leads to the same linearized value, here λ, μ are ordinary numbers.
Case (ii): A, B anti-commute; that is, $AB = -BA$, which will be the case when A and B are fermion operators. In that case, consistency demands that λ and μ anti-commute with one another, and also with the operators A, B, then λ, μ may be taken as Grassmann or Clifford numbers.

Thus a general hamiltonian, after linearization by this method, will look naturally like an element of a superalgebra, with $A_{\bar{1}}$-type elements multiplied by Grassmann (or Clifford) numbers, just as in the simple example above. This approach has recently been used to give a superlagebraic model of superconductivity [**6**].

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Integrals of Subharmonic Functions

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This article reviews a problem concerning potential theory in $\mathbf{R}^n$ which has its roots in classical complex analysis. One of the interesting features of the problem is the way in which the solution has gradually emerged, sometimes in a surprising fashion. The article is based on a lecture given at the First September Meeting of the Society, held at Trinity College, Dublin.

1 Background in C

Let $N(f, r)$ denote the maximum modulus of an analytic function f on the circle $\{z \in \mathbf{C} : |z| = r\}$. The starting point for our discussion is provided by the following facts from elementary complex analysis.

Hadamard's Three Circles Theorem. *If f is analytic on $\{|z| < R\}$ and $f \not\equiv 0$, then $\log N(f, r)$ is convex as a function of $\log r$.*

Principle of Removable Singularities. *If f is analytic on $\{0 < |z| < R\}$ and $rN(f, r) \to 0$ as $r \to 0+$, then f has an analytic continuation to $\{|z| < R\}$.*

The latter result is saying that either $N(f, r)$ behaves badly near 0 or else 0 is a removable singularity for f, in which case $N(f, r)$ is continuous at 0. The Three Circles Theorem has the following analogue for suprema over lines. (See [14, p.180] for an important application of this result in the proof of the M. Riesz convexity theorem.)

Three Lines Theorem. *Let f be bounded and analytic on $\mathbf{R} \times (0, 1)$, continuous on $\mathbf{R} \times [0, 1]$, and let $f \not\equiv 0$. Then*

$$y \mapsto \sup\Big\{\log\big|f(x + iy)\big| : x \in \mathbf{R}\Big\}$$

defines a convex function on $[0, 1]$.

We will be concerned with analogues of the above results for integrals of subharmonic functions. We recall that a function s defined on a connected open subset ω of $\mathbf{R}^n$ ($n \geq 1$) and taking values in $[-\infty, +\infty)$ is called *subharmonic* if $s \not\equiv -\infty$ and:

(i) s is upper semicontinuous (u.s.c.), i.e. $\limsup_{Y \to X} s(Y) = s(X)$ for all $X \in \omega$;

(ii) the mean of s over the boundary of any closed ball in ω is greater than or equal to its value at the centre.

Notes. (I) A function h is harmonic (i.e. h satisfies Laplace's equation) if and only if both h and $-h$ are subharmonic.
(II) If f is analytic on $\mathbf{C}$ and $f \not\equiv 0$, then $\log|f|$ is subharmonic. (Here we are identifying $\mathbf{C}$ with $\mathbf{R}^2$ in the usual way).
(III) Condition (ii) above can be replaced by (ii′): for any open set W with compact closure in ω, and for any continuous function h on $\overline{W}$ which is harmonic on W and satisfies $h \geq s$ on ∂W, we have $h \geq s$ on W.
(IV) Although it is usual to work with subharmonic functions on open subsets of $\mathbf{R}^n$, where $n \geq 2$, the definition also makes sense for $n = 1$. We discuss this further at the end of Section 3.

2 Convexity Theorems

If s is a non-negative subharmonic function on $\mathbf{R}^{n-1} \times (0,1)$, put

$$M(x_n) = \int_{\mathbf{R}^{n-1}} s(x_1, \ldots, x_n)\, dx_1 \ldots dx_{n-1} \quad (0 < x_n < 1).$$

The following analogue of the Three Lines Theorem is essentially due to Hardy, Ingham and Pólya [8] in the case $n = 2$. (See also [13, 9]).

Theorem 1 *If $M(\cdot)$ is locally bounded on $(0,1)$, then it is convex.*

Proof ($n = 2$). Let $0 < \alpha < \beta < 1$, and choose a, b such that $ay + b = M(y)$ for $y = \alpha, \beta$. Now define

$$h_\epsilon(x, y) = ay + b + \epsilon \cosh(\pi x) \sin(\pi y)$$

(a harmonic function), and

$$u_\ell(x, y) = \int_{-\ell}^{\ell} s(x + t, y)\, dt,$$

which is subharmonic because it is finite valued, u.s.c. (by Fatou's Lemma) and submeanvalued (by Tonelli's Theorem). Also $u_\ell \leq h_\epsilon$ on $\mathbf{R} \times \{\alpha, \beta\}$ and

$$u_\ell(x, y) - h_\epsilon(x, y) \to -\infty \quad (|x| \to \infty,\ \alpha \leq y \leq \beta),$$

so (cf. (ii′) above) $u_\ell \leq h_\epsilon$ on $\mathbf{R} \times [\alpha, \beta]$. Letting $\epsilon \to 0+$ and $\ell \to \infty$, we get $M(y) \leq ay + b$ for $y \in [\alpha, \beta]$, proving convexity.

Question. Is local boundedness the "right" condition?

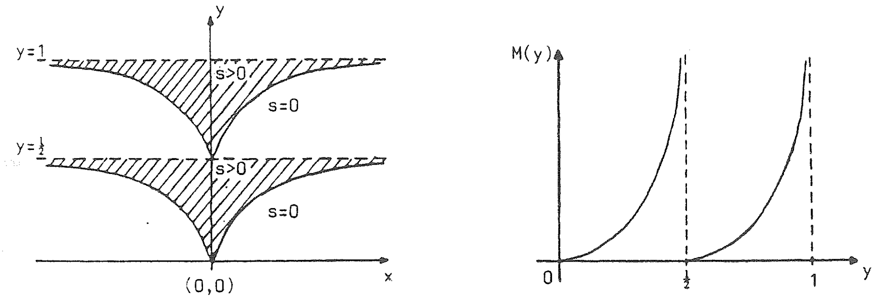
The hypothesis cannot be dispensed with entirely. To give some idea of possible behaviour we give below a few simple examples when $n = 2$.

Examples (i) $s(x, y) \equiv 1$; $\qquad M(y) \equiv +\infty$.

(ii) $s(x, y) = e^{2\pi x}|\sin \pi y|$; $\qquad M(y) = \begin{cases} 0 & \text{if } y \in \{0, \frac{1}{2}, 1\} \\ +\infty & \text{otherwise.} \end{cases}$

(iii) $s(x, y) = \dfrac{e^y}{x^2 + (y+1)^2}$; $\qquad M(y) = \dfrac{\pi e^y}{y + 1}$.

(iv)



Thus $M(\cdot)$ may be everywhere infinite, or everywhere finite, or neither. Even if $M(\cdot)$ is always finite, it need not be convex.

Theorem 2. *If $M(\cdot)$ is locally integrable on $(0,1)$, then it is finite and convex.*

This result, due to Kuran [10], shows that convexity holds provided we restrict the type of discontinuity that is allowed to occur. It was substantially improved when Rippon [12] applied a result of Domar to obtain the following.

Theorem 3. *If $\log^+ M(\cdot)$ is locally integrable on $(0,1)$, then $M(\cdot)$ is finite and convex.*

It was also shown in [12] that the hypothesis here is best possible, so the convexity property of $M(\cdot)$ is now satisfactorily described. However, we will mention a recent generalization [7] which shows what happens when integration of s is carried out with respect to fewer of the co-ordinates.

3 A Generalization

A subset E of ω is called *polar* if there is a subharmonic function on ω which takes the value $-\infty$ on E. A function s is said to be *quasi-subharmonic* if the function $\widehat{s}(X) = \limsup_{Y\to X} s(Y)$ is subharmonic, and $\widehat{s}$ equals s except on a polar set.

Let $X = (X', X'') \in \mathbf{R}^{n-m} \times \mathbf{R}^m$, $(2 \le m \le n-1)$, and put

$$P(s, X'') = \int_{\mathbf{R}^{n-m}} s(X', X'')\,dX',$$

$$P_\infty(s, X'') = \sup\{s(X', X'') : X' \in \mathbf{R}^{n-m}\}.$$

Theorem 4. *Let s be subharmonic on $\mathbf{R}^{n-m} \times (0,1)^m$.*

(i) If $\{\log^+ P(s^+,\cdot)\}^{m+\epsilon}$ is locally integrable on $(0,1)^m$, then $P(s,\cdot)$ is either subharmonic on $(0,1)^m$ or identically valued $-\infty$.

(ii) If $\{\log^+ P_\infty(s^+,\cdot)\}^{m+\epsilon}$ is locally integrable on $(0,1)^m$, then $P_\infty(s,\cdot)$ is quasi-subharmonic on $(0,1)^m$.

Notes . The hypotheses can be weakened slightly [7]. A version of (i) with stronger hypotheses was proved independently by Aikawa [1].

Example. To see that, in (ii), quasi-subharmonicity is the best that can be said, let E be a polar subset of $(0,1)^m$ $(m \ge 2)$, and let u be a negative subharmonic function taking the value $-\infty$ on E. Then the function $s(X) = -\{-u(X'')\,|X|^{2-n}\}^{1/2}$ can be shown to be subharmonic on $\mathbf{R}^{n-m} \times (0,1)^m$, and clearly

$$P_\infty(s, X'') = \begin{cases} -\infty & \text{if } u(X'') = -\infty \\ 0 & \text{elsewhere on } (0,1)^m. \end{cases}$$

Consider now the notions of harmonicity and subharmonicity for functions of one real variable. A "harmonic" function h must satisfy $d^2h/dx^2 \equiv 0$, so $h(x) = ax + b$. From condition (ii′) of §1, if a "subharmonic" function s satisfies $s(x) \le h(x)$ at $x = \alpha, \beta$, the same inequality holds for $x \in (\alpha, \beta)$, so s is convex. Since it is impossible for a convex function to take the value $-\infty$, the only polar subset of $\mathbf{R}$ is the empty set. Hence, in $\mathbf{R}$, the terms "convex", "subharmonic" and "quasi-subharmonic" are synonymous. Thus Theorem 4 generalizes (in different ways) Theorems 1 – 3 and the Three Lines Theorem.

4 Growth Theorems

We now consider analogues for $M(\cdot)$ of the Principle of Removable Singularities. In what follows, we assume that s is a non-negative subharmonic function on the half-space $\mathbf{R}^{n-1} \times (0, +\infty)$, and that $M(\cdot)$ is finite and convex on $(0,+\infty)$. We also note that, if $M(\cdot)$ is bounded on $(a, +\infty)$ for some $a > 0$, then $M(\cdot)$ is decreasing (wide sense).

The following is due to Flett [6].

Theorem 5. *If $M(y) = O(y^{n-1})$ as $y \to +\infty$, then $M(\cdot)$ is decreasing.*

Proof Let $B(X, r)$ denote the open ball of centre X and radius r, and let ν denote the volume of $B(O,1)$. By hypothesis there exists $c > 0$ such that $M(y) \le cy^{n-1}$ for all $y \ge \frac{1}{2}$. If $x_n \ge 1$, then

$$\begin{aligned} s(X) &\le \frac{1}{\nu(x_n/2)^n} \int_{B(X,x_n/2)} s(Y)\,dY \quad (\text{cf } \S1,\ (\text{ii})) \\ &\le \frac{1}{\nu(x_n/2)^n} \int_{\mathbf{R}^{n-1}\times(x_n/2,3x_n/2)} s(Y)\,dY \\ &= \frac{1}{\nu(x_n/2)^n} \int_{x_n/2}^{3x_n/2} M(y)\,dy \end{aligned}$$

$$\leq \frac{c}{\nu(x_n/2)^n}\int_{x_n/2}^{3x_n/2} y^{n-1}\,dy = \text{constant}.$$

Thus s is bounded on $\mathbf{R}^{n-1}\times[1,+\infty)$, and it follows that s is majorized by its Poisson integral I_s on $\mathbf{R}^{n-1}\times(1,\infty)$. Hence

$$M(x_n) = \int_{\mathbf{R}^{n-1}} s(X)dx_1\ldots dx_{n-1} \leq \int_{\mathbf{R}^{n-1}} I_s(X)dx_1\ldots dx_{n-1} = M(1) \qquad (x_n > 1)$$

by Tonelli's theorem, and so $M(\cdot)$ is bounded on $(1,+\infty)$.

In fact, Kuran [10] showed that the exponent in Theorem 5 can be increased.

Theorem 6. *If $M(y) = o(y^n)$, then $M(\cdot)$ is decreasing.*

Example . To see that the exponent cannot be further increased in the case $n=2$, let $\alpha > 1$ and

$$s(re^{i\theta}) = \begin{cases} r^\alpha \cos\alpha(\theta-\frac{\pi}{2}) & (|\theta-\frac{\pi}{2}|<\frac{\pi}{2\alpha}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then s is subharmonic on $\mathbf{R}\times(0,+\infty)$ and $M(y) = \text{const.}\, y^{\alpha+1}$ for $y>0$. (For $n\geq 3$, a similar example is based on Legendre functions).

However, Nualtaranee [11] was able to refine Kuran's hypothesis.

Theorem 7. *If $M(y) = O(y^n)$, then $M(\cdot)$ is decreasing.*

The problem of finding the "correct" condition is now clearly down to a matter of "fine tuning". A contribution in this direction was obtained by Rippon [12] using a result of Dahlberg about minimally thin sets in half-spaces.

Theorem 8. *If s has a harmonic majorant on $\mathbf{R}^{n-1}\times(0,+\infty)$ and*

$$\int_1^\infty \min\left[1, \{y/M(y)\}^{1/(n-1)}\right] dy = +\infty, \qquad (*)$$

then $M(\cdot)$ is decreasing.

Condition $(*)$ was also shown to be the best possible. Using the convexity of $M(\cdot)$ it can be seen that $(*)$ is implied by the condition $\liminf_{y\to+\infty} y^{-n}M(y) < +\infty$. It is now not difficult to obtain the following improvement of Theorem 7.

Corollary. *If $\liminf_{y\to+\infty} y^{-n}M(y) < +\infty$ then $M(\cdot)$ is decreasing.*

Open Question . Can the hypothesis about the harmonic majorant be removed from Theorem 8?

This question appears to be difficult. If the answer is "yes", then Rippon's condition $(*)$ is best possible [12].

5 An Extension

We mention now a recent result [3] which shows what can be said about the growth of $M(\cdot) = M(s,\cdot)$ when we drop the requirement that s be non-negative. Again, s denotes a subharmonic function on $\mathbf{R}^{n-1}\times(0,+\infty)$.

Theorem 9. *If $\log^+ M(s^+,y) = o(y)$ and*

$$\int_1^\infty y^{-n-1}M(s,y)\,dy < +\infty,$$

then $M(s,\cdot)$ and $M(s^+,\cdot)$ are decreasing, and $M(s^-,y) = o(y)$.

The proof of Theorem 9 begins by estimating the distributional Laplacian of s on strips and using this to show that s has a harmonic majorant on $\mathbf{R}^{n-1}\times(0,+\infty)$. With regard to the sharpness of the result we mention the following. (i) If $\log^+ M(s^+,y) = O(y)$, then all three conclusions fail. (ii) If we replace y^{-n-1} by $y^{-n-1-\epsilon}$, the counterexample of §4 (involving Legendre functions) applies. (iii) The conclusion about $M(s^-,\cdot)$ is best possible in that, if $\phi(y)$ decreases to 0 as $y\to+\infty$, then there is a negative subharmonic function s such that $M(s^-,y)\geq y\phi(y)$.

6 Other Results

A number of papers have dealt with $M(\Phi\circ s,\cdot)$, where Φ is an increasing, convex function (whence $\Phi\circ s$ is subharmonic). We mention here only the case $\Phi(x) = x^p$, where $p>1$. The following is a refinement of a result of Brawn [4] in the light of Theorem 3.

Theorem 10. *If s is non-negative and subharmonic on $\mathbf{R}^{n-1}\times(0,1)$ and $\log^+ M(s^p,\cdot)$ is locally integrable on $(0,1)$, then $\{M(s^p,\cdot)\}^{1/p}$ is finite and convex.*

The convexity property here is replaced by subharmonicity if we integrate only over $\mathbf{R}^{n-m}$ as in §3, (see [6]). With regard to growth theorems, we mention the following result of Armitage [2].

Theorem 11. *If s is non-negative and subharmonic on $\mathbf{R}^{n-1} \times (0, +\infty)$ and $M(s^p, y) = O(y^{n+p-1})$ as $y \to +\infty$, then $M(s^p, y)$ decreases to 0 as $y \to +\infty$.*

Thus, with s replaced by the "strongly subharmonic" function s^p, we can weaken the hypotheses of Theorem 7 and strengthen the conclusion.

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