PROBLEM PAGE

Editor: Phil Rippon

My first problem this time arose from a computer experiment in iteration theory. Over the last few years, one of the commomest purposes to which computers have been put has been the plotting of the Mandelbrot set, defined as follows.

First put

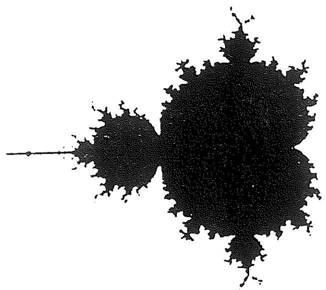
$$z_{n+1}(c) = z_n(c)^2 + c, \quad n = 0, 1, \ldots$$

where c is a complex number and $z_0 = 0$. Thus $z_1(c) = c$, $z_2(c) = c^2 + c$, and so on. The Mandelbrot set is

$$M = \{\, c : z_n(c) \not\to \infty \,\},$$

or equivalently, by an elementary argument,

$$M = \{\, c : |z_n(c)| \leq 2, \quad n = 1, 2, \ldots \}.$$



M

To plot M therefore, we calculate for each value of c corresponding to a screen pixel, the sequence $z_n(c)$, $n = 1, 2, \ldots, N$ (where $N = 30$, say) and we plot the pixel if $|z_n(c)| \leq 2$, for $n = 1, 2, \ldots, N$. The familiar set M appears below. Increasing the value of N should in theory give a better approximation to M, but in practice there is an optimal N depending on the screen resolution.

Recently, my colleagues David Grave, Robert Hassan and Peter Strain-Clark were using a transputer system to plot M when they came upon an interesting relation of M, obtained by using the iteration formula

$$z_{n+1}(c) = \overline{z_n(c)}^2 + c, \quad n = 1, 2, \ldots \tag{1}$$

where $z_0(c) = 0$. This relation of the Mandelbrot set has a rather unexpected property.

22.1 *Let $z_n(c)$ be defined by (1). Prove that the Mandelbar set*

$$M_{\text{bar}} = \{\, c : |z_n(c)| \leq 2, \quad n = 1, 2, \ldots \}$$

has rotational symmetry.

The set M_{bar} has many other intriguing properties; for example, it seems to contain many small copies of itself as well as small copies of the Mandelbrot set! Anyone who has a program for plotting the Mandelbrot set should be able to plot M_{bar} by inserting a minus sign in the appropriate place.

Just one other problem this time. I've forgotten where I heard this, and would appreciate any reference to it.

22.2 *Prove that it is impossible to tile the plane with triangles in such a way that at most 5 triangles meet at each vertex.*

Now to earlier problems. First a remark about problem **11.2** (in the new notation), which concerned sequences of the form

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots$$

If a_0, a_1, are real, then a_n is periodic with period 9. It has now been proved, by Dov Aharanov and Uri Elias, that if a_0, a_1 are complex, then such sequences are always bounded (the proof looks very complicated and uses a theorem of Moser concerning 'twist mappings').

The first problem of Issue 20 was as follows:

20.1 Find a formula whose value is 64, which uses the integer 4 twice, and no operations other than: $+$, $-$, $\times$, $/$, $\uparrow$, $\sqrt{\ }$ and !.

The answer is

$$64 = \left(\sqrt{\sqrt{\sqrt{4}}}\right)^{4!}$$

This problem is reminiscent of the "four fours" problem: how many of the integers can you express using the integer 4 four times, and the above operations? I remember wasting idle moments in my youth expressing all the integers from 1 to 100 in this way, but one of my OU students last year showed me a notebook he had completed some forty years ago which contained expressions for all integers up to 1000 and many beyond!

20.2 Given positive integers a_k, $k = 1, 2, \ldots N$ (not necessarily distinct), prove that some sum of the form

$$a_{k_1} + a_{k_2} + \ldots + a_{k_m}, \quad 1 \le k_1 < k_2 < \ldots < k_m \le n$$

is equal to 0 mod n.

My colleague John Mason, who showed me this problem, calls it the "some sum" problem. There is a very neat solution, which shows that some sum of the form

$$a_p + a_{p+1} + \ldots + a_q, \quad 1 \le p \le q \le n \tag{2}$$

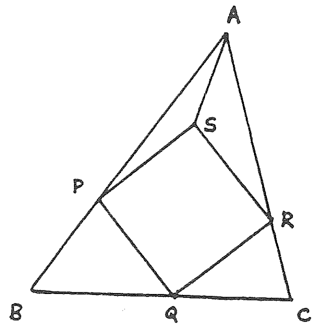
is equal to 0 mod n. Indeed, if the n integers

$$a_1 + a_2 + \ldots + a_m, \quad m = 1, 2, \ldots n$$

are distinct mod n, then one of them is equal to 0 mod n. Otherwise, two of them are equal mod n and so their difference is equal to 0 mod n. Either way, we get a sum of the form (2).

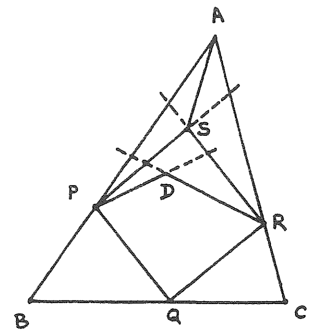
20.3 Show that if a square lies within a triangle, then its area is at most half the area of the triangle.

Here is a proof by paperfolding! We may assume, by scaling the triangle while keeping the square fixed, that at least three vertices of the square lie on the triangle.



We now claim that the reflections of the four triangles APS, ARS, BPQ, CQR, across their respective sides of the square $PQRS$ combine to completely cover the square.

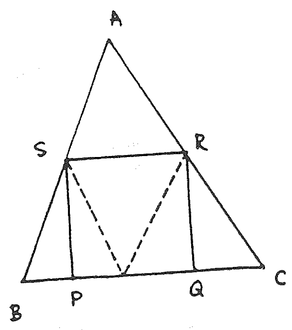
To see this, note that the angles ASP, ASR are non-acute, so that the reflections of the triangles ASP and ASR together cover the quadrilateral $PSRD$.



It is then clear that the reflections of triangles BPQ, CRQ across their respective sides of the square together cover the quadrilateral $PQRD$, since the reflection of PB lies along PD, the reflection of RC lies along RD, and the reflections of QB, QC lie along a common line through Q.

In fact this proof assumes that angles APS, ARS are both less than 45°. If one of them exceeds 45° (and at most one of them can), then a similar argument applies, but the point D lies on the edge of the square.

Further consideration of this paperfolding approach shows that the case of equality occurs precisely when all four vertices lie on sides of the triangle and the side of the triangle which contains two vertices of the square is twice as long as the side of the square.



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