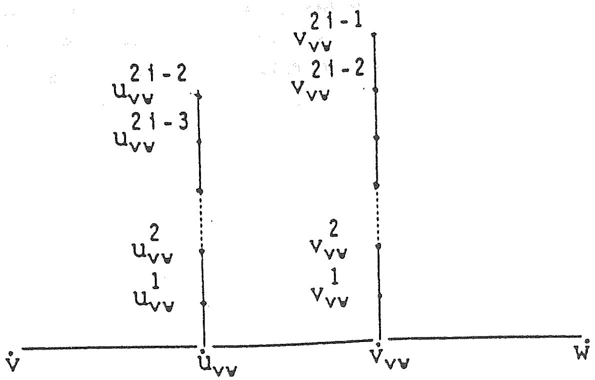
The fact that Γ is considered as a labelled directed graph may spoil the pleasure of some people so it is worth while pointing out that there is a simple means to replace a labelled directed graph by an unlabelled undirected graph with the same automorphism group, as follows:

We suppose the graph is labelled by a finite set, $\{x_1, \ldots x_n\}$. If the edge from v to w is labelled with x_i replace that edge by the subgraph:



Thus we replace the edge by a path v, u_{vw}, v_{vw}, w, at vertex u_{vw}, we attach a new path of length $2i-2$ and at v_{vw} a new path of length $2i-1$. This labels the edge in a purely graph theoretic way.

References

[1] J. McDermott, *Groups and trees*, Irish Mathematical Society Newsletter **10** (1984), 46–52.

[2] J. Stillwell, *Classical and Combinatorial Group Theory*, Graduate Texts in Maths., No. 72, Springer-Verlag, 1980.

School of Mathematics, University College of North Wales,
Bangor, Gwynedd LL57 1UT, Wales.

Algebra and Superalgebra

Allan Solomon

This article is based on a talk given by the author to the Dublin University Mathematical Society on April 30, 1987.

On this 210th Anniversary of Gauss' birthday, I should like to start off by referring to the Fundamental Theorem of Algebra, for which Gauss gave four proofs. The assertion of this theorem—that every polynomial equation has a root—must be interpreted by extending the real numbers to the complex, i.e., every polynomial over the real numbers has a (complex) root. In fact, the theorem remains true if we extend to polynomials over the complex field; the complex numbers are algebraically closed. We may, however, extend the complex numbers in an elegant and non-trivial way to the quaternions. The system of quaternions $\mathcal{H}$ provides us with our first example of an algebra. An algebra $\mathcal{A}$ is a linear space over a field $\mathbf{F}$, on which a multiplication, having the usual distributive properties, is defined. Essentially, we have

$$\alpha x + \beta y \in \mathcal{A} \quad \text{and} \quad xy \in \mathcal{A} \qquad (x, y \in \mathcal{A}, \alpha, \beta \in \mathbf{F}),$$

with

$$(\alpha x)y = \alpha(xy),$$
$$(\alpha x + \beta y)z = \alpha xz + \beta yz,$$
$$z(\alpha x + \beta y) = \alpha zx + \beta zy$$

Here α, β are elements of the field $\mathbf{F}$ over which the linear space $\mathcal{A}$is defined; in the usual applications this will be the real field $\mathbf{R}$ on the complex field $\mathbf{C}$. An algebra may, or may not, have the associative property;

$$(xy)z = x(yz) \qquad \forall x, y, z \in \mathcal{A}.$$

If it does, it is called an ***associative algebra.*** In general, the algebras and superalgebras of my title are ***not*** associative algebras. However, most algebras that have applications may be represented by matrices; and since matrices multiply associatively, we may effectively ***embed*** our algebras in associative algebras.

However, our algebras will not be *commutative*; that is $xy \neq yx$ is general. The quaternions are a non-trivial extension of the complex numbers because they form a non-commutative system, perhaps the earliest such example. Let us look more closely at this example.

A quaternion q is written $q = \alpha + \lambda i + \mu j + \nu k \quad (\alpha, \lambda, \mu, \nu \in \mathrm{R})$; so, as a vector space, $\mathcal{H}$ is 4-dimensional, with basis $\{1, i, j, k\}$. In order to define $\mathcal{H}$ as an algebrà, we must give a multiplication table for the basis elements; and these are the famous relations of Hamilton:

$$i^2 = j^2 = k^2 = -1; \quad ij = -j \cdot i = k, \quad j \cdot k = -k \cdot j = i, \quad h \cdot i = -i \cdot k = j.$$

So we see the non-commutativity in, for example $i \cdot j = -j \cdot i$.

How would such a non-commutativity, so non-intuitive from our experience with school alebra, arise naturally,? It arose naturally in a geometric context.

Consider the rotations of a sphere in R^3 with its centre at the origin. First we have rotate the sphere by an angle $\pi/2$ about the (fixed in space) K-axis, followed by a $\pi/2$ rotation about the J-axis. Now we perform these two operations in reverse order; a $\pi/2$ roration about J followed by a $\pi/2$ rotation about K. It is easy to see that the resulting position of the sphere is not he same in the two cases. These operations, of rotating the sphere, do not *commute.*

In fact, the operation of rotating the sphere through an angle θ about an axis through its centre may be represented by a quaternion

$$q = \alpha + \lambda i + \mu j + \nu k,$$

where $\alpha = \tan\theta/2$, and the axis has direction cosines $(\cos f, \cos g, \cos h)$ given by

$$\lambda = \tan\frac{\theta}{2}\cos f \quad \mu = \tan\frac{\theta}{2}\cos g \quad \nu = \tan\frac{\theta}{2}\cos h.$$

The rotation of the sphere is given by

$$xi + yj + zj \mapsto q(xi + yj + zk)q^{-1}$$

where $xi + yj + zk$ is a *unit* quaternion $(x^2 + y^2 + z^2 = 1)$ and so represents a point on the surface of the sphere.

For example, the first of the two rotations used above is given by $q_1 = 1 + k$ $(\theta = \pi/2$, and the axis contains $(0, 0, 1))$ The second is given by $q_2 = 1 + j$. If

we take the point k on the sphere, we have:
(a) Applying first q_1 and then q_2: $k \mapsto q_1 k q_1^{-1} = (1+k)k(1-k)/2 = k \mapsto (1+j)k(1-j)/2 = i$
(b) Applying first q_2 and then q_1: $k \mapsto q_2 k q_2^{-1} = (i+j)k(1-j)/2 = i \mapsto q_1 i q_1^{-1} = (1+h)i(1-h)/2 = j$.

This discovery of the connection between quaternions and rotations was made by Gauss in 1819, pre-dating Hamilton's work by almost a quarter of a century. However, Gauss did not publish it. The result was given in a paper of Olinde Rodrigues (1840) again pre-dating Hamilton, by three years. (I am indebted to my Open University colleague, Jeremy Gray for referring me to his article in the Archive for History of Exact Sciences where the details on this discovery of quaternions are laid out.)

Quaternions are not much used nowadays, mainly because they can be represented by the more familiar matrices with complex or real entries (see Diagram (a)). This is a pity. They do have their enthusiasts still, however.

A MATRIX REPRESENTATION OF THE QUATERNIONS

$$i = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$k = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$

$$ij = k, \quad jk = i, \quad ki = j$$

$$i^2 = j^2k^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

These matrices are equivalent to the Pauli Spin Matrices

Diagram (a)

Symplectic groups—much used in dynamics and physics—have their most elegant expression in terms of quaternionic matrices—although again, these groups may be expressed otherwise. The fact that a quaternion is al element of a 4-dimensional space would seem to indicate a possible use in relativity;

however, the quaternion

$$q = \alpha + \lambda i + \mu j + \nu k$$

has a natural norm

$$||q||^2 = \alpha^2 + \lambda^2 + \mu^2 + \nu^2;$$

while the relativistic theory would require a form such as $-\alpha^2+\lambda^2+\mu^2+\nu^2$. For this reason, Professor John Synge introduced *Minkowski quaternions* [2] or *minquats* for short (called physical quaternions by Silberstein in 1912),

$$q = q_4 + q_1 i + q_2 j + q_3 k$$

where q_4 is a pure imaginary ($q_4 = \sqrt{-1}\,\alpha$, α real). Now it is easy to see that if we multiply two minquats q, q' together, the scalar term $q_4 q_4'$ is no longer pure imaginary, and so the result is not a minquat. Thus minquats do not form an algebra. However, the general quaternions with complex coefficients *do* form an algebra, called biquaternions by Hamilton. The most amusing use of biquaternions I know is due to Louis Kaufman [3]: Define

$$H = H_1 i + H_2 j + H_3 k, \quad E = E_1 i + E_2 j + E_3 k,$$
$$J = J_1 i + J_2 j + J_3 k, \quad F = H + \sqrt{-1}E,$$

and define the operators D and ∇ by:

$$\nabla = \frac{\partial}{\partial x} + i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k,$$

$$D = \sqrt{-1}\frac{\partial}{\partial t} + \nabla$$

Then the six electrodynamics equations of Maxwell may be written as the single biquaternionic equation

$$DF = \sqrt{-1}\,\rho + J$$

The quaternions form a system very like ordinary numbers, in that we may add and subtract, multiply and divide. Technically, they form a *Division Ring*. But we have lost the commutativity of the real and of the complex numbers. If we relax associativity, we may define one final division ring, the *octonions* of Cayley. I say a *final* division ring, for it may be shown that there are no others. The property which you cannot preserve is the absence of divisors of zero; that is when two non-zero elements are multiplied together, the result cannot be zero. Quaternions have this property, which is equivalent to saying that every non-zero element has an inverse: Thus, if

$$q \in \mathcal{H}, \quad q = \alpha + \lambda i + \mu j + \nu k \neq 0$$

then

$$q^{-1} = \frac{(\alpha - \lambda i - \mu j - \nu k)}{(\alpha^2 + \lambda^2 + \mu^2 + \nu^2)}.$$

Biquaternions to *not* have this property' that is, they *have* divisors of zero:

$$(\sqrt{-1} + k)(\sqrt{-1} - k) = 0.$$

A more important contribution of Cayley is the idea of a matrix. Sets of matries (over $\mathbf{R}$ or $\mathbf{C}$, for examples) form Algebras; and the Algebras and Superalgebras I wish to consider in the sequel may all be represented by matrices.

Clifford Algebras and Grassman Algebras

In passing, I should like to refer to two sorts of algebra which have many applications nowadays Clifford (1845–1879) Algebras and Grassmann (1809–1877) Algebras. These are both associative algebras.

First of all, *Clifford Algebras*. We take a basis $\{e_1, e_2, \ldots, e_k\}$ for a real k-dimensional vector space, and then define a multiplication of the basis vectors:

$$e_i e_j = -e_j e_i \quad (i \neq j)$$
$$e_i^2 = -1.$$

By this means we define an algebra—we *generate* an algebra, since we are allowed products. But due to the reduction we can make if two elements in a product are equal (e.g., $e_1e_2e_1e_3 = -e_1e_1e_2e_3 = e_2e_3$) we need only consider products in which all the basis vectors are unequal; so the algebra has for basis

	1 (no e's),	e_i,	e_ie_j,	...	$e_1e_2\ldots e_k$
number of elements:	1	k	${}^kC_2$		1

The total number of elements in the basis for the algebra is

$$1 + {}^kC_1 + {}^kC_2 + \ldots + {}^kC_k = (1+1)^k = 2^k$$

Let us consider some cases of Clifford Algebras:

1. $k = 1$:· 2^1 elements,basis $\{1, e_1, e_1^2 = -1\}$, $\mathcal{A} \cong \mathbf{C}$, the Complex Numbers
2. $k = 2$: 2^2 elements, basis $\{1, e_1, e_2, e_1e_2\}$, $\mathcal{A} \cong \mathcal{H}$, the Quaternions.
3. $k = 3$: 2^3 elements, basis $\{1, e_i, e_i, e_j, e_1e_2e_3\}$, $\mathcal{A} \cong \mathcal{H} \oplus \mathcal{H}$.

The 16-element case ($k = 4$) is related to an algebra introduced by Dirac (1902–85) to describe electromagnetism.

If instead of taking $e_1^2 = -1$, we assume $e_i^2 = 0$, we obtain the *Grassmann Algebras*, again of dimension 2^k.

Jordan Algebras and Lie Algebras

To introduce the remaining algebras, I wish to talk about, we turn to Quantum Mechanics. In one formulation, the basic laws of Quantum Mechanics are algebraic in character; this is the matrix mechanics of Heisenberg (1901–76). The dynamical quantities Q and P for position and momentum respectively are to be thought of as Hermitian matrices—since these correspond to *real* physical observables.

Hermitian conjugation is a complex conjugation which also reverses the order of matrices: thus

$$(AB)^\dagger = B^\dagger A^\dagger; \quad (\sqrt{-1}\,A)^\dagger = -\sqrt{-1}\,A^\dagger.$$

The operators representing real physical quantities, such as P and Q, are Hermitian, that is

$$P = P^\dagger, \quad Q = Q^{\iota}.$$

It would therefore be very nice to form an *Algebra* of Hermitian matrices. Ordinary addition is no problem:

$$(A+B)^\dagger = A^\dagger + B^\dagger + A + B. \qquad (A = A^\dagger, B = B^\dagger)$$

But

$$(AB)^\dagger = B^\dagger A^\dagger = BA \neq AB \qquad (A = A^\dagger, B = B^\dagger).$$

in general. Nevertheless, it is not too difficult to devise multiplication rules which preserve hermiticity. These correspond to ***Jordan Algebras*** and ***Lie Algebras***.

Jordan Algebras:

$$A * B \equiv AB + BA$$

so that

$$(A*B)^\dagger = (AB+BA)^\dagger = B^\dagger A^\dagger + B^\dagger A^\dagger = BA + AB = A*B.$$

(Multiplication is always *commutative*.)

Lie(1842–1899) Algebras:

$$A * B = AB - BA$$

This multiplication actually preserves anti-hermiticity. If $A^\dagger = -A$ and $B^\dagger = -B$, then

$$(A*B)^{\iota} = (AB - BA)^\dagger = (B^\dagger A^\dagger - A^\dagger B^\dagger) = BA - AB = -A*B.$$

But if we consider our physical operator P, Q etc., to be $\sqrt{-1}$ times an anti-hermitian operator, this amounts to preserving hermiticity. (Multiplication is always *alternating* or *anti-commutative*.)

Neither the Jordan nor the Lie Algebras are associative; but for the Lie Algebras associativity is replaced by the Jacobi identity:

$$(A*B)*C + (B*C)*A + (C*A)*B = 0$$

It is conventional to write the $*$ operation for Lie algebras as a bracket:

$$[A, B] = AB - BA$$

This implies the possibility of embedding the Lie Algebra in an associative algebra, where $(AB)C = A(B)$ — always possible for Lie Algebras (the Poincare-Birkhoff-Witt Theorem). However, not every abstract Jordan Algebra is thus obtainable.

Lie Algebras are the most frequently met in Physics, since the basic operator Q and the momentum operator P, may be expressed as a Lie Algebra bracket:

$$[Q,P] \equiv QP - PQ = \sqrt{-1}\,\hbar\,1 \qquad (h = 10^{-27}\text{erg sec})$$

This relation gives rise to the famous Heisenberg uncertainty principle, which imposes limits on the simultaneous accuracy of measurement of the observables Q and P. And the above Lie Algebra, consisting of $\{Q, P, 1\}$, is a very elementary and very famous Lie Algebra, sometimes called the Heisenberg algebra. This leads to the very physical *Boson and Fermion Algebras.*

If we define

$$b = Q + iP/\sqrt{2}, \quad b^{\dagger} = Q - iP/\sqrt{2},$$

then, taking units for which $\hbar = 1$,

$$[b, b^{\dagger}] \equiv bb^{\dagger} - b^{\dagger}b = 1$$

gives an even simpler form of this Lie algebra. It is found in applicatins that the operator b is associated with a particle in Physics with zero spin (or an even number of spin units of $(1/2)\hbar$)). Such a particle is called a *boson.* Examples are mesons in nuclear physics and, most important of all, the photon in Quantum Optics.

If by analogy, we assume a similar Jordan algebra for a different operator f, we get the basic anti-commutation relation

$$\{f, f^{\dagger}\} \equiv ff^{\dagger} + f^{\dagger}f = 1 \quad (\hbar = 1)$$

Such a relation is satisfied by particles in physics which possess an odd number of spin-units. Examples are the particles of the nucleus, neutrons and protons, and, most importantly, the electron.

The property of a particle obeying either commutation or anti-commutation relations is called its "statistics", and can have a profound effect on the observed properties. For example, Helium Four consists of bosons, and becomes superfluid at about two degrees above absolute zero. The very similar isotope Helium Three, on the other hand, is a gas of Fermions and becomes superfluid, only at about under a thousandth of a degree above absolute zero.

If we wish to consider algebras in which both types of statistics are simultaneously present, we are led to superalgebras.

Superalgebras

Superalgebras are simply mixtures of a Lie Algebra with a Jordan Algebra; or, an algebra which incorporates both the commutation relation of a Lie Algebra and the anti-commutation of a Jordan Algebra. Physically, we culd call them boson-fermion algebras. Thus the basic operation is neither commutation $[x,y] \equiv xy - yx$, or anti-commutation $\{x,y\} = xy + yx$, but an operation which can be either, depending on the elements x and y.

Abstractly, we write our superalgebra $\mathcal{A}$ as a sum of algebras

$$\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}};$$

that is, every element x in $\mathcal{A}$ belongs either to $\mathcal{A}_{\bar{0}}$ or $\mathcal{A}_{\bar{1}}$;, and

$$[x,y\} = -(-1)^{\alpha\beta}[y,x\}$$

where $x \in \mathcal{A}_{\alpha}$, $y \in \mathcal{A}_{\beta}$; thus $\alpha,\beta = \bar{0}$: $[x,y\} = -[y,x\}$ (Lie type)
$\alpha,\beta = \bar{1}$: $[x,y\} = [y,x\}$ (Jordan type)
$\alpha = \bar{0}, \beta = \bar{1}$: $[x,y\} = -[y,x\}$ (Lie type)
and

$$[\mathcal{A}_{\alpha}, \mathcal{A}_{\beta}\} \subset \mathcal{A}_{\alpha+\beta}.$$

The supersymmetry associated with superalgebras provides a theoretical framework for some current theories of Particle Physics [4] (although I am informed it has not been observed to date experimentally) and this idea has been used in Nuclear Physics and, more recently, in Condensed Solid State Physics.

We give a simple example of a superalgebra in Diagram (b), representing the elements of the 4–dimensional algebra $\mathcal{A}$ by 2×2 matrices over $\mathbf{R}$. Note that although the example may be simple, the algebra $\mathcal{A}$ is not 'simple' in the technical sense, in that it possesses a (non-trivial) ideal; in fact $\{\alpha 1 : \alpha \in \mathbf{R}\} \subset \mathcal{A}$ is such an ideal. Just as a complete classification of all the simple Lie Algebras (finite dimensional over fields of characteristic zero) has been given by E. Cartan (1869–1951) and others, a similar classification has been made for superalgebras by Victor Kac of M.I.T. (1977).

An interesting confluence of the ideas of Clifford and Grassmann Algebras with those of Lie Algebras and Superalgebras arises when we consider representations of the latter by matrices [5]. We may reduce both types of bracket (Lie and Jordan) to a single type (Lie) by introducing a representation in terms of matrices over a Grassmann algebra instead of, say, the reals. We

AN EXAMPLE OF A SUPERALGEBRA

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x, y \in \mathcal{A}_{\bar{1}}, \quad h, 1 \in \mathcal{A}_{\bar{0}}$$

$\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ is generated by $x, y, h, 1$.

$$[x, y\} = xy + yx = 1$$

$$[h, x\} = hx - xh = x$$

$$[h, y\} = hy - yh = -y$$

In all other cases, $[a, b\} = 0$.

Diagram (b)

illustrate this using our small superalgebra $\mathcal{A}$ above. If every element of $\mathcal{A}_{\bar{1}}$ is assumed to be multiplied by an element (of the odd part) of a Grassmann algebra then, for example, since

$$[e_1 x, e_2 y] \equiv e_1 x e_2 y - e_2 y e_1 x = e_1 e_2 (xy + yx) = e_1 e_2 [x, y\}$$

we obtain closure by consideration only of the commutator (Lie) bracket. (We assumed in the above that the elements of the Grassmann algebra commuted with the elements of $\mathcal{A}$; we may alternatively assume that the e_i anti-commute with $\mathcal{A}_{\bar{1}}$, commute with $\mathcal{A}_{\bar{0}}$. And since we only used the property $e_1 e_2 = -e_2 e_1$, a Clifford algebra would also provide a convenient representation for a superalgebra.)

We conclude this note by indicating how these algebras may arise when considering physical systems. The dynamics of such systems are governed by a hamiltonian H, an operator expressed in terms of other operators of the theory. The time evolution of an operator A is given by

$$\sqrt{-1}\,\frac{d}{dt} A = [A, H] \equiv AH - HA$$

where we have a Lie Bracket on the right-hand side. This bracket is a natural operation when both A and H belong to a Lie Algebra, or a Superalgebra (with H in the even part $\mathcal{A}_{\bar{0}}$). This would occur when, for example, the operators are linear or bilinear in boson or fermion operators (b, f) described above. Otherwise, an approximation process ("linearization") may be used (called 'Mean Field Theory' in Many Body Physics). Suppose $H = AB$, where A, B are some operators. We may write the identity

$$H = AB \equiv (A - \lambda)(B - \mu) + A\mu + \lambda B + \lambda\mu.$$

Typically, λ, μ are thought of as the expectation values of the operators A, B respectively in some state ω of the system. In the event that we may neglect the $(A - \lambda)(B - \mu)$ term—rationalizing this by assuming we do not consider states for which operators A, B stray far from the ω values—we may approximate:

$$H_{\mathbf{approx}} \sim A\mu + \lambda B - \lambda\mu.$$

This approximation is only consistent if
Case (i): A, B commute; that is, $AB = BA$ and so the approximation for BA

leads to the same linearized value, here λ, μ are ordinary numbers.
Case (ii): A, B anti-commute; that is, $AB = -BA$, which will be the case when A and B are fermion operators. In that case, consistency demands that λ and μ anti-commute with one another, and also with the operators A, B, then λ, μ may be taken as Grassmann or Clifford numbers.

Thus a general hamiltonian, after linearization by this method, will look naturally like an element of a superalgebra, with $A_{\bar{1}}$-type elements multiplied by Grassmann (or Clifford) numbers, just as in the simple example above. This approach has recently been used to give a superlagebraic model of superconductivity [**6**].

References

[1] J.J. Gray, Archive for History of Exact Sciences 21 (375), 1980.

[2] J.L. Synge, Communications of the Dublin Institute for Advanced Studies, Series A, No. 21, 1972.

[3] L. Kaufmann, Private Communication.

[4] For an Elementary introduction see P.G.O. Freund, *Introduction to Supersymmetry*, Cambridge Univ. Press, 1986.

[5] I. Bars, *Supergroups and their Representations*, in "Applications of Group Theory and Physics and Mathematical Physics", M. Flato et. al. (eds.), A.M.S. Lectures in Applied Mathematics 21, Providence, 1988.

[6] A. Nontarsi, M. Rasetti and A.I. Solomon, *Dynamical Superalgebra and Supersymmetry for a Many-Fermion System* (to be published).

Faculty of Mathematics
Open University, Milton Keynes, UK

Integrals of Subharmonic Functions

Stephen J. Gardiner

This article reviews a problem concerning potential theory in $\mathbf{R}^n$ which has its roots in classical complex analysis. One of the interesting features of the problem is the way in which the solution has gradually emerged, sometimes in a surprising fashion. The article is based on a lecture given at the First September Meeting of the Society, held at Trinity College, Dublin.

1 Background in C

Let $N(f,r)$ denote the maximum modulus of an analytic function f on the circle $\{z \in \mathbf{C} : |z| = r\}$. The starting point for our discussion is provided by the following facts from elementary complex analysis.

Hadamard's Three Circles Theorem. *If f is analytic on $\{|z| < R\}$ and $f \not\equiv 0$, then $\log N(f,r)$ is convex as a function of $\log r$.*

Principle of Removable Singularities. *If f is analytic on $\{0 < |z| < R\}$ and $rN(f,r) \to 0$ as $r \to 0+$, then f has an analytic continuation to $\{|z| < R\}$.*

The latter result is saying that either $N(f,r)$ behaves badly near 0 or else 0 is a removable singularity for f, in which case $N(f,r)$ is continuous at 0. The Three Circles Theorem has the following analogue for suprema over lines. (See [14, p.180] for an important application of this result in the proof of the M. Riesz convexity theorem.)

Three Lines Theorem. *Let f be bounded and analytic on $\mathbf{R} \times (0,1)$, continuous on $\mathbf{R} \times [0,1]$, and let $f \not\equiv 0$. Then*

$$y \mapsto \sup\left\{\log|f(x+iy)| : x \in \mathbf{R}\right\}$$

defines a convex function on $[0,1]$.