Taylor Exactness
and The Apostol Jump

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Abstract

The middle exactness condition of Joseph Taylor is related to the zero-jump condition of Constantin Apostol and used to derive Kaplansky's lemma.

0. If $T : X \to Y$ and $S : Y \to Z$ are linear operators between complex spaces we shall call the pair (S, T) *exact* iff

0.1
$$S^{-1}(0) \subseteq T(X),$$

whether or not the *chain condition*

0.2
$$ST = 0$$

is satisfied. For example if $T = 0$ this means that S is *one-one*; if $S = 0$ this means that T is *onto*. When S and T are bounded operators between normed spaces we shall call the pair (S, T) *weakly exact* if

0.3
$$S^{-1}(0) \subseteq \operatorname{cl} T(X),$$

and *split exact* if there are bounded $T' : Y \to X$ and $S' : Z \to Y$ for which

0.4
$$S'S + TT' = I.$$

It is clear at once that

0.5
$$(S, T) \text{ split exact} \implies (S, T) \text{ exact} \implies (S, T) \text{ weakly exact};$$

conversely if S and T are both *regular* in the sense that there are bounded $T^\wedge : Y \to X$ and $S^\wedge : Z \to Y$ for which

0.6
$$T = TT^\wedge T \text{ and } S = SS^\wedge S$$

then there is implication

0.7
$$(S, T) \text{ weakly exact} \implies (S, T) \text{ split exact} :$$

indeed if (0.3) and (0.6) both hold then ([10] Theorem 10.3.3)

0.8
$$(I - TT^\wedge)(I - S^\wedge S) = 0,$$

giving two candidates for T' and S' to satisfy (0.4).

Lemma 1 *If $U : W \to X$, $T : X \to Y$ and $V : Y \to Z$ are linear there is implication*

1.1
$$(V, TU) \text{ exact}, (T, U) \text{ exact} \implies (VT, U) \text{ exact}$$

and

1.2
$$(VT, U) \text{ exact}, (V, T) \text{ exact} \implies (V, TU) \text{ exact}.$$

If U, T and V are bounded there is implication

1.3
$$(V, TU), (T, U) \text{ split exact} \implies (VT, U) \text{ split exact}$$

and

1.4
$$(VT, U), (V, T) \text{ split exact} \implies (V, TU) \text{ split exact}.$$

Proof. These are beefed up versions of parts of Theorem 10.9.2 and Theorem 10.9.4 of [10]: for example if $V^{-1}(0) \subseteq TU(W)$ and $T^{-1}(0) \subseteq U(W)$ then

$$VTx = 0 \implies Tx \in V^{-1}(0) \subseteq TU(W) \implies x - Uw \in T^{-1}(0) \subseteq U(W) \bullet$$

Lemma 1 does not extend to weak exactness: to violate the weak analogue of (1.2) take $U = 0$, T dense but not onto and $V^{-1}(0) = \mathbb{C}e$ with $e \in Y \setminus T(X)$.

Lemma 2 *If $U : W \to X$ and $V : Y \to Z$ are bounded and linear, and $T = TT^\wedge T : X \to Y$ is regular, then*

2.1
$$V^{-1}(0) \subseteq T(X) \implies T^\wedge V^{-1}(0) \subseteq (VT)^{-1}(0)$$

and

$$2.2 \qquad T^{-1}(0) \subseteq U(W) \Longrightarrow T^\wedge TU(W) \subseteq U(W).$$

Also

$$2.3 \qquad V'V + TT' = I \Longrightarrow VTT^\wedge = V''V$$

and

$$2.4 \qquad T'T + UU' = I \Longrightarrow T^\wedge TU = UU''.$$

Proof. The first part of this is essentially given by Mbekhta ([16] Proposition 2.4): to see (2.1) argue

$$Vy = 0 \Longrightarrow VTT^\wedge y = VTT^\wedge Tx = VTx = Vy = 0.$$

For (2.3) take $V'' = VTT^\wedge V' + I - VV'$ •

It is familiar that the product of regular operators need not be regular ([10](7.3.6.17);[3]§2.8), and that regularity of the product need not imply regularity of the factors ([10](7.3.6.16);[3]§2.8):

Theorem 3 *If* $T : X \to Y$ *and* $S : Y \to Z$ *are bounded and linear and* (S, T) *is split exact then*

$$3.1 \qquad ST \text{ regular} \iff S, T \text{ regular}.$$

Proof. If $ST = STUST$ and $S'S + TT' = I$ then

$$(I - TT')T(I - UST) = 0 = (I - STU)S(I - S'S).$$

Conversely if $S = SS^\wedge S$ and $T = TT^\wedge T$ and $S^{-1}(0) \subseteq \text{cl } T(X)$ then ([10] Theorems 3.8.3, 2.5.4)

$$STT^\wedge S^\wedge ST = S(TT^\wedge + S^\wedge S - I)T = ST \bullet$$

When $T : X \to X$ and $S : X \to X$ are complex linear operators on the same space X we shall call the pair (S, T) *left non-singular* if

$$3.2 \qquad S^{-1}(0) \cap T^{-1}(0) = \{0\},$$

right non-singular if

$$3.3 \qquad S(X) + T(X) = X,$$

and *middle non-singular* if, in matrix notation,

$$3.4 \qquad (-S \quad T)^{-1}(0) \subseteq \begin{pmatrix} T \\ S \end{pmatrix}(X).$$

This last condition means of course that whenever $Sy = Tx$ there is z for which $y = Tz$ and $x = Sz$, and is a special case of (0.1). Each of these conditions is symmetric in S and T, and not restricted to pairs (S, T) which are *commutative* in the sense that

$$3.5 \qquad ST = TS.$$

Gonzalez ([7] Proposition) has essentially shown

Theorem 4 *Necessary and sufficient for middle non-singularity of* (S, T) *are the following three conditions:*

$$4.1 \qquad S^{-1}(0) \subseteq T\, S^{-1}(0);$$

$$4.2 \qquad T^{-1}(0) \subseteq S\, T^{-1}(0);$$

$$4.3 \qquad S(X) \cap T(X) \subseteq (ST)\, (TS - ST)^{-1}(0).$$

If (4.1) *and* (4.2) *hold then also*

$$4.4 \qquad (ST)^{-1}(0) + (TS)^{-1}(0) \subseteq S^{-1}(0) + T^{-1}(0).$$

Proof. Suppose first that middle non-singularity (3.4) holds: then

$$Sy = 0 \Longrightarrow (-S \quad T)\begin{pmatrix} y \\ 0 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} T \\ S \end{pmatrix}x,$$

giving $y = Tx$ with $x \in S^{-1}(0)$; this proves (4.1), and similarly (4.2). Also

$$w = Tx = Sy \Longrightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} T \\ S \end{pmatrix}z \Longrightarrow w = STz = TSz,$$

giving (4.3). Conversely if these conditions hold then, using first (4.3),

$$\begin{pmatrix} y \\ x \end{pmatrix} \in \begin{pmatrix} -S & T \end{pmatrix}^{-1}(0) \Longrightarrow Sy = Tx = STz = TSz,$$

giving $y - Tz \in S^{-1}(0) \subseteq T\, S^{-1}(0)$ and $x - Sz \in T^{-1}(0) \subseteq S\, T^{-1}(0)$, so that there are u and v for which

$$y - Tz = Tu \quad \text{with} \quad Su = 0 \quad \text{and} \quad x - Sz = Sv \quad \text{with} \quad Tv = 0 :$$

but now $\begin{pmatrix} T \\ S \end{pmatrix}(z + u + v) = \begin{pmatrix} y \\ x \end{pmatrix}$, as required by (3.4). Towards the last part we assume only (4.1), and claim

4.5 $$(ST)^{-1}(0) \subseteq S^{-1}(0) + T^{-1}(0) :$$

for if $(ST)x = 0$ then $Tx \in T\, S^{-1}(0)$, giving $Tx = Tz$ with $Sz = 0$, and hence

$$x = (x - z) + z \in T^{-1}(0) + S^{-1}(0) \bullet$$

The conditions (4.3) and (4.4) are not together sufficient for either (4.1) or (4.2), even in the presence of commutivity: if for example

4.6 $$S = T = P = P^2 \neq I$$

is a non-trivial idempotent then both (4.3) and (4.4), and of course also (3.5), hold, while neither (4.1) nor (4.2) are satisfied. The conditions (4.1) and (4.2) are not together sufficient for (4.3): for example take $S = T$ to be one-one with $T(X) \neq T^2(X)$. Specifically if $X = \ell_2$ we can take $S = T = U$ the forward shift with $(Ux)_{n+1} = x_n$ and $(Ux)_0 = 0$. Curto ([5] pp 71-72) has shown essentially that, in the presence of commutivity (3.5), middle non-singularity (3.4) is equivalent to (4.1) together with

4.7 $$T^{-1}S(X) \subseteq S(X),$$

and therefore also (4.2) together with

4.8 $$S^{-1}T(X) \subseteq T(X).$$

"Duality" considerations then suggest that (4.7), (4.8) and (4.4) might together be equivalent to (3.4). This however fails without commutivity : if for example $X = \ell_2$ we can take $T = V$, the backward shift with $(Vx)_n = x_{n+1}$, and $S = W$ with $(Wx)_n = (1/n)x_n$, to satisfy both (4.7) and (4.8), and also (4.4), but not (3.4). Sufficient for the non-singularity conditions (3.2)-(3.4) are the corresponding invertibility conditions: we call the pair (S, T) *left invertible* if there is another pair (S', T') for which

4.9 $$S'S + T'T = I,$$

right invertible if there is another pair (S'', T'') for which

4.10 $$SS'' + TT'' = I,$$

and *middle invertible* if there are pairs (S', T') and (S'', T'') for which, in matrix notation,

4.11 $$\begin{pmatrix} -S'' \\ T'' \end{pmatrix}\begin{pmatrix} -S & T \end{pmatrix} + \begin{pmatrix} T \\ S \end{pmatrix}\begin{pmatrix} T' & S' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

In the context of pure linear algebra it is clear that "invertibility" and "non-singularity" are equivalent, by the argument for (0.7); for bounded linear operators between normed spaces we require that the "left", "right" and "middle" inverses be made out of bounded operators. When the operators S and T commute and the space X is a Hilbert space then non-singularity implies invertibility; for Banach spaces this question appears to be still open ([9] pp 73-74). In general it is sufficient for left, right and middle invertibility that (4.9) holds for a pair (S', T') such that

4.12 $$(S', S), (S', T), (T', T), (T', S) \quad \text{are commutative.}$$

The reader may suspect that there is an analogue for Theorem 4 with "invertibility" in place of "non-singularity": the author has been unable to find it. The invertible analogues of the conditions (4.1) and (4.2), and of (4.7) and (4.8), are not hard to find - each consists of either a column or a row from (4.11): the reader is invited to think up invertible analogues for (4.3) and (4.4). Theorem 4 should also have an analogue for "weak exactness": thus (3.2) is equivalent to implication

4.13
$$SU = TU = 0 \Longrightarrow U = 0,$$

the weakly exact analogue of (3.3) is

4.14
$$VS = VT = 0 \Longrightarrow V = 0,$$

and the weakly exact analogue of (3.4) is

4.15 $\quad (-S \quad T)\begin{pmatrix} -U' \\ U \end{pmatrix} = (V \quad V')\begin{pmatrix} T \\ S \end{pmatrix} = 0 \Longrightarrow (V \quad V')\begin{pmatrix} -U' \\ U \end{pmatrix} = 0.$

It is not hard, starting from the "invertible" versions of (4.1) and (4.2), and of (4.7) and (4.8), to write down corresponding weak versions of these four conditions .

The next observation is again based on Gonzalez ([7] Theorem), and has also been noted by Curto ([5] p 72):

Theorem 5 *If (S_1, S_2, T) is commutative then there is equivalence*

5.1 $\quad (S_1 S_2, T) \quad$ *non-singular* $\quad \Longleftrightarrow (S_1, T) \quad$ *and* $\quad (S_2, T) \quad$ *non-singular*

and equivalence

5.2 $\quad (S_1 S_2, T) \quad$ *invertible* $\quad \Longleftrightarrow (S_1, T) \quad$ *and* $\quad (S_2, T) \quad$ *invertible.*

Proof. Consider first invertibility : if $S_1' S_1 + T_1' T = I = S_2' S_2 + T_2' T$ then

$$I = S_2'(S_1' S_1 + T_1' T)S_2 + T_2' T = (S_2' S_1')S_1 S_2 + (S_2' T_1' S_2 + T_2')T;$$

conversely if $S_{12}' S_1 S_2 + T_{12}' T = I$ then

$$(S_{12}' S_2)S_1 + T_{12}' T = I = (S_{12}' S_1)S_2 + T_{12}' T.$$

This proves (5.2) for left invertibility, and similarly for right invertibility . Towards middle invertibility, suppose that

5.3 $\quad R'S' + SR = I \quad$ with $\quad S'U = U'T' \quad$ and $\quad W'T' + WU = I :$

then
$$U = R'S'U + SRU = R'U'T' + UTRU$$

giving
$$WR'U'T' = WU(I - TRU) = (I - W'T')(I - TRU)$$

and hence

5.4
$$(WR'U' + W')T' + T(RU) = I.$$

The implication (5.3)$\Longrightarrow$(5.4), which we have just proved, gives forward implication in (5.2) for middle invertibility if we take

5.5 $\quad \mathbf{T} = \begin{pmatrix} T \\ S_1 \end{pmatrix}, \mathbf{T}' = (-S_1 \quad T), \mathbf{S} = \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} \quad$ and $\quad \mathbf{S}' = (-S_1 S_2 \quad T)$

with

5.6
$$\mathbf{U} = \begin{pmatrix} I & 0 \\ 0 & S_2 \end{pmatrix}, \quad \mathbf{U}' = S_2, \quad \mathbf{W} = \begin{pmatrix} I & 0 \\ T_{12}' S_1 & S_{12}' S_1 \end{pmatrix},$$

$$\mathbf{W}' = \begin{pmatrix} 0 \\ T_{12}' \end{pmatrix}, \quad \mathbf{R} = (T_{12}' \quad S_{12}'), \quad \mathbf{R}' = \begin{pmatrix} -S_{12}'' \\ T_{12}'' \end{pmatrix}.$$

Conversely if

5.7
$$\begin{pmatrix} -S_1' \\ T_1' \end{pmatrix}(-S_1 \quad T) + \begin{pmatrix} T \\ S_1 \end{pmatrix}(S_1'' \quad T_1'') = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

then since again

$$\begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & S_2 \end{pmatrix}\begin{pmatrix} T \\ S_1 \end{pmatrix}$$

we have

$$\begin{pmatrix} I & 0 \\ 0 & S_2 \end{pmatrix} = \begin{pmatrix} -S_1' \\ S_2 T_1' \end{pmatrix}(-T \quad S_1) + \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix}(T_1'' \quad S_1'')$$

giving

$$\begin{pmatrix} 0 \\ S_2 \end{pmatrix} = \begin{pmatrix} -S_1' \\ S_2 T_1' \end{pmatrix}T + \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix}S_1''$$

and hence

$$5.8 \qquad \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} S_1'' = \begin{pmatrix} S_1' & 0 \\ -S_2 T_1' & I \end{pmatrix} \begin{pmatrix} T \\ S_2 \end{pmatrix}$$

and also

$$5.9 \qquad \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} -S_1' \\ S_2 T_1' \end{pmatrix} (-S_1) + \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} T_1''.$$

Combining (5.8) with

$$5.10 \qquad \begin{pmatrix} -S_2' \\ T_2' \end{pmatrix} \begin{pmatrix} -S_2 & T \end{pmatrix} + \begin{pmatrix} T \\ S_2 \end{pmatrix} \begin{pmatrix} T_2'' & S_2'' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

gives

$$5.11 \qquad \begin{pmatrix} S_1'' & 0 \\ -S_2 T_1' & I \end{pmatrix} = \begin{pmatrix} S_1' S_2' \\ S_2 T_1' S_2' + T_2' \end{pmatrix} \begin{pmatrix} -S_2 & T \end{pmatrix} + \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} S_1'' \begin{pmatrix} T_2'' & S_2'' \end{pmatrix},$$

which combines with (5.9) to give

$$5.12 \qquad \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} (T_1'' + S_1'' T_2 S_1) + \begin{pmatrix} S_1' S_2' \\ S_2 T_1' S_2' + T_2' \end{pmatrix} (-S_2 S_1).$$

From (5.11) and (5.12) we get

$$5.13 \qquad \begin{pmatrix} S_1' & 0 & I \\ -S_2 T_1' & I & 0 \end{pmatrix} = \begin{pmatrix} S_1' S_2' \\ S_2 T_1' S_2' + T_2' \end{pmatrix} \begin{pmatrix} -S_2 & T & -S_2 S_1 \end{pmatrix}$$
$$+ \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} \begin{pmatrix} S_1'' T_2'' & S_1'' S_2'' & T_1'' + S_1'' T_2'' S_1 \end{pmatrix},$$

from which we can read off

$$5.14 \qquad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} S_1' S_2' \\ S_2 T_1' S_2' + T_2' \end{pmatrix} \begin{pmatrix} -S_1 S_2 & T \end{pmatrix} + \begin{pmatrix} T \\ S_1 S_2 \end{pmatrix} \begin{pmatrix} T_1'' + S_1'' T_2'' S_1 & S_1'' S_2'' \end{pmatrix}.$$

This is backward implication in (5.2) for middle invertibility, and completes the proof of (5.2). With the information displayed in the proof of (5.2), the argument for (5.1) can be left to the reader •

For bounded linear operators between Banach spaces, (5.1) follows from the spectral mapping theorem for the Taylor spectrum, and then (5.2) from the corresponding theorem for the "Taylor split spectrum" ([10] Theorems 11.9.10, 11.9.11). Our derivation of forward implication in (5.2) is based on the corresponding argument for non-singularity ([9] Theorem 4.3; [6]); our derivation of backward implication in (5.2) also follows from the corresponding argument for non-singularity, which is what is given by Gonzalez [7]. The reader may find it entertaining to try and carry out the matrix juggling in terms of operator calculations; he may also like to try and do the non-singularity argument (5.1) in a general ring, using conditions (4.13)-(4.15).

If $T : X \to X$ is linear then its *hyperrange* and *hyperkernel* are the subspaces

$$5.15 \qquad T^\infty(X) = \bigcap_{n=1}^\infty T^n(X)$$

and

$$5.16 \qquad T^{-\infty}(0) = \bigcup_{n=1}^\infty T^{-n}(0);$$

when T is continuous on a normed space X neither of these need be closed. If we write

$$5.17 \qquad \mathrm{comm}(T) = \{ S \in BL(X,X) : ST = TS \}$$

for the *commutant* of T and

$$5.18 \qquad \mathrm{comm}^{-1}(T) = \mathrm{comm}(T) \cap BL^{-1}(X,X)$$

for the *invertible commutant* of T, then we can collect the following

Lemma 6 *If $T \in BL(X,X)$ is arbitrary then*

$$6.1 \qquad T^{-1} T^{-\infty}(0) \subseteq T^{-\infty}(0)$$

and

6.2 $\qquad T \text{ essentially one}-\text{one} \Longrightarrow T^{\infty}(X) \subseteq T\, T^{\infty}(X).$

If $S \in \text{comm}(T)$ *then*

6.3 $\qquad S\, T^{-\infty}(0) \subseteq T^{-\infty}(0) \text{ and } S\, T^{\infty}(X) \subseteq T^{\infty}(X).$

If $S \in \text{comm}^{-1}(T)$ *then*

6.4 $\qquad (T-S)^{-1}(0) \subseteq T^{\infty}(X) \text{ and } T^{-\infty}(0) \subseteq (T-S)(X).$

Proof. This is Theorem 7.8.3 of [10]●

We shall call the operator $T : X \to X$ *self-exact* if the pair (T, T) satisfies (0.1):

6.5 $\qquad\qquad\qquad T^{-1}(0) \subseteq T(X),$

n-exact if (T, T^n) satisfies (0.1):

6.6 $\qquad\qquad\qquad T^{-1}(0) \subseteq T^n(X),$

and *hyperexact* if

6.7 $\qquad\qquad\qquad T^{-1}(0) \subseteq T^{\infty}(X).$

There are various equivalent forms of these conditions:

Theorem 7 *If $T : X \to X$ is linear and $n \in \mathbf{N}$ and $m + k = n + 1$ then*

7.1 $\qquad T^{-1}(0) \subseteq T^n(X) \Longleftrightarrow T^{-k}(0) \subseteq T^m(X) \Longleftrightarrow T^{-n}(0) \subseteq T(X)$

and

7.2 $\quad T^{-1}(0) \subseteq T^{\infty}(X) \Longleftrightarrow T^{-\infty}(0) \subseteq T(X) \Longleftrightarrow T^{-\infty}(0) \subseteq T^{\infty}(X).$

If $T = TT^{\wedge}T$ *is regular then*

7.3 $\qquad T^{\wedge}T^{\infty}(X) \subseteq T^{\infty}(X) \text{ and } T^{\wedge}T^{-\infty}(0) \subseteq T^{-\infty}(0).$

If $S \in \text{comm}^{-1}(T)$ *then*

7.4 $\qquad (T-S)^{-\infty}(0) \subseteq T^{\infty}(X) \text{ and } T^{-\infty}(0) \subseteq (T-S)^{\infty}(X)$

and

7.5 $\qquad\qquad T^{-\infty}(0) \cap (T-S)^{-\infty}(0) = \{0\}$

and for each $m, n \in \mathbf{N}$

7.6 $\qquad\qquad\qquad T^m(X) + (T-S)^n(X) = X.$

Proof. Most of this comes from Lemma 1 and Lemma 2, taking U and V to be powers of T. For the last part factorise $(T^m - S^m)^n$ in two ways to see that $((T-S)^n, T^n)$ satisfies (4.9)-(4.11) for each n:

7.7 $\qquad\qquad S^{mn} - r_{m,n}(T,S)T^n = (T-S)^n q_m(T,S)^n$

for certain polynomials q_m and $r_{m,n}$●

We cannot replace m and n by ∞ in (7.6): for a counterexample take $T = U$ to be the forward shift on $X = \ell_2$ and $S = I$.

Definition 8 *Call $T \in BL(X,X)$* hyper-regular *if it is regular and hyperexact. We shall say that T is* "consortedly regular" *if there are sequences (S_n) in $\text{comm}^{-1}(T)$ and $(T_n^{\wedge})$ in $BL(X,X)$ for which*

8.1 $\qquad \|S_n\| + \|T_n^{\wedge} - T^{\wedge}\| \longrightarrow 0 \text{ and } T - S_n = (T - S_n)T_n^{\wedge}(T - S_n),$

and "holomorphically regular" *if there is $\delta > 0$ and a holomorphic mapping $T_z^{\wedge} : \{|z| < \delta\} \longrightarrow BL(X,X)$ for which*

8.2 $\qquad\qquad T - \lambda I = (T - \lambda I)T_{\lambda}^{\wedge}(T - \lambda I) \text{ for each } |\lambda| < \delta.$

Mbekhta ([16] Theorem 2.6) has essentially proved

Theorem 9 *If X is complete and $T \in BL(X,X)$ then*

9.1

T *consortedly regular* $\Longrightarrow T$ *hyper* $-$ *regular* $\Longrightarrow T$ *holomorphically regular.*

Proof. If T is consortedly regular then, using (6.4), there is inclusion $T^{-k}(0) \subseteq (T - S_n)(X)$ for arbitrary k and n, where S_n satisfies (8.1), and hence if $T^k x = 0$ then $x = (T - S_n)T_n^\wedge x$ giving

$$(I - TT^\wedge)x = ((T - S_n)T_n^\wedge - TT^\wedge)x \longrightarrow 0 \text{ as } n \longrightarrow 0,$$

and hence $x = TT^\wedge x \in T(X)$. This gives, without completeness, the first implication of (9.1). Conversely suppose $T = TT^\wedge T$ is hyper-regular and $S \in \text{comm}(T)$ with $\|S\|\|T^\wedge\| < 1$: using (6.3) and (7.3) and expanding $(I - T^\wedge S)^{-1}$ in the geometric series gives

$$S(I - T^\wedge S)^{-1}T^{-1}(0) \subseteq \text{cl } T^{-\infty}(0) \subseteq \text{cl } T(X)$$

and hence

$$(I - TT^\wedge)S(I - T^\wedge S)^{-1}(I - T^\wedge T) = 0,$$

which by (3.8.4.3) from the proof of Theorem 3.8.4 [10] says

9.2 $$T - S = (T - S)(I - T^\wedge S)^{-1}T^\wedge(T - S).$$

Specialising to scalar $S = \lambda I$ gives the second implication of (9.1)•

The derivation of (9.2) is based on Caradus [4]; cf also Theorem 3.9 of Nashed [18]. If we observe

9.3 $$T^\wedge(T - S) + (I - T^\wedge T) = I - T^\wedge S$$

that $I - T^\wedge S$ sends the null space of $T - S$ into the null space of T, then we can see that for Fredholm T and one-one $I - T^\wedge S$ we have $\dim(T - S)^{-1}(0) \leq \dim T^{-1}(0)$ ([10] Theorem 6.4.5). Conversely if $T = TT^\wedge T$ is hyperregular and $S \in \text{comm}(T)$ has small enough norm,
9.4
$$(T - S)^\wedge T + I - (T - S)^\wedge(T - S) = I + (T - S)^\wedge S \text{ with } (T - S)^\wedge = (I - T^\wedge S)^{-1}T^\wedge,$$

furnishing an invertible operator which sends the null space of T into the null space of $T - S$. In the Fredholm case this is the Apostol *zero jump* condition [1],[22],[19].

Theorem 9 says that the hyper-regular operators form an open subset of $BL(X, X)$, and hence that a certain kind of "spectrum" is closed in $\mathbb{C}$ We

may also observe that the topological boundary of the spectrum is contained in this "hyper-regular spectrum":

9.5 $$\{T \in \text{cl}_{\text{comm}} BL^{-1}(X, X) : T \text{ hyper} - \text{regular}\} \subseteq BL^{-1}(X, X).$$

We are claiming that if hyper-regular T is the limit of a sequence $T - S_n$ of invertible operators which commute with T then T must also be invertible. It follows from (9.4) that if $(T - S)^\wedge$ and $I - T^\wedge S$ are both invertible then so is $T^\wedge$; since this argument extends to $T^\wedge TT^\wedge$ this also makes T invertible.

The spectral mapping theorem for polynomials extends to the "hyper-regular spectrum":

Theorem 10 *If $ST = TS$ then*

10.1 $$ST \text{ self} - \text{exact} \implies S, T \text{ self} - \text{exact}$$

and

10.2 $$ST \text{ hyper} - \text{regular} \implies S, T \text{ hyper} - \text{regular}$$

If $ST = TS$ and (S, T) is middle exact then

10.3 $$S, T \text{ self} - \text{exact} \implies ST \text{ self} - \text{exact}$$

and

10.4 $$S, T \text{ hyper} - \text{regular} \implies ST \text{ hyper} - \text{regular}.$$

Proof. The first part is an extension of Mbekhta ([17] Lemma 4.15): if $(ST)^{-1}(0) \subseteq (ST)(X)$ then

$$T^{-1}(0) \subseteq (ST)^{-1}(0) \subseteq (ST)(X) = (TS)(X) \subseteq T(X),$$

and similarly for S and powers T^n and S^m. This gives (10.1) and most of (10.2): the regularity of S and T come from (3.1). Conversely, for (10.3), use (4.1)-(4.4):

$$(ST)^{-1}(0) \subseteq S^{-1}(0) + T^{-1}(0) \subseteq S(X) \cap T(X) \subseteq (ST)(X).$$

This gives (10.3) and most of (10.4): the regularity of ST is (3.1) again•

One situation in which all the invertibility and non-singularity conditions for (S, T) are satisfied is when we can write

10.5
$$S = q(A), T = r(A)$$

for an operator $A : X \to X$ and polynomials q and r without non-trivial common factor. In general polynomials q and r have a unique "highest common factor" $\mathrm{hcf}(q, r)$ determined by the logical equivalence

10.6
$$\{q, r\} \subseteq (\mathrm{Poly})p \iff p \in (\mathrm{Poly})\mathrm{hcf}(q, r),$$

together with the requirement that it be "monic" (unless either q or r is 0, in which case also $\mathrm{hcf}(q, r) = 0$). It is now familiar that, by the Euclidean algorithm,

10.7
$$\mathrm{hcf}(q, r) \in (\mathrm{Poly})q + (\mathrm{Poly})r,$$

so that there are polynomials q' and r' for which $\mathrm{hcf}(q, r) = q'q + r'r$. If in particular $\mathrm{hcf}(q, r) = 1$, so that q and r have no common non-trivial common factors, then (in the algebra *Poly*) the pair (q, r) satisfies all the invertibility conditions (4.9)-(4.11) (since the analogue of (4.12) holds). This extends to the pair $(S, T) = (q(A), r(A))$, with $(S', T') = (q'(A), r'(A))$ whenever $A : X \to X$ is an operator: thus if (10.5) holds then the non-singularity conditions (3.2)-(3.4) are satisfied.

Lemma 11 *If $A : X \to X$ is linear there is equality*

11.1
$$\sum_{\lambda \in \mathbf{C}} (A - \lambda I)^{-\infty}(0) = \bigcup_{0 \neq p \in \mathrm{Poly}} p(A)^{-1}(0)$$

and

11.2
$$\bigcap_{\lambda \in \mathbf{C}} (A - \lambda I)^{\infty}(X) = \bigcap_{0 \neq p \in \mathrm{Poly}} p(A)(X).$$

Proof. The left hand side of (11.1) is obviously included in the right; conversely if $p = qr \in \mathrm{Poly}$ with $\mathrm{hcf}(q, r) = 1$ then by (4.4)

11.3
$$p(A)^{-1}(0) = q(A)^{-1}(0) + r(A)^{-1}(0).$$

More generally if $p = q_1 q_2 \ldots q_n$ is a finite product of factors q_j of which no pair has any common factors then the null space of $p(A)$ is the sum of the null spaces $q_j(A)^{-1}(0)$; but by the fundamental theorem of algebra p is a product of polynomials of the form $(z - \lambda)^k$ for distinct complex numbers λ. This proves (11.1). Similarly, the right hand side of (11.2) is included in the left, and the opposite inclusion follows by the inductive extension of equality (4.3):

11.4
$$p(A)(X) = q(A)(X) \cap r(A)(X)$$

if $p = qr$ with $\mathrm{hcf}(q, r) = 1\bullet$

The operator $A : X \to X$ is described as *algebraic* if there exists a non-trivial polynomial $p \in \mathrm{Poly}$ for which

11.4
$$p(A)^{-1}(0) = X,$$

and as *locally algebraic* if

11.5
$$\bigcup_{0 \neq p \in \mathrm{Poly}} p(A)^{-1}(0) = X.$$

The intermediate notion is that A is *boundedly locally algebraic* if there is $k \in \mathbf{N}$ for which

11.6
$$X = \bigcup \{p(A)^{-1}(0) : 0 \neq p \in \mathrm{Poly}, \mathrm{degree}(p) \leq k\}.$$

For bounded linear operators between Banach spaces, an application of Baire's theorem says that $(11.5) \implies (11.6)$ ([13] Theorem 15; [20] Theorem 4.8 ; [21] (3.4)); our interest here is to expound *Kaplansky's lemma* ([13] Lemma 14; [20] Theorem 4.8; [21](3.5)), which says that $(11.6) \implies (11.4)$:

Theorem 12 *If $A : X \to X$ is boundedly locally algebraic then it is algebraic.*

Proof If A is locally algebraic in the sense of (11.5) then by (11.1) there is equality

12.1
$$\sum_{\lambda \in \mathbf{C}} (A - \lambda I)^{-\infty}(0) = X;$$

for a locally algebraic operator to be algebraic it is necessary and sufficient that it have a finite set of *eigenvalues*

12.2
$$\pi^{\text{left}}(A) = \{\lambda \in \mathbb{C} : (A - \lambda I)^{-1}(0) \neq \{0\}\} = \{\lambda \in \mathbb{C} : (A - \lambda I)^{-\infty}(0) \neq \{0\}\}.$$

We claim that if the set $\pi^{\text{left}}(A)$ is infinite then the condition (11.6) must fail: for if $\lambda_1, \lambda_2, \ldots, \lambda_m$ are pairwise distinct eigenvalues of A, with corresponding eigenvectors $x_1, x_2, \ldots, x_m$, and $p \in \text{Poly}$ is a polynomial, we claim that there is implication

12.3
$$p(A)(\sum_{j=1}^{m} x_j) = 0 \Longrightarrow \{x_1, x_2, \ldots, x_m\} \subseteq p(A)^{-1}(0)$$

$$\Longrightarrow \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subseteq p^{-1}(0),$$

forcing degree$(p) \geq m$. To see why the first part of (12.3) holds argue that

12.4 $\text{hcf}(q, r) = 1, q(A)y = r(A)z = 0, y + z = 0 \Longrightarrow y = z = 0$:

this is because the pair $(S, T) = (q(A), r(A))$ satisfies the condition (3.2) so that equality $y + z = 0$ puts $y = -z$ in $q(A)^{-1}(0) \cap r(A)^{-1}(0) = \{0\}$. To apply (12.4) to the first part of (12.3) take

$$y = x_j, z = \sum_{i \neq j} x_i, q = z - \lambda_j, r = \prod_{i \neq j} z - \lambda_i.$$

To see why the second part of (12.3) holds observe that

12.5 $p(\lambda_j) \neq 0 \Longrightarrow \text{hcf}(p, z - \lambda_j) = 1 \Longrightarrow p(A)^{-1}(0) \cap (A - \lambda_j I)^{-1}(0) = \{0\}$•

When the operator A is algebraic then (12.1) becomes a finite direct sum decomposition of the space X : this decomposition makes it clear, as is observed by Aupetit [2], that if $k \in \mathbb{N}$ satisfies the condition (11.6) then (11.4) can be satisfied with degree$(p) \leq k$. When A is algebraic then each existing inverse $(A - \lambda I)^{-1}$ is expressible as a polynomial in A : when X is finite dimensional this is one of the familiar applications of the Cayley-Hamilton theorem. We may also observe, as in the finite dimensional case [12], that when $(A - \lambda I)^{-1}$ does not exist, then all the eigenvectors $x \in (A - \lambda I)^{-1}(0)$ lie in the range

of a related polynomial in A : the simple observation is that if $p(A) = 0$ and $p = qr$ with $\text{hcf}(q, r) = 1$ then

12.6 $q(A)^{-1}(0) = r(A)(X).$

We conclude by expounding another generalization of Kaplansky's lemma; the unpublished argument is due to Laffey ([15] Lemma 1; [20] (3.5)):

Theorem 13 *If the operator $A : X \to X$ is boundedly locally algebraic modulo a finite dimensional subspace $Y \subseteq X$, then it is algebraic.*

Proof The assumption is that there is $k \in \mathbb{N}$ for which, for each $x \in X$, there is a non-trivial polynomial $p_x \in \text{Poly}$ for which

13.1 $\text{degree}(p_x) \leq k$ and $p_x(A)x \in Y.$

We are not assuming that the finite dimensional subspace Y is "invariant" under A in the sense that $A(Y) \subseteq Y$, but immediately replace it with the (possibly infinite dimensional) invariant subspace

13.2
$$\hat{Y} = \bigcup_{p \in \text{Poly}} p(A)(Y) = Y + \sum_{n \in \mathbb{N}} A^n(Y)$$

generated by it, together with the induced quotient operator $\hat{A} : X/\hat{Y} \longrightarrow X/\hat{Y}$. Applying Kaplansky's lemma (Theorem 12) to $\hat{A}$ gives a non-trivial polynomial (of degree $\leq k$) $p_0 \in \text{Poly}$ for which

13.3 $p_0(A)(X) \subseteq \hat{Y}.$

If Y is of dimension m , with basis $y = (y_1, y_2, \ldots, y_m)$, then again by assumption there are non-trivial polynomials $q_1, q_2, \ldots, q_m$ (of degree $\leq k$) for which

13.4 $q_i(A)Ay_i \in Y$ $(i = 1, 2, \ldots, m),$

and hence complex numbers $(\lambda_{ij}) = B$ for which

13.5 $q_i(A)Ay_i = \sum_{j=1}^{m} \lambda_{ij} y_j$ $(i = 1, 2, \ldots, m).$

This gives

13.6 $$Q(A)y = 0 \in Y^m,$$

treating the basis $y \in Y^m$ as a column matrix, where $Q(A) \in L(X, X)^m$ is the "operator matrix"

13.7 $$Q(A) = B \otimes I - q(A)(I \otimes A)$$

$$= \begin{pmatrix} \lambda_{11}I - q_1(A)A & \lambda_{12}I & \ldots & \lambda_{1m}I \\ \lambda_{21}I & \lambda_{22}I - q_2(A)A & \ldots & \lambda_{2m}I \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{m1}I & \lambda_{m2}I & \ldots & \lambda_{mm}I - q_m(A)A \end{pmatrix}.$$

It follows ([8] Problem 70; [11](2.0.4)p.108)

13.8 $\quad q_0(A)y_j = 0 \quad (j = 1, 2, \ldots, m) \quad$ with $\quad q_0(A) = \det Q(A) \in L(X, X):$

since all the entries of $Q(A)$ commute we can write $\mathrm{adj}Q(A).Q(A) = q_0(A) \otimes I$, exactly as in the numerical case. It now follows

$$q_0(A)Y = 0 \quad \text{and hence} \quad q_0(A)\mathrm{Poly}(A)Y = \{0\}$$

and hence

13.9 $$q_0(A)p_0(A)X \subseteq q_0(A)\hat{Y} = \{0\}.$$

Our final observation explains why (13.5) was not replaced by something simpler : by construction

13.10 $$\mathrm{degree}(q_0) \geq m,$$

which ensures $0 \neq q_0 p_0 \in \mathrm{Poly} \bullet$

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Dimension Theory and Stable Rank

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1. Topological Dimension Theory

The theory of dimension in topology grew from attempts to establish the topological invariance of the dimension of Euclidean spaces. The first proof that the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ are homeomorphic only if n and m are equal was given by Brouwer in 1911. His proof did not explicitly involve a property that might serve as a topological definition of n, but in the same year Lebesgue suggested an approach which led to the covering dimension. If I is the closed unit interval of $\mathbf{R}$ it was observed by Lebesgue that the cube I^n can be covered by arbitrarily small closed sets in such a manner that not more than $n+1$ of them meet (in a common point). This is illustrated in the 2-dimensional case by the usual pattern of brickwork, where a maximum of 3 bricks can meet.

To define the covering dimension we introduce a preliminary concept. If $\Phi = (U_\lambda)_{\lambda \in \Lambda}$ is a family of subsets of a topological space X and $x \in X$ the *order* of Φ at x, denoted $ord_x(\Phi)$, is defined to be the number of elements λ of Λ such that U_λ contains x (if there are infinitely many such elements λ then $ord_x(\Phi) = +\infty$). The *order* of Φ is defined to be the supremum of all $ord_x(\Phi)$ where x runs over X. Thus for the brickwork family of sets mentioned above the order is 3. If X is a topological space the *(covering) dimension* of X, denoted $dim(X)$, is the least integer n such that every finite open covering of X has an open refinement of order not greater than $n+1$. If no such integer n exists then we set $dim(X) = +\infty$. Here is an alternative, very useful, formulation: For any topological space X, the inequality $dim(X) \leq n$ holds if and only if for each open covering $U_1, \ldots, U_{n+2}$ of X there is an open covering $V_1, \ldots, V_{n+2}$ such that $V_j \subseteq U_j$ for $j = 1, \ldots, n+2$ and $V_1 \cap \ldots \cap V_{n+2} = \emptyset$. For the classical spaces such as $\mathbf{R}^n$, I^n and S^n (the n-sphere) the covering dimension is the number one would expect. It is however non-trivial to show that $dim(\mathbf{R}^n) = n$. The proof involves the well known theorem of Brouwer which asserts that for all n the sphere S^{n-1} is not a retract of the closed unit