW' = I \text{ and } TW = 0$$

then

2.3 $$Tx = b \Longrightarrow x = T'b + WW'b \in T'b + T^{-1}(0)$$

and

2.4 $$x = T'b \Longrightarrow Tx = (I - S'S)b.$$

Proof. Clear•

The operations of differentiation and integration fit together in the pattern of (0.2) and (0.3): if $0 \in \Omega$ define operators D, D' and J on the space $X = C^\infty(\Omega)$ by setting

2.5 $$(Dx)(t) = \frac{dx(t)}{dt} \ ; \ (D'x)(t) = \int_{s=0}^{t} x(s)ds \ ; \ (Jx)(t) = x(0) \ ;$$

then evidently

2.6 $$DD' = I = D'D + J \text{ with } DJ = 0 = JD' \ .$$

If $f \in X$ is arbitrary define multiplications L_f and E_f by setting

2.7 $$(L_f x)(t) = f(t)x(t) \ ; (E_f x)(t) = e^{f(t)}x(t) \ ;$$

then also

2.8 $$L_g L_f = L_{gf} = L_f L_g \ ; \ JL_f = L_{Jf}J \ ; \ JE_f = E_{Jf}J$$

and

2.9 $$\begin{aligned} E_{-f}E_f &= I = E_f E_{-f} \ ; \\ DL_f &= L_f D + L_{Df} \ ; \\ DE_f &= E_f(D + L_{Df}). \end{aligned}$$

It is clear that we can take $T = D$ in Theorem 2 to obtain the familiar form of the solution of the equation $Dx = f$; the same extends to the *first order linear* equation:

Theorem 3. *If* $T = D + L_{Df}$ *then*

3.1 $$TT' = I = T'T + WW' \text{ with } TW = 0$$

with

3.2 $$T' = E_{-f}D'E_f \ ; \ W = E_{-f}J \ ; \ W' = E_f \ .$$

Proof. Again clear•

For second and higher order linear equations there is the technique of *variation of parameters*: we claim that this also can be described by Theorem 2 . The ideas are clear from equations of order two:

Theorem 4. *If* $T = D^2 + L_p D + L_q$ *is second order linear with*

4.1 $$T^{-1}(0) = D^{-1}(0)f + D^{-1}(0)g$$

then

4.2 $$TT' = I = T'T + WW' \text{ with } TW = 0,$$

where

$$T' = (\,L_f \quad L_g\,)\begin{pmatrix} D'L_h & 0 \\ 0 & D'L_h \end{pmatrix}\begin{pmatrix} -L_g \\ L_f \end{pmatrix};$$

4.3 $$W = (\,L_f J \quad L_g J\,);$$

$$W' = L_h\begin{pmatrix} L_{Dg} & -L_g \\ -L_{Df} & L_f \end{pmatrix}\begin{pmatrix} I \\ D \end{pmatrix}$$

with

4.4 $$1/h = \det H \ \textit{with}\ H = \begin{pmatrix} f & g \\ Df & Dg \end{pmatrix}.$$

Proof. We follow the usual "variation of parameters" argument, noting that the *Wronskian matrix* H must be invertible, and in effect make the familar substitution:

4.5 $$L_H\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} I \\ D \end{pmatrix}\ ,\ \text{giving}\ L_H\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ T \end{pmatrix}.$$

It follows

$$\begin{pmatrix} D'DU \\ D'DV \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix} L_H^{-1}\begin{pmatrix} 0 \\ T \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} L_\phi \\ L_\psi \end{pmatrix} T$$

giving

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} L_\phi \\ L_\psi \end{pmatrix} T + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

and hence

$$I = (\,L_f \quad L_g\,)\begin{pmatrix} U \\ V \end{pmatrix}$$

4.6 $$= (\,L_f \quad L_g\,)\begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix}\begin{pmatrix} L_\phi \\ L_\psi \end{pmatrix} T +$$

$$(\,L_f \quad L_g\,)\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}.$$

This gives $T'T + WW' = I$; it is left to the reader to check that $TW = 0$ and $TT' = I$•

Of course the coefficients p and q in the operator T are determined by the complementary functions f and g:

4.7 $$L_H\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -D^2 & 0 \\ 0 & -D^2 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}.$$

We can make a similar analysis of *recurrence relations*. Define operators U and V on the space of all real sequences by setting

4.8 $$(Vx)_n = x_{n+1},\ (Ux)_0 = 0 \text{ and } (Ux)_{n+1} = x_n\ :$$

these are the backward and forward *shifts*, and satisfy

4.9 $$VU = I = UV + K \text{ with } KU = 0 = VK\ ,$$

where $(Kx)_0 = x_0$ and $(Kx)_{n+1} = 0$. If we introduce operators L_p, E_p and M by setting

4.10 $$\begin{aligned} (L_p x)_n &= p_n x_n, \\ (E_p x)_0 &= x_0, \\ (E_p x)_{n+1} &= p_0 p_1 \ldots p_n x_{n+1}, \\ (Mx)_n &= x_0 x_1 \ldots x_n \end{aligned}$$

then

4.11 $$L_p E_p = L_{Mp},\ VL_p = L_{Vp}V,\ UL_p = L_{Up}U,\ VE_p = L_p E_p V$$

and hence

4.12 $$(V - L_p)E_p = L_p E_p (V - I).$$

The *first order linear* recurrence relation is the equation $(V - L_p)x = q$:

Theorem 5. *If* $T = V - L_p$ *then*

5.1 $$TT' = I = T'T + E_p J \text{ with } TE_p J = 0,$$

where $(Jx)_n = x_0$ *and*

5.2 $$T'(x_0, x_1, x_2, \ldots) = (0, x_0, p_1 x_0 + x_1, p_2 p_1 x_0 + p_2 x_1 + x_2, \ldots).$$

Proof. Clear•

We can see analogy with differential equations if we write

5.3 $D = V - I,\ D' = SU = S - I$ where $(Sx)_n = x_0 + x_1 + \ldots + x_n$,

giving

5.4 $$DD' = I = D'D + J \text{ with } DJ = 0 = JD' .$$

If p_n never vanishes we have

5.5 $$T' = E_p S E_p^{-1} U = WU$$

where

5.6 $$\begin{aligned} &W(x_0, x_1, x_2, x_3, \ldots) = \\ &(x_0, p_0x_0 + x_1, p_1p_0x_0 + p_1x_1 + x_2, \\ &\quad p_2p_1p_0x_0 + p_2p_1x_1 + p_2x_2 + x_3, \ldots) \end{aligned}$$

Note also

5.7 $$VJ = J,\ JU = 0,\ SU = US,\ (V - I)S = V.$$

In higher dimensions suppose $\Omega \subseteq \mathbf{R}^3$ is open connected and "starlike" with respect to $0 \in \mathbf{R}^3$, and look at ([1] Ch 5 §3) *differential forms*

5.8 $$w = w_0 + \sum_{r=1}^{3} \sum_{|j|=r} w_j dx_j \ (w_j \in X = C^\infty(\Omega))$$

and *differentiation*

5.9
$$D : w_0 + \sum_{r=1}^{3} \sum_{|j|=r} w_j dx_j \to \sum_{i=1}^{3} D_i w_0 dx_i + \sum_{i=1}^{3} \sum_{r=1}^{2} \sum_{|j|=r} D_i w_j dx_{ij} \ ;$$

here $(D_1 f)(a) = \lim_{t \to 0}(f(a_1 + t, a_2, a_3) - f(a_1, a_2, a_3))/t$ etc. (partial differentiation) and $dx_{ij} = dx_i \wedge dx_j$ (exterior multiplication). Diagrammatically:

5.10
$$X \xleftarrow{(D_3\ \ -D_2\ \ D_1)} \begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}-D_2 & D_1 & 0\\ -D_3 & 0 & D_1\\ 0 & -D_3 & D_2\end{pmatrix}} \begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}D_1\\D_2\\D_3\end{pmatrix}} X$$

Since the D_j commute, this sequence forms a "chain". The *homotopy* H is derived from *multiplications* L_j and *moments* S_j (weighted radial averages), given by

5.11 $$(L_j f)(a) = a_j f(a),\quad (S_j f)(a) = \int_{t=0}^{1} t^{j-1} f(ta)dt \ ;$$

specifically

$$Hw_0 = 0 \ ;$$

5.11
$$H(w_1dx_1 + w_2dx_2 + w_3dx_3) = (S_1w_1)x_1 + (S_1w_2)x_2 + (S_1w_3)x_3 \ ;$$

$$\begin{aligned} H(w_{12}dx_{12} + w_{13}dx_{13} + w_{23}dx_{23}) = {} & \\ &(S_2w_{12})(x_1dx_2 - x_2dx_1) \\ &+ (S_2w_{13})(x_1dx_3 - x_3dx_1) \\ &+ (S_2w_{23})(x_2dx_3 - x_3dx_2) \ ; \end{aligned}$$
$$H(w_{123}dx_{123}) = (S_3w_{123})(x_1dx_{23} - x_2dx_{13} + x_3dx_{12}) .$$

Diagramatically:

$$\begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}-L_2S_2 & -L_3S_2 & 0\\ L_1S_2 & 0 & -L_3S_2\\ 0 & L_1S_2 & L_2S_2\end{pmatrix}} \begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}L_3S_3\\ -L_2S_3\\ L_1S_3\end{pmatrix}} X$$

and

5.12 $$X \xleftarrow{(L_1S_1\ \ L_2S_1\ \ L_3S_1)} \begin{pmatrix}X\\X\\X\end{pmatrix}$$

Theorem 6

6.1 $$HD + DH = I - J$$

where

$$J(w_0 + \sum_{r=1}^{3} \sum_{|j|=r} w_j dx_j) = w_0(0)\underline{1}. \tag{6.2}$$

Proof. Note the commutation rules

$$D_iD_j - D_jD_i = S_iS_j - S_jS_i = L_iL_j - L_jL_i = 0 \;;$$

$$\sum_{i=1}^{3} L_iS_1D_i = I - J \;;\; \sum_{i=1}^{3} L_iS_{k+1}D_i = I - kS_k \text{ if } k \geq 1 \;; \tag{6.3}$$

$$D_iL_j = \delta_{ij}I + L_jD_i \;;\; D_iS_k = S_{k+1}D_i \;;\; L_jS_{k+1} = S_kL_j \;.$$

References

[1.] D. G. Edelen, *Applied exterior calculus* (Wiley 1985).
[2.] M. Gonzalez, Null spaces and ranges of polynomials, *Pub. Mat. Univ. Barcelona* **32**, (1988), 167-170
[3.] R. E. Harte, *Invertibility and singularity* (Dekker 1988)
[4.] R. E. Harte, Cayley-Hamilton for eigenvalues, *Bull. Irish Math. Soc.* **22**, (1989), 66-68.
[5.] R. E. Harte, *Taylor exactness and the Apostol jump* Bull. Irish Math. Soc. **24** (1990), 20-40.

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ON SOME MATHEMATICAL WORKS IN THE LIBRARY OF THE ROYAL IRISH ACADEMY.

G.L. Huxley, M.R.I.A.

In the course of two centuries the Academy's library has grown steadily. In Irish subjects it has become one of the best collections of manuscripts and printed matter in the country. Other fields of knowledge are also well represented: this short paper draws attention to our mathematical holdings in the hope that mathematicians and historians of mathematics will be encouraged to make greater use of the books and periodicals at Academy House.

Most of the mathematical texts and periodical articles have been obtained by gift or by exchange. Consequently coverage of domains within the subject is far from complete or coherent. Many areas are, however, represented, and the geographical range of periodicals is remarkable. There are long runs of the *Acta* and other serial publications of leading European academies—for example, from Paris, Berlin, Rome, Göttingen, Heidelberg, and Helsinki. Our holdings of the *Philosophical Transactions* of the Royal Society of London extend back to Volume 1 (1665/6). The St. Petersburg *Commentarii* begin in the age of Euler and the set of Liouville's *Journal de Mathématique pures et appliqueés* continues until 1924 from the first issue in 1836. There are strong collections of Japanese periodicals published in English, among them numerous editions of *Tensor* and the *Hiroshima Mathematical Journal*. Current work in the United States is well represented: we receive the *Proceedings of the National Academy of Sciences*, the Princeton *Annals of Mathematics*, the *Duke Mathematical Journal*, and, among other leading periodicals, *Studies in Applied Mathematics* (Cambridge, Massachusetts).

From Eastern Europe come *Colloquium Mathematicum* (Warsaw), *Acta Mathematica Hungarica* (Budapest), the *Czechoslovak Mathematical Journal*; and our intakes from the U.S.S.R. and associated territories are substantial. Outstanding are the *Doklady* of the Academy of Sciences (Moscow and Leningrad) and the Academy's Mathematical *Isvestiya*.

From Georgia and Armenia numerous publications have been received. Irish exchanges with the University of Kazan are of long standing. Matter arrives steadily from Holland and New Zealand, from Spain and from India, and from many other countries. The accessions reflect the international standing of the Academy and the wide distribution of section A of the *Proceedings*. A cherished property is a series of Crelle's *Journal* from Volume 1 (1836) to 144 (1914).

There are few modern mathematical textbooks in the library, but mathematical classics are of permanent value—and not only to historians; and our assemblages benefited greatly from the bequest in 1987 of modern works belonging to the late Professor J. G. Semple. The many books of Geometry reflect the strength of Ireland in the subject in the nineteenth century. The powerful mind of George Salmon is well attested by treatises—for instance, *A Treatise on Higher Plane Curves* (second edition, Dublin 1873); we possess a signed third edition of his *Treatise on the Analytic Geometry of Three Dimensions* (Dublin 1874) and a sixth edition of his *Treatise on Conics* (London 1879). Euclidean Geometry appears in a classic historical exposition by G.J. Allman, *Greek Geometry from Thales to Euclid* (Dublin and London 1889). John Casey's *The First Six Books of the Elements of Euclid* (Dublin and London 1882) contains at p. 249 M'Cullagh's proof of the minimum property of the line named after Philo of Byzantium (expressing two mean proportionals between two given lines). Casey's *Sequel* to the previous work (fourth edition, Dublin and London 1886) exhibits a rigour such as would be welcome in some more recent school-geometries. The Academy possess a copy of G. Monge's *Géometrie descriptive* (fourth edition, Paris 1820) and a Russian translation, with commentary, of the same treatise by A.I.Kargin (Moscow and Leningrad 1947).

Also present is A.M.Legendre's *Éléments de Géometrie* (twelfth edition, Paris 1823).

The history of non-Euclidean Geometry can be studied in several works: we have V.F.Kagan's biography of Lobachevsky (Moscow and Leningrad 1944); Kagan also edited a selection of Lobachevsky's studies in the theory of parallel lines (*Issledovania*, M.-L. 1945). A *Libellus* (Claudiopolis (Clug) 1902) commemorating the centenary of the birth of the younger Bolyai includes an *index* (by R. Bonola) *operum ad Geometriam absolutam spectantium*. Among the books of the Semple bequest is H.S.M. Coxeter, *Non-Euclidean Geometry* (Toronto 1942).

The holdings in logic and foundations are significant. Aptly prominent is George Boole: we have a reprint of his *Mathematical Analysis of Logic* (Oxford 1951, originally Cambridge 1847) and a collection, edited by R. Rhees, of Boole's studies in logic and probability (London 1952). Instructive also for historians is Desmond MacHale, *George Boole. His Life and Work* (Dublin 1985). In the same domain is Kurt Gödel, *On Formally Undecidable propositions of Principia Mathematica and Related Systems* (English translation, Edinburgh and London 1962). Another classic of mathematical logic is Boole's *An Investigation of the Laws of Thought*; the Academy is fortunate to possess a first edition (London 1854). Jan Lukasiewicz lectured in the Academy from 1946 onwards on mathematical logic: we have his *Aristotle's Syllogistic from the Standpoint of modern Formal Logic* (Oxford 1951).

Historians of the theory of numbers will find much of interest in the Library. A rare work is B.N. Delone, *The Petersburg School of the Theory of Numbers* (Moscow and Leningrad 1947). Another valued property is the collected *Oeuvres* of P.L.Tchebychef in two volumes (St. Petersburg 1907 and 1899). (The Academy's Russian links, of long standing, are indicated also by the presence of L.Euler's *Opuscula Analytica* I (Petropoli 1783)). Happily the Library has Euler's three volumes of *Dioptrica* also published at St. Petersburg (1769, 1770, 1771). Not only antiquarians will be pleased to find a copy of C.F.Gauss, *Disquisitiones Arithmeticae* (Leipzig 1801) and, perhaps, not only analysts will study with pleasure the papers on Abelian functions and on differential equations in Karl Weierstrass, *Mathematische Werke* I (Berlin 1894); but few nowadays are likely to find easy the notation in Edward Waring's *Miscellanea Analytica de Aequationibus Algebraicis et*

Curvarum Proprietatibus (Cambridge 1762). The translation into Russian of mathematical treatises by Al-Farabi and their publication in Kazakhstan at Alma Ata in 1972 serve to emphasize that the study of the history of mathematics knows no frontier.

A few details in conclusion, chosen not quite at random from selective explorations, will show that the intellectual profit from the use of the Library can be great. Here is C. Huygens studying the ancient problem of the quadrature of the circle (see *Oeuvres Complètes*, Volume 20, The Hague 1940). Here is evidence that Newton supposed the Creator to have made parts of absolute space impenetrable (see his *Opuscula*, Volume I, ed. J. Castillioneus, p.xxxiii, Lausanne and Geneva 1734). Here is Salmon handsomely giving credit to Boole for his part in originating the principles of linear transformation in modern algebra (George Salmon, *Lessons introductory to the Modern Higher Algebra*, third edition, Dublin 1876, p.103). Here is a copy of the third edition of Newton's *Principia* that once graced a library in Ballinlough. And here are collected works of W. Rowan Hamilton, J.J. Larmour, F. Severi and others.

Mathematical practitioners who wish to study in the Library are invited to ask the staff about rules and registration. The Academy believes that the mathematical aspects of its Library deserve to be better known; accordingly an increase in the number of mathematically interested readers would be a welcome development.

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Honorary Librarian.

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PROBABILITY IN FINITE SEMIGROUPS

Desmond MacHale

Let S be a finite non-empty set and let $*$ be a closed binary operation on S. For $x \in S$ let $C(x) = C_S(x)$, the centralizer of x in S, be $\{y \in S | x * y = y * x\}$, the set of all elements of S which commute with x. We define $Pr.(S)$ to be $\sum_{x \in S} |C(x)|/|S|^2$ so that $Pr.(S)$ is the probability that a pair of elements of S, chosen at random, will commute with each other.

Clearly, for x, $y \in S$, $x \in C(x)$, and $x \in C(y)$ if and only if $y \in C(x)$, but apart from these trivial restrictions there are no other restrictions on the values $Pr.(S)$ may have. Thus $1 \geq Pr.(S) \geq 1/|S|$ and the size of $Pr.(S)$ is a good indication of "how commutative" $\{S, *\}$ is, since $Pr.(S) = 1$ if and only if S is commutative.

If $\{G, *\}$ is a group then there are severe restriction on the values that $Pr.(G)$ may assume. For example we have the following (see [1] and [2])

(i) If $Pr.(G) > \frac{5}{8}$ then $Pr.(G) = 1$.
(ii) If $Pr.(G) > \frac{1}{2}$ then $Pr.(G) = \frac{1}{2} + \frac{1}{2^{2k+1}}$ for some k.
(iii) It is not possible to have $\frac{7}{16} < Pr.(G) < \frac{1}{2}$.

The bound given in (i) is the best possible and is attained for example by D_4, the group of all symmetries of the square.

At the lower end of the scale it is possible to make $Pr.(G)$ as small as we please in absolute terms, though not as small as $1/|G|$ unless G is trivial.

An easy calculation shows that $Pr.(S_3) = \frac{1}{2}$, where S_3 is the group of all permutations on three objects, and it is not difficult to show that $Pr.(A \times B) = Pr.(A).Pr(B)$ for the direct product of groups A and B. Consider then $G = S_3 \times S_3 \times S_3 \times \cdots \times S_3$, the direct product of n copies of S_3. $Pr.(G) = \frac{1}{2^n}$, which tends to zero as n gets large.

If $\{R, *\}$ is the multiplicative semigroup of a ring $\{R, +, *\}$, then again there are severe restrictions on the values $Pr.(R)$ can assume (see [2]).

Among these restriction we mention the following:

(i) $Pr.(R) \leq \frac{5}{8}$ for a non-commutative ring R.
This bound is attained by the following two rings of matrices over $\mathbf{Z}_2$

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad \text{for all } a, b, c \in \mathbf{Z}_2 \right\}.$$

Note that the second of these rings is a ring with unity.

(ii) If p is the least prime dividing $|R|$ then

$$Pr.(R) \leq \frac{1}{p^3}(p^2 + p - 1),$$

with equality if and only if $(R : Z(R)) = p^2$, where $Z(R)$ is the centre of R.

In this note we concentrate on the case where $\{S, *\}$ is a semigroup and we show that a finite semigroup can be as commutative or as noncommutative as we like.

For each $n \geq 4$ we show that there is a semigroup T_n of order n with $Pr.(T_n) = (n^2 - 2)/n^2$, which is as large as possible. For $n \geq 4$ let $T_n = \{a_1, a_2, \ldots, a_n\}$ and define a binary operation $*$

on T_n by $a_n * a_{n-1} = a_2$, with all other products equal to a_1. For example, $\{T_4, *\}$ looks like this.

$*$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_2	a_1	a_1	a_1
a_3	a_1	a_2	a_1	a_1
a_4	a_1	a_1	a_2	a_1

$\{T_n, *\}$ is closed and $(x * y) * z = a_1 = x * (y * z)$ for all x, y, $z \in T_n$, so $\{T_n, *\}$ is a semigroup. It is easy to see that $Pr.(T_n) = (n^2 - 2)/n^2$ and this fraction can be made as close as we like to 1 by taking n large enough.

At the lower end of the scale we show that for each n there exists a semigroup W_n with $Pr.(W_n) = \frac{1}{n}$, which is as small as possible.

Let $W_n = \{b_1, b_2, \ldots, b_n\}$ and define a binary operation $\bullet$ on W_n by $b_i \bullet b_j = b_j$, for all i, j. For example $\{W_3, \bullet\}$ is given by the following table.

$\bullet$	b_1	b_2	b_3
b_1	b_1	b_2	b_3
b_2	b_1	b_2	b_3
b_3	b_1	b_2	b_3

$\{W_n, \bullet\}$ is closed and for all x, y, $z \in W_n$

$$(x \bullet y) \bullet z = y \bullet z = z = x \bullet z = x \bullet (y \bullet z),$$

So $\{W_n, \bullet\}$ is a semigroup. Further $b_i \bullet b_j = b_j \bullet b_i \iff b_i = b_j$. So $Pr.(W_n) = \dfrac{1}{n}$.

Problems

1. Given a rational number $0 < \frac{m}{n} < 1$ does there exist a semigroup S with $Pr.(S) = \frac{m}{n}$? Note that $Pr.(S_1 \times S_2) = Pr.(S_1)Pr.(S_2)$ for the direct product of semigroups so that given some values of $\frac{m}{n}$ we can generate others.

2. Are there any restrictions on $Pr.(S)$ for other algebraic structures such as inverse semigroups, near-rings, bands or groupoids?

There are many other questions that can be asked about probability in finite algebraic systems. For example, we ask "what is the probability that an element of a semigroup S has an inverse?" We call this probability $I(S)$. If S does not have identity then $I(S) = 0$ so we assume S has identity e. We ask the following question: For each N, is it possible to choose semigroups of order n which satisfy $I(S) = \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots \frac{n}{n}$?

It is easy to show that for each n we can achieve the value $I(S) = \frac{1}{n}$, as follows. The case $n = 1$ is trivial, so suppose that $n \geq 2$. Consider a semigroup K of order $n-1$ such that k does not have identity (for example, put every product equal to a fixed element). Then adjoin to K an element e such that $e \bullet x = x \bullet e = x$ for all $x \in K$ and $e \bullet e = e$.

Then $I(K \cup \{e\}) = \frac{1}{n}$.

Also, we can achieve $\frac{n}{n} = 1$ because there is always a group of order n, namely the cyclic group $\{\mathbf{Z}_n, \oplus\}$. Thus for $n = 2$ we can achieve $\frac{0}{2}, \frac{1}{2}, \frac{2}{2}$. We show that for $n = 3, 4$ the other values are also achievable.

For $n = 3$, $\{\mathbf{Z}_3, \otimes\}$ gives $I(S) = \frac{2}{3}$.

(In fact, for p a prime $I\{\mathbf{Z}_p, \otimes\} = \frac{p-1}{p}$, since $\mathbf{Z}_p - \{0\}$ is a group under $\otimes$.)
For $n = 4$, $\{\mathbf{Z}_4, \oplus\}$ gives $I(S) = \frac{2}{4}$ while $\{GF(4), \otimes\}$ gives $\frac{3}{4}$.

In fact $\{GF(p^n), \otimes\}$ gives $\frac{p^n-1}{p^n}$ for any prime power p^n. In general $\{\mathbf{Z}_n, \otimes\}$ gives $\phi(n)/n$, where $\phi(n)$ is the Euler ϕ-function.

3. Find semigroups of order 5 for which the values $\frac{2}{5}$ and $\frac{3}{5}$ are achieved.

Note that $I(A \times B) = I(A)I(B)$ for the direct product of semigroups A and B and this fact can be of use in generating the *values* required in problem 3, though not necessarily among semigroups of order 5. Finally, we can consider $I(R)$ where R is a finite ring with unity. Let $[n]$ be the greatest integer function. We quote the following theorem found in [3].

Theorem. *If* $I(R) > \frac{1}{|R|^2}(|R| - [\sqrt{R}])$, *the* $I(R) = \frac{1}{|R|}(|R| - 1)$ *and R is a finite field.*

$\mathbf{Z}_{p^2}$ for p a prime shows that this result is best possible.

References

[1]. W. H. Gustafson, *What is the probability that two group elements commute?* American Math. Monthly **80** (1973) 1031–1034.
[2]. D. MacHale, *Commutativity in finite rings,* American Math. Monthly **83** (1976) 30–32.
[3]. D. MacHale, *Wedderburn's Theorem revisited.* Bull. Irish Math. Soc **17** (1986) 44–46.

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BOOK REVIEW

Introduction to Linear Algebra (2nd Edition)

L. W. Johnson, R. D. Riess & J. T. Arnold
Addison-Wesley ISBN 0-201-16833-2, 1989

In the preface to "Introduction to Linear Algebra," the authors stress the importance of the book's subject matter as a component of undergraduate mathematics particularly for scientific, engineering and social science undergraduates. This book is a carefully planned and well presented text book on linear algebra. Containing 7 chapters and almost 600 pages, it strives to approach the subject at two levels. At the practical level, matrix theory and the related vector-space concepts provide a language and a powerful computational framework for posing and solving important problems. Beyond the practical level, its treatment of the subject contains a valuable introduction to mathematical abstraction and logical development.

It is at this practical level that the book is particularly attractive. It contains a variety of (optional) applications in the first

three (core) chapters. In addition, the final chapter (Chapter 7) gives a reasonable treatment of a selection of numerical methods in linear algebra and includes Gaussian elimination, the power and inverse power methods for eigenvalue problems, reduction to Hessenberg form and estimation of the eigenvalues of Hessenberg matrices. This "mix" of theory, applications and numerical methods makes the book a very attractive proposition for students taking both elementary and advanced modules in linear algebra as well as a basic course in numerical linear algebra. In the latter case, there is the added attraction of having a range of FORTRAN programs listed in the final chapter.

Chapter 1, entitled *Matrices and Linear Equations* contains a standard but well-presented exposition of Gaussian elimination and matrix algebra. However, it is refreshing to note that the chapter contains a number of illustrating applications. These include the use of matrix methods in data fitting (polynomial interpolation), numerical integration and differentiation.

Chapter 2, *The Vector Space,* R^n provides an introduction to vector-space ideas (subspace, basis, dimension, etc.) in the familiar setting of R^n. In this chapter, the applications include the least-squares problem in R^n, data fitting and least-squares solutions of overdetermined linear systems.

Because of the two-level approach adopted by the authors, certain material such as that in Chapter 3, *The Eigenvalue Problem*, is revisited and considered in greater depth in later chapters. For example, there is a necessity to provide a brief introduction to determinants in Chapter 3 to facilitate the early treatment of eigenvalues but a more complete treatment of determinants (even repeating some of the earlier discussion) is given in Chapter 5. Chapter 3 also contains some (optional) applications. These include difference equations and Markov Chains.

Chapters 4, *Vector Spaces and Linear Transformations* and 5, *Determinants*, follow along traditional lines. For example, the former concentrates on vector spaces and subspaces, linear independence and bases, inner product spaces, linear transformations and their matrix representations. However, this is more than just a traditional approach. The topics are organized so that they flow logically and naturally from the concrete and computational to the more abstract. There is also a wealth of examples to enable the student to gain even further insight into the various concepts.

Chapter 6, *Eigenvalues and Applications*, begins with an introduction to quadratic forms and this is followed, using the treatment of eigenvalues from Chapter 3, by a review of systems of differential equations. The remainder of the chapter is devoted to Hessenberg matrices, Householder transformations, the QR factorization and least-squares solutions, matrix polynomials and the Cayley-Hamilton theorem. The final section considers generalized eigenvectors and solutions of systems of differential equations.

The final chapter, Chapter 7, *Numerical Methods in Linear Algebra*, the contents of which were mentioned earlier, is a welcome addition to a general text on Linear Algebra. If a foundation course on Numerical Linear Algebra is offered in conjunction with more theoretical modules, then this book is sufficiently self-contained for both aspects. The provision of listings of FORTRAN programs is also welcome in relieving the student from time consuming and error prone programming.

The authors advise that an instructor's manual and a student solutions manual are now available. The book itself contains solutions to all the odd-numbered computational exercises while the student solutions manual includes detailed solutions for these exercises. The instructor's manual contains solutions to all of the exercises.

I consider the book to be an attractive proposition for undergraduate training. There are a number of reasons why this is so. It provides a gradual increase in the level of abstraction, it contains an early introduction to eigenvalues. The first three chapters could themselves constitute a core (single-term) module on linear algebra at first or second year undergraduate level.

At all levels in the book, the material is presented very clearly and it is augmented continuously by numerous examples and, in particular sections, by applications of very general appeal. The exercise sets are themselves graduated (or spiralling) and many sections contain exercises that hint at ideas which are developed later in the text. The inclusion of computer awareness and reliable computer programs provide the numerical flavour which, in my opinion, enhances the appeal of a text book on one of the most important components of undergraduate mathematics in our third level colleges.

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PROBLEM PAGE

Editor: Phil Rippon

The first problem this time is elegant, simple to state and yet rather surprising. I heard it first from Tom Laffey, who indicated that it may have an application to checking the accuracy of computer calculations!

25.1 *Let*

$$m * n = mn + [\varphi m][\varphi n],$$

where m, n are positive integers, φ is the golden ratio $\frac{1}{2}(1+\sqrt{5})$ and $[x]$ denotes the integer part of x. Prove that $*$ is associative.

The next problem came from John Toland at the University of Bath. By checking special cases one can 'guess' the solution, but producing a proof is a different matter!

25.2 *What are the eigenvalues of the matrix*

$$\begin{pmatrix} 0 & -(n-1) & & & \\ 1 & 0 & -(n-2) & & 0 \\ & 2 & 0 & & \\ & & & \ddots & \\ & 0 & & 0 & -1 \\ & & & n-1 & 0 \end{pmatrix} ?$$

Finally, here is a tantalising 'find the next term in the sequence' problem, which I heard first from Derek Goldrei here at the OU.

25.3 *Find the next term in the sequence*

$$2, 4, 16, 37, 58, 89, \ldots$$

How do such sequences behave in general?

Now here are the solutions to the problems in Issue 22. Problem 22.1 was concerned with a relative of the Mandelbrot set, which was found by my colleagues David Crowe, Robert Hasson and Peter Strain-Clark. To define this set, consider the recurrence relation

$$z_{n+1}(c) = \overline{z_n(c)}^2 + c, \quad n = 0, 1, 2, \ldots,$$

where c is complex and $z_0(c) = 0$. Without the complex conjugate, such sequences are used to define the Mandelbrot set and so it makes sense to give the name Mandelbar set to

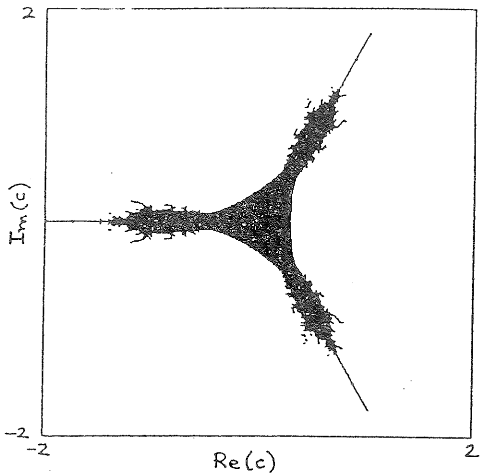
$$M_{\text{BAR}} = \{c : z_n(c) \to \infty \text{ as } n \to \infty\}.$$

By a simple argument, this is equivalent to

$$M_{\text{BAR}} = \{c : |z_n(c)| \le 2, \quad \text{for } n = 1, 2, \ldots\}.$$

22.1 *Prove that M_{BAR} has a rotational symmetry.*

When M_{BAR} is plotted using (1) the following picture appears.



The picture suggests that M_{BAR} is symmetric under a rotation through $2\pi/3$ about the origin, and this can be verified as follows. Let

$$f_c(z) = \overline{z}^2 + c,$$

so that

$$z_n(c) = f_c^n(0),$$

where f_c^n denotes n applications of the function f_c. Now put $\omega = e^{2\pi i/3}$, so that $\overline{\omega}^2 = \omega$, and hence

$$f_{\omega c}(z) = \overline{z}^2 + \omega c = \omega((\overline{z/\omega}^2 + c)) = \omega f_c(z/\omega).$$

By induction, therefore,

$$f_{\omega c}^n(z) = \omega f_c^n(z/\omega),$$

and so, on putting $z = 0$, we obtain

$$z_n(\omega c) = \omega z_n(c).$$

Thus $z_n(\omega c)$ tends to ∞ if and only if $z_n(c)$ tends to ∞, as required.

In fact it turns out that M_{BAR} has been studied for some time by John Milnor, who uses the more descriptive name of **tricorn**, T, for it. The set arose first in connection with the Mandelbrot set for cubics (a subset of $\mathbf{C}^2$), which has been studied in great depth by Milnor, John Hubbard and Bodil Branner. More details about $M_{BAR} = T$ and further references can be found in:

D. Crowe, R. Hasson, P.E.D. Strain-Clark and P.J. Rippon, 'On the structure of the Mandelbar set', Non-linearity **2**(1989), 541-553.

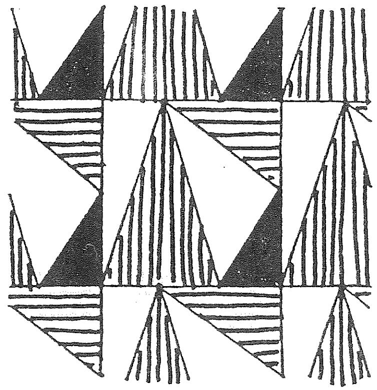
J. Milnor, 'Remarks on iterated cubic maps', Preprint, Stonybrook Institute for Mathematical Sciences.

R. Winters, 'Bifurcations in families of antiholomorphic functions in biquadratic maps', Ph.D Thesis, Boston University, 1990.

22.2 *Prove that it is impossible to tile the plane with triangles in such a way that at most 5 triangles meet at each vertex.*

I came across this problem many years ago while living for a time in West Africa. The local library had just a few maths books, including a Hungarian problem book, which included 22.2. Unfortunately, my wording of 22.2 was not quite precise in view of

examples like the following, which I found when writing up these solutions.



In this example, vertices of some of the triangles are allowed to lie on sides of others. If such a configuration is forbidden, then the problem is correctly posed and the solution goes as follows.

Suppose, if possible, that there does exist a tiling of the plane in which at most 5 triangles meet at each vertex. Consider a large circle and form the union of those triangles in the tiling all of whose vertices lie in or on the circle. This union forms a polygon and we note that at most 3 of the polygon's constituent triangles can meet at one of its boundary vertices. Indeed if 4 of its constituent triangles were to meet at one of its boundary vertices, then there would be exactly one triangle from the tiling outside the polygon at this vertex and this triangle would have to lie in the polygon (because its 3 vertices would be in or on the circle).

Suppose now that the polygon consists of t triangles from the tiling, and that it has e edges and v interior vertices. Summing all the angles of the t triangles and using the fact that the interior angles around the boundary of the polygon sum to $(e-2)\pi$, we obtain the equation

$$t\pi = (e-2)\pi + 2v\pi,$$

so that

$$t = e - 2 + 2v. \tag{1}$$

Counting all the angles of the t triangles, we obtain the inequality

$$3t \leq 3e + 5v, \tag{2}$$

and, from (1) and (2), it follows that $v \leq 6$. This is a contradiction, however, if the circle is large enough, since we also have $t \leq 5v$ because each triangle of the polygon has at least one interior vertex.

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