NOTES

Venn Diagrams: A Combinatorial Comment

T. B. M. McMaster

When, in tender years, we all first learned how to draw Venn diagrams, those in charge of our education insisted that these be depicted as in Fig. 1; and if, through inquisitiveness, amnesia or sheer cussedness, we produced a deviant hieroglyph such as those in Fig. 2, they generally informed us that
(i) We were silly, and
(ii) even though some examples could be devised which fitted into our 'wrong' diagram, the vast majority of instances could only be accomodated on the 'general case' picture which we had been told to emulate.

Now a detailed analysis of proposition (i) may not perhaps be appropriate at this juncture, but assertions such as (ii) have a habit of surfacing in the mind after lying dormant for years. So it has come to pass that several members of our Department have recently been exploring some of the combinatorial/probabilistic questions which are raised by subjecting it to scrutiny and generalization. This brief note presents a report on one of these investigations.

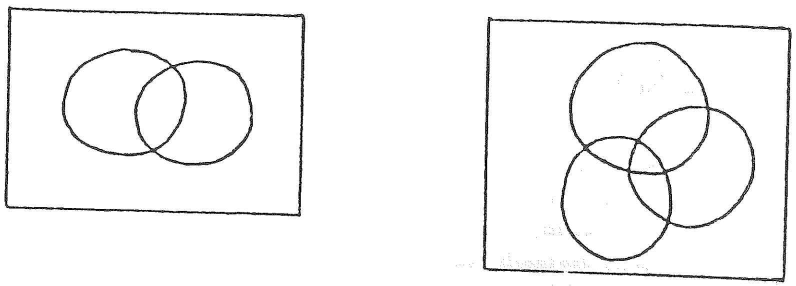

Figure 1: 'right' Venn diagrams

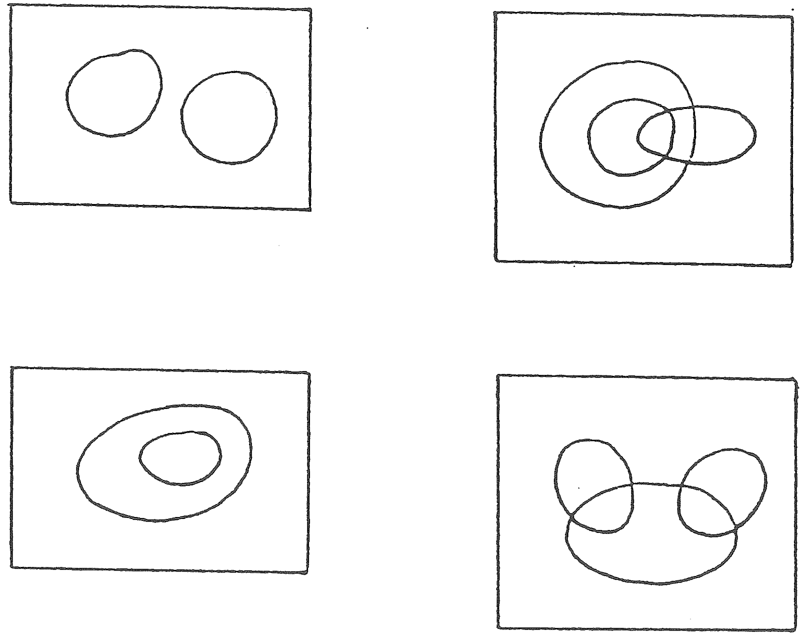

Figure 2: 'wrong' Venn diagrams

The problem to which we here address ourselves is this: given a finite set X (with n elements say), a (small) positive integer s, and a random selection of s distinct subsets of X, how likely is it that the 'general case' Venn diagram is the only correct one to describe their relationships? As the number of such random selections is easily obtained, namely

$$T(s) = 2_n(2_n - 1)(2_n - 2)\ldots(2_n - s + 1)/s!$$

the equivalent combinatorial problem is: how many selections (of s subsets) does the general case diagram alone depict?

Since the feature which distinguishes the general case here from the various degenerate ones is non-emptiness of the disjoint 'regions' on the diagram (except perhaps for the 'outer zone'), it is convenient to begin with a formula for the number of ways of choosing a specified number of pairwise disjoint non-empty subsets of X (not necessarily covering the whole of X).

Lemma *Consider a positive integer $k \le n$. The number $\Omega(k)$ of ordered choices of k pairwise disjoint non-empty subsets of X is given by*

$$\Omega(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k+1-j)^n ,$$

and the number of unordered choices is $\Omega(k)/k!$

Proof The formula claimed for $\Omega(1)$ gives $2^n - 1$, which is evidently correct. Assuming now its validity for integers from 1 to $k-1$, observe that there are $(k+1)^n$ ways of distributing the elements of X across k boxes and one wastepaper basket, and that we shall determine $\Omega(k)$ by subtracting from this total the number of distributions in which one, two, three, ...,k of the boxes remain empty. This gives

$$\Omega(k) = (k+1)^n - \binom{k}{1}\Omega(k-1) - \binom{k}{2}\Omega(k-2) - \ldots \binom{k}{k-1}\Omega(1) - 1 ,$$

and when we substitute in the assumed formulae for $\Omega(k-1)$, $\Omega(k-2)$, ...,$\Omega(1)$, the coefficient of $(k+1-j)^n$ in the resulting expansion is

$$(-1)^j \left\{ \binom{k}{1}\binom{k-1}{j-1} - \binom{k}{2}\binom{k-2}{j-2} + \ldots + (-1)^{j+1}\binom{k}{j}\binom{k-j}{0} \right\}$$

which is easily evaluated as $(-1)^j \binom{k}{j}$. Induction completes the demonstration.

Let us now return to the simplest case ($s = 2$) of the original problem. There are $T(2) = 2^{n-1}(2^n - 1)$ ways of selecting an (unordered) pair A, B of subsets of X, and this selection will be non-degenerate in the Venn diagram sense if and only if none of the three sets $A \cap B$, $A \cap B'$, $A' \cap B$ is empty. Now there are $\Omega(3)/3!$ ways of selecting three non-empty disjoint subsets (call them K, L, M) of X, but each such selection resolves itself into three distinct choices of $\{A, B\}$ when we try to identify K, L and M with $A \cap B$, $A \cap B'$ and $A' \cap B$; for A and B could be

either $K \cup L$ and $K \cup M$,
or $K \cup L$ and $L \cup M$,
either $K \cup M$ and $L \cup M$;

hence we see that:

Proposition 1 *The number of non-degenerate (in the present sense) choices of two subsets of X is*

$$\frac{\Omega(3)}{3!} = \frac{1}{2}(4^n - 3.3^n + 3.2^n - 1) .$$

In the same way, there are $T(3)$ ways of choosing three subsets A, B, C, of X, and the choice is non-degenerate precisely when none of the seven sets $A \cap B \cap C$, $A' \cap B \cap C$, $A \cap B' \cap C$, $A \cap B \cap C'$, $A \cap B' \cap C'$, $A' \cap B \cap C'$, $A' \cap B' \cap C$ is empty. There are $\Omega(7)/7!$ choices of seven disjoint non-empty subsets (call them K, L, M, N, O, P, Q) of X, each resolving itself into several distinct choices of $\{A, B, C\}$. To be precise, for each of the 7! permutations of $K, L, \ldots, Q$ we could identify those sets in order with $A \cap B \cap C$, $A' \cap B \cap C$, ... $A' \cap B' \cap C$, thus constructing an ordered triple (A, B, C); it will be necessary to divide by 3! to disregard the order and so $\{K, L, \ldots, Q\}$ actually resolves itself into 7!/3! unordered combinations of $\{A, B, C\}$. Thus we have shown that:

Proposition 2 *The number of non-degenerate choices of three subsets of X is*

$$\frac{\Omega(7)}{3!} = \frac{1}{6}(8^n - 7.7^n + 21.6^n - 35.5^n + 35.4^n - 21.3^n + 7.2^n - 1)$$

Although Venn diagrams themselves cease to be of much use for $s > 3$, the above analysis requires no significant change to cope with larger values. Thus one reaches the following conclusion:

Theorem *Let n and s be positive integers, and let X be a set with n elements. The number of ways of choosing s distinct subsets $A_1, A_2, \ldots, A_s$ of X (irrespective of order), subject to the condition that every one of the sets*

$$C_1 \cap C_2 \cap \ldots \cap C_s$$

(where for $1 \le i \le s$, C_i is either A_i or A_i', but $C_i = A_i$ for at least one value of i) is non-empty, is

$$\frac{1}{s!} \sum_{j=0}^{2^s - 1} (-1)^j \binom{2^s - 1}{j} (2^s - j)^n$$

The following table records, for $1 \leq n \leq 10$, the calculated values of $T(2)$ and $T(3)$, of $\Omega(3)/2!$ and $\Omega(7)/3!$, and of the probabilities p_2 and p_3 that a randomly chosen pair or trio of sets is non-degenerate, recorded to four decimal places. It shows as expected that the probabilities, though small for small values of n, rise as n does. Better information on their behaviour for large n is easy to extract from the above formulae, which yield that

$$p_2 = 1 - 3(3/4)^n(1 + o(1))$$
$$p_3 = 1 - 7(7/8)^n(1 + o(1))$$

and in general, where p_s is defined as $\Omega(2^s - 1)/s!T(s)$, that

$$p_s = 1 - (2^s - 1)(1 - 2^{-s})^n(1 + o(1))$$

as $n \to \infty$. Since these probabilities tend to 1, we are obliged to concede that they told us the truth all those years ago. Rather a pity, really.

n	$T(2)$	$\Omega(3)/2!$	p_2	$T(3)$	$\Omega(7)/3!$	p_3
1	1	0	0	0	0	0
2	6	0	0	4	0	0
3	28	3	.1071	56	0	0
4	120	30	.2500	560	0	0
5	496	195	.3931	4,960	0	0
6	2,016	1,050	.5208	41,664	0	0
7	8,128	5,103	.6278	341,376	840	.0025
8	32,640	23,310	.7142	2,763,520	30,240	.0109
9	130,816	102,315	.7821	22,238,720	630,000	.0283
10	523,776	437,250	.8348	178,433,024	9,979,200	.0559

Department of Pure Mathematics
The Queen's University of Belfast
Belfast BT7 1NN, Northern Ireland

HISTORY OF MATHEMATICS

Giovanni Frattini 1852–1925

Maurizio Emaldi

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A little over one hundred years ago, between 1885 and 1886 three papers by the Roman mathematician, Giovanni Frattini *"On the generators of groups of operations"* appeared in the proceedings of the Royal Academy of Lincei. In the first of these the author introduced the subgroup Φ of a finite group of operations consisting of the set of all operations which "cannot effectively contribute to the generators" of the group. This can be characterized as the intersection of all proper maximal subgroups. He demonstrated that the group in question is nilpotent and in doing so used a most elegant argument which today is called "the Frattini argument". The results contained in these three papers, the full scope of which were not fully grasped at the time of their publication, are amongst the most significant contributions of Italian mathematicians to the theory of groups in the latter half of last century. The definition of the subgroup Φ of a finite group given by Frattini has been extended to groups in general and today is generally called "the Frattini-subgroup". (As far as we can determine, this name appeared explicitly for the first time in a paper by G. Zacher: "Construction of finite groups with trivial Frattini-subgroup." Rend. Sem. Mat. Padova, vol. 21, 1952). In group theory the Frattini-subgroup and more generally the analogous notion in algebraic structures play a central role in many questions. Thus it seems opportune to give a brief biography of the author and document his mathematical interests. While our investigations have led to a complete list of his publications, we shall give but a selection here. We have used the writings of R. Marcolongo "Bollettino di Matematica (1926)", of P. Teofilato "Memorie della Pontificia Accademia dei Nuovi Lincei (1926)," G. Zappa "Supplemento ai Rendiconti