THE SUBGROUP STRUCTURE OF THE FINITE CLASSICAL GROUPS

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The classification of finite simple groups revealed that non-abelian finite simple groups fall into three distinct families:

(1) the alternating groups A_n, with $n \geq 5$;

(2) the simple groups of Lie type;

(3) 26 sporadic simple groups.

The alternating groups are well known to anyone who has studied group theory at the most elementary level but the sporadic groups are less accessible to non-specialists. Chevalley showed in 1955 how certain simple groups (including finite simple groups) can be constructed as automorphism groups of Lie algebras over arbitrary fields. Chevalley's construction was modified by Steinberg, Ree, Hertzig and others to provide further simple groups (so-called twisted groups). The groups obtained by these procedures are called simple groups of Lie type. While certain of the finite simple groups of Lie type were unknown until these constructions were introduced in the 1950's and early 1960's others turned out to be versions of groups that had been well known since the work of C. Jordan in 1870 and L. E. Dickson in 1901. These are the finite classical simple groups, which are derived from certain groups of automorphisms of vector spaces over finite fields. The groups of automorphisms in question fall into four families: special linear, symplectic, unitary, orthogonal. It may be argued that these four families of linear groups provide the best introduction to the study of finite simple groups and to finite groups in general. Certainly, the techniques of linear algebra, field theory and permutation groups learnt in most undergraduate courses find wide application in the analysis of the classical groups.

The term finite classical group encompasses various groups derived from certain progenitors that we shall now try to describe. Let V be a vector space of dimension $n \geq 2$ over the finite field $\mathbf{F}_q$ of order q, where q is a power of a prime. The group of all automorphisms of V is called the *general linear group* of degree n over $\mathbf{F}_q$ and is denoted by $GL(n,q)$. The normal subgroup of $GL(n,q)$ consisting of all automorphisms of determinant 1 is called the *special linear group* of degree n over $\mathbf{F}_q$ and is denoted by $SL(n,q)$. The centre Z of $SL(n,q)$ consists of all scalar matrices of determinant 1 and the factor group $SL(n,q)/Z$ is a non-abelian simple group unless $n=2$ and $q=2$ or 3. This group is called the *projective special linear group* of degree n over $\mathbf{F}_q$ and is denoted by $PSL(n,q)$. Suppose now that f is a non-degenerate alternating form defined on $V \times V$. In this case, n must be even, say $n = 2m$. An *isometry* of f is an automorphism σ of V that satisfies

$$f(\sigma u, \sigma v) = f(u,v)$$

for all u and v in V. The set of all isometries of f forms a group called the *symplectic group* of degree $2m$ over $\mathbf{F}_q$ and it is denoted by $Sp(2m,q)$. Since all non-degenerate alternating forms defined on $V \times V$ are equivalent, different choices of f lead to conjugate groups of isometries. The centre of $Sp(2m,q)$ has order 2 if q is odd and order 1 if q is even and the group obtained by factoring out the centre is called the *projective symplectic group* of degree $2m$ over $\mathbf{F}_q$. It is denoted by $PSp(2m,q)$ and it is a non-abelian simple group unless $n=2$ and $q=2$ or 2 or $n=4$ and $q=2$. As $Sp(2,q) = SL(2,q)$, the exceptions to simplicity when $n=2$ are explained by the results for $PSL(2,q)$. $Sp(4,2)$ is isomorphic to the symmetric group S_6, which contains the simple group A_6 as a subgroup of index 2. Suppose we now replace $\mathbf{F}_q$ by $\mathbf{F}_{q^2}$ and let f be a non-degenerate hermitian form defined on $V \times V$. We may define isometries of f as in the alternating case and the group of all isometries of f is called the *unitary group* of degree n over $\mathbf{F}_{q^2}$. It is denoted by $U(n,q)$ or $U(n,q^2)$ (the differences are occasionally confusing). We define the special unitary group $SU(n,q)$ to be those isometries of determinant 1 and the *projective special unitary group* $PSU(n,q)$ is obtained from $SU(n,q)$ by factoring out its centre. As $SU(2,q)$ is isomorphic to $SL(2,q)$, we have the usual exceptions to simplicity for $PSU(2,q)$. If $n \geq 3$, $PSU(n,q)$ is a non-abelian simple group unless $n=3$ and $q=2$.

Finally we turn to the orthogonal groups. Let Q be a non-degenerate quadratic form defined on V. An isometry of Q is an automorphism σ of V that satisfies $Q(\sigma v) = Q(v)$ for all v in V. If $n = 2m$, there are two inequivalent classes of quadratic forms defined on V and their corresponding isometry groups are denoted by $O^+(2m,q)$ and $O^-(2m,q)$. The groups have different orders. $O^+(2m,q)$ is called the *split orthogonal group* of degree $2m$ over $\mathbf{F}_q$ and $O^-(2m,q)$ is called the *non-split orthogonal group* of degree $2m$ over $\mathbf{F}_q$. Suppose now that $n = 2m+1$ is odd. If q is a power of 2, there is a single equivalence class of non-degenerate quadratic forms defined on V and the corresponding isometry groups turn out to be isomorphic to $Sp(2m, q)$. If q is odd, there are two equivalence classes of non-degenerate quadratic forms defined on V, but their corresponding isometry groups are isomorphic and are denoted by $O(2m+1, q)$. Suppose now that n is arbitrary but q is odd. It can be shown that the commutator subgroup $\Omega^{\pm}(2m, q)$ or $\Omega(2m+1, q)$ of a finite orthogonal group has index 4 in the group and the centre of the Ω subgroup has order 1 or 2 if $\dim V \geq 3$. On factoring out the centre, we obtain projective groups $P\Omega^{\pm}(2m, q)$ and $P\Omega(2m+1, q)$. If $\dim V \geq 5$, the groups so obtained are non-abelian simple groups. Suppose next that $n = 2m$ is even and q is a power of 2. It can be shown that $O^{\pm}(2m, q)$ has a subgroup $SO^{\pm}(2m, q)$ of index 2. If $2m \geq 6$, $SO^{\pm}(2m, q)$ is a non-abelian simple group. It might be added that if q is a power of 2, both orthogonal groups $O^{\pm}(2m, q)$ are $Sp(2m, q)$ and there is a rich interplay between orthogonal and symplectic geometry in this case. Unfortunately, this material is often omitted from standard texts, such as 'Geometric Algebra' by Artin, although it is not intrinsically harder than the odd characteristic theory. The book 'Linear Groups' by L. E. Dickson (1901) develops most of this theory and indeed a significant number of results concerning classical groups are due to Dickson. The proofs in Dickson's book are rather computational for modern tastes, and some of his nomenclature has become obsolete, but the book still remains a remarkable source of information on the finite classical groups.

After this rather long introduction, we turn now to the book under review. The aim of the book is to determine the maximal subgroups of the finite classical simple groups. This is clearly an extremely difficult problem, especially when one observes that every finite group must occur eventually as a subgroup of some finite classical group. Indeed, it does not seem likely that the problem is even approachable without invoking the classification of finite simple groups. In an analogous piece of work, E. B. Dynkin (1952) classified the maximal subgroups of the classical complex linear groups $SL(n, \mathbf{C})$, $Sp(2m, \mathbf{C})$ and $O(n, \mathbf{C})$. Dynkin's work made essential use of the classification of simple Lie groups over $\mathbf{C}$ and of the theory of their irreducible finite dimensional complex representations. While some analogies with Dynkin's technique may be drawn for the finite classical groups, the finite problem seems to be considerably harder. Dickson's book made a first inroad into the classification of maximal subgroups of finite classical groups by including a chapter listing all subgroups of $PSL(2, q)$. H. H. Mitchell (1911, 1914) and R. W. Hartley (1926) extended these investigations to certain three and four dimensional classical groups.

The authors' starting point is a paper of Aschbacher (On the maximal subgroups of the finite classical groups, Invent. Math. 76 (1984), 469–514). In this paper, Aschbacher introduces a natural collection of geometrically defined subgroups $\mathcal{C}(G)$ of a finite simple classical group G. These fall into eight families, $\mathcal{C}_1 - \mathcal{C}_8$, which include maximal parabolic subgroups (well known from permutation actions), certain classical groups of smaller degree over extension fields of $\mathbf{F}_q$, tensor products of classical groups acted on by symmetric groups (related to wreath products) and extensions of symplectic-type r-groups (r being a prime) by symplectic groups. Aschbacher also introduces a family $\mathcal{S}$ of almost simple groups that have an irreducible projective representaion on the underlying vector space V. His main result is that if H is a subgroup of G, then either H is contained in $\mathcal{C}(G)$ or in $\mathcal{S}$. Moreover, the great majority of subgroups lie in $\mathcal{C}(G)$. The authors undertake an intricate analysis of the collection $\mathcal{C}(G)$ and their main theorem is as follows:

(A) the group-theoretic structure of each member of $\mathcal{C}(G)$ is known;
(B) the conjugacy amongst members of $\mathcal{C}(G)$ is known;
(C) for $H \in \mathcal{C}(G)$, all overgroups of H in $\mathcal{C}(G) \cup \mathcal{S}$are known.

In fact, a more general result is proved, as G is allowed to be a group satisfying $G_0 \triangleleft G \leq \mathrm{Aut}(G_0)$, where G_0 is a classical group and $\mathcal{C}(G)$ is a collection of subgroups of G obtained from $\mathcal{C}(G_0)$. The precise details of the main theorem are difficult to summarize and Chapter 3 is devoted to explanation. Various tables are required to present the information. Because of the complexity of the solution to the problem, it requires a certain amount of effort to interpret these tables and some instructive examples are provided. Determination of the maximal subgroups of the classical groups still requires the knowledge of when a subgroup in $\mathcal{S}$ is maximal in G. This is an area where much work is in progress. However, it does not seem at present that a non-specialist can expect a quick answer to such questions as whether the Conway group is a maximal subgroup of $SO^{\pm}(24, 2)$.

The second chapter of the book provides an introduction to the classical groups and their properties. This material might prove useful to someone wishing to have a rapid survey of these groups. Chapter 5 is a particularly welcome summary of less familiar properties of finite simple groups. There are tables giving information on the minimal degree of a non-trivial permutation representation of a finite classical group, the containment of alternating and classical groups in sporadic simple groups and lower bounds for the degree of a non-trivial irreducible projective representation of a group of Lie type over a field of characteristic coprime to the underlying characteristic of the group.

There is also substantial information about representations in the defining characteristic, with special emphasis on spin modules for orthogonal groups. Chapters 6, 7 and 8 are concerned with finding the maximal overgroups of the subgroups in $\mathcal{C}(G)$. These chapters are densely written and are probably of interest mainly to specialists.

I feel that this book would be a valuable asset for anyone interested in finite groups, geometry over finite fields or linear algebra. The book contains a substantial body of new results, and constitutes a major research achievement for the authors. It also fulfils a valuable encyclopedic role, which may be its main function for the majority of readers. Considering the quantity of writing, I did not detect an excessive number of typos. I noticed a confusion over a reference to a paper of McLaughlin, two papers being mixed up. I can thoroughly recommend this book to anyone who needs to know about finite classical groups, and its price makes it accessible to virtually everyone.

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