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ARTICLES

Group Presentations, Topology and Graphs

Timothy Porter

A few years ago John McDermott wrote a short article [1] for the Newsletter, as it was then called. This article takes up the relationship studied briefly in his article and looks at several other uses of simple graph-theoretic ideas in the study of group presentations. The level of graph theory involved is not much deeper than that used in his article The material is used quite successfully in both a three year course in Knot Theory and in an M.Sc. course in algebra at U.C.N.W., Bangor.

1 Group Presentations

As examples of group presentations, we will use a few very simple ones such as:
C_6, the cyclic group oforder 6, having an obvious presentation $(a : a^6)$, but also another slightly more subtle one, $(x, y : x^2, y^3, [x, y])$. D_3, the dihedral group of order 6, with a presentation $(a, b : a^3, b^2, (ab)^2)$.
In each case we specify a set of generators and some relations between them. To be slightly more precise, we recall:

$$X \subset G \text{ generates } G \text{ if } X \subset H \leq G \text{ implies } H = G$$

i.e., if there is no proper subgroup of G containing X. In this case every $g \in G$ can be written nonuniquely as a word in elements from $X \cup X^{-1}$.

The relations in the presentation are there to handle the *problem of nonuniqueness of representative words.* This is simply illustrated by the following example.

In C_6, $X = \{a\}$, $a^8 = a.a.a.a.a.a.a.a$, and $a^2 = a.a$ representing the same element. This makes it awkward to talk about the relationships between different words; in some way we want to say that a^8 and a^2 are different words but at the same time they are equal as elements of $C3$. The solution to the conundrum is to form the free group on a set $Y \cong X$, satisfying $Y \cap G = \emptyset$ to so that no confusion of symbols can arise. More generally, if Y is a set, we denote by $F(Y)$ the free group on Y, that is the group formed from all words in y's and y^{-1}'s (with any occurences of yy^{-1} etc. cancelled). If Y has n elements we say $F(Y)$ has rank n and write

$$r_{F(Y)} = n.$$

Any $\omega \in F(Y)$ can be written *uniquely* in the form

$$\omega = y_{i_1}^{\alpha_1} \dots y_{i_n}^{\alpha_n}$$

where $y_{i_1}, \dots y_{i_n} \in Y$, $\alpha_1 \dots \alpha_n \in \{-1, 1\}$ and $i_j \neq i_{j+1}$ $(1 \leq j \leq n)$. Now pick $f: Y \to G$. This will induce $\varphi: F(Y) \to G$ defined recursively by

$$\varphi(y\omega) = f(y)\varphi(\omega)$$

If φ is onto then $f(Y)$ generates G, so $\ker \varphi$ measures the *nonuniqueness* of representative words, i.e., the relations between the generators. To be able to study $\ker \varphi$ we pick $R \subset \ker \varphi$ so that $R \subset N \lhd F(Y) \Rightarrow N \geq \ker \varphi$, i.e., so that $\ker \varphi$ is the normal closure of R in $F(Y)$.

Example Let $\mathcal{P}$ be the presentation $(x, y : x^3, y^2, (xy)^2)$ of D_3, $F(Y) = F(x, y) = F_2$, free of rank 2

$$f: Y \to D_3 \quad \text{is given by} \quad \begin{cases} f(x) = a, & \text{rotation} \\ f(y) = b, & \text{reflection} \end{cases}$$

Any relation between a and b is a consequence of $r = x^3$, $s = y^2$, and $t = (xy)^2$ i.e. is a product of conjugates of r, s and t.

We note that $F = F(Y)$ acts on $N = N(R)$:

$$F \times N \to N, \quad (\omega, c) \mapsto {}^{\omega}c = \omega c \omega^{-1}$$

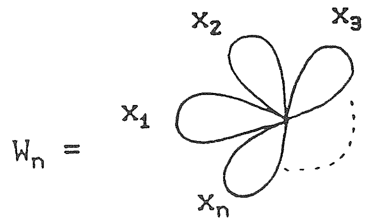
and any element, c, of N can be written in the form:

$$c = {}^{\omega_1}(r_1^{\varepsilon_1}) \dots {}^{\omega_n}(r_n^{\varepsilon_n})$$

with $r_i \in R$, $\varepsilon_i = \pm 1$, $\omega_i \in F$, $i = 1, 2, \dots n$.

2 Graphs

As explained in [1], a graph, Γ, consists of a set of vertices joined by edges; the edges may be directed, labelled (coloured) etc., as required. A path from a vertex, x, to a vertex, y, in a graph consists of a sequence, $x_0, x_1, \dots, x_k$, of vertices such that $x = x_0$, $y = x_k$ and each pair x_{i-1}, x_i is joined by an edge; if there are several edges joining x_{i-1} to x_i we must specify which of the edges is being used. As an example consider the following graph having a single vertex:



A path in W_n is exactly a word in the symbols x_i and their "inverses" x_i^{-1}; however, unlike the elements of a free group, in the paths we can have occurences of $x_i x_i^{-1}$. To construct a group from these edge paths in a graph, Γ, one first proves that any path in Γ determines a unique reduced path, i.e., a path in which such pairs do not occur; one then picks some vertex, v, and looks at the reduced paths that start and end at that vertex. Composition is given by putting two reduced paths next to each other and then forming the reduced path determined by them. The group one gets is called the fundamental group of the graph, and it is denoted $\pi_1(\Gamma, v)$. This group will in general depend on the choice of vertex, v, but if the graph is connected, i.e., if any two vertices can be joined by some path in Γ, then any two choices of base vertex give isomorphic groups, so if Γ is connected we can write $\pi_1(\Gamma)$ without serious risk of ambiguity. By picking a maximal spanning tree as indicated in [1] one can prove:

If G is a connected finite graph with α_0 vertices and α_1 edges, then $\pi_1(\Gamma)$ is free of rank $\alpha_1 - \alpha_0 + 1$.

In fact a basis for $\pi_1(\Gamma)$ can easily be found. Each element of the basis corresponds to an edge not in the spanning tree. The corresponding reduced path goes out in the tree to the start-vertex of the edge, crosses that edge in the given direction and returns to v again in the spanning tree.

We can model a graph, Γ, by a topological space, X, say, by taking a set of points in $\mathbf{R}^n$ as the vertices and for each edge an arc in $\mathbf{R}^n$ joining the corresponding vertices. With this model $\pi_1(\Gamma, v)$ is of course isomorphic to the topologically defined fundamental group $\pi_1(X)$ of the space X based at the vertex corresponding to v, (c.f. [2]).

3 Covering spaces and covering graphs

From topology we next take some results from the theory of covering spaces. These we really only need in the case that all the spaces concerned are graphs so the theory of "covering graphs" would suffice This can we found in the book by Stillwell, [2], which is an excellent source for much of this material.

If X is a space, and $H \leq \pi_1(X)$ then to H there corresponds a covering space, $p\colon X_H \to X$ with

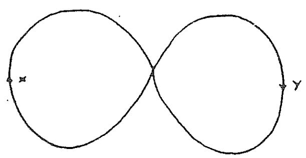
$$p_*\colon \pi_1(X_H) \to \pi_1(X)$$

a monomorphism with $\operatorname{Im} p_* = H$. If X is a graph, so is X_H.

To illustrate the idea in the case we want, take X to be W_m so that $\pi_1(X) \cong F(x_1, \ldots, x_m) = F$, say, and take $H \leq F$. Then X_H is the the Schreier diagram of H in F having the cosets of H in F as vertices and for each generator, x_i, and coset xH an edge labelled x_i from xH to $x_i xH$.

The map from X_H to X maps all the vertices to the one vertex of X and maps edges according to their labels. e.g. $\mathcal{P} = (x, y, : x^3, y^2, (xy)^2)$, the presentation of D_3 given earlier. Then

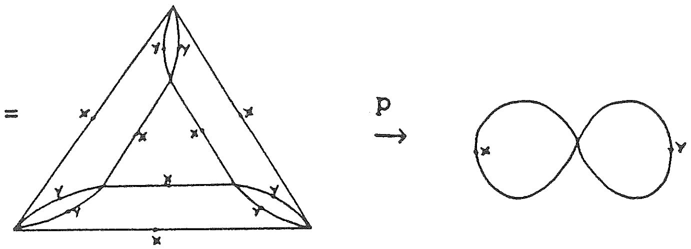
$X =$



$$H = N(x^3, y^2, (xy)^2)$$

Then

$X_H =$



In this case the Schreier diagram is easily seen to be the Cayley graph of the group with respect to the given generators.

The above discussion provides the basis for a proof of the following famous theorem.

Neilsen-Schreier: *If F free, and $H \leq F$ then H is free.*

Together with our remark earlier on the rank of the fundamental group of a graph and basic, easily verified, facts about induced maps between fundamental groups we get:

The Schreier Index Formula: *If r_F and r_H are finite and $|F : H| = i < \infty$ then*

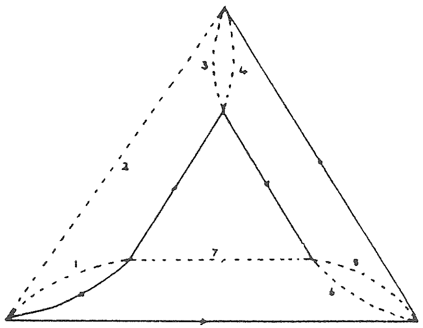
$$r_H = r_F i - i + 1$$

"Proof": X_H has i vertices and $r_F i$ edges.

4 Bases for $N(R)$

We saw that $H \cong \pi_1(X_H)$ is free. Can we find a basis? In other words, can we find elements freely generating H? Rather than looking at this in general we will consider our previous example in more detail. In that case $r_F = 2$, $|F : H| = |D_3| = 6$, so $r_H = 2 \times 6 - 6 + 1 = 7$, so we want a seven element basis for H.

We need to calculate $\pi_1(X_H)$ so we choose a maximal spanning tree T, in X_H; say

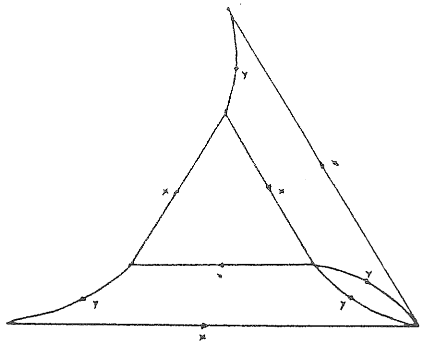


Generators of $\pi_1(X_H)$ correspond to the edges not in the tree as earlier. For our choice of tree they are:

1. $\underline{y}y$	2. $xx\underline{x}$	3. $xx\underline{y}^{-1}x^{-1}y$
4. $xx\underline{y}x^{-1}y$	5. $y^{-1}xx\underline{y}x^{-1}$	6. $x\underline{y}x^{-1}x^{-1}y$
	7. $y^{-1}xx\underline{xy}$	

in each word the underlined element corresponds to the edge not in T.

Each of these is a consequence of the relations $r = x^3$, $s = y^3$ and $t = (xy)^2$. This is not only a result of the overall theory but that can be "visualised" in the following way:



Take for instance the fourth basis element in our list:

$$\begin{aligned} 4.\ xxyx^{-1}y &= (xy)^2(y^{-1}x^{-1}y^{-2}xy)(y^{-1}x^{-1}y(xy)^2y^{-1}xy)(y^{-1}x^{-3}y) \\ &= t.^{y^{-1}x^{-1}}(s^{-1}).^{y^{-1}x^{-1}y}t.^{y^{-1}}(r^{-1}) \end{aligned}$$

Thus by interpreting the Cayley graph as a covering graph of the graph W_n one can find bases for the subgroup of relations and also one can express those basis elements as products of conjugates of the chosen relations. This method also enables one to identify certain identities among relations, but as that subject really needs another article to do it justice. I will not say more here.

5 Automorphisms of Graphs

From the theory of covering spaces one has the idea of a deck automorphism. This is as follows:

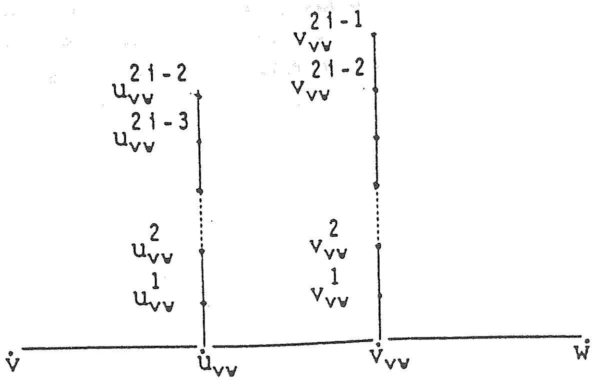
Suppose that $p: Y \to X$ is a covering graph, then a deck automorphism of p is an automorphism φ of the graph Y such that $p\varphi = p$. Of course the deck automorphisms of a covering graph (Y, p) form a group, $A(Y, p)$. In the case of the covering graph constructed from a presentation, this group is the subgroup of the automorphism group of the Cayley graph consisting of those automorphisms that preserve the labels. (We will look at this in our example shortly.) As the image of $\pi_1(X_H)$ in this case is normal, the covering graph is "regular" and the theory of covering spaces gives an isomorphism between $A(X_H, p)$ and $\pi_1(X_H)/\mathrm{Im}\{p^*: \pi_1(X_H) \to \pi_1(X)\}$. As this latter group is exactly the group that was being presented, we have the following result:

Theorem: (Frucht,1938) *If G is a finite group, X a generating set for G, Γ the labelled graph of G relative to X then G is isomorphic to the group of label preserving automorphisms of Γ.*

Example In our example in §3 with $G = D_3$, there are two obvious automorphisms of the graph that preserve labels. One rotates the graph through 120°; if the rotation is clockwise, this corresponds to the element x in D_3, if anticlockwise to x^{-1}. The other exchanges the inner and outer triangles; this corresponds to y. These two generate the deck/label automorphism group of the graph and the relations are clearly those given in the presentation of D_3.

The fact that Γ is considered as a labelled directed graph may spoil the pleasure of some people so it is worth while pointing out that there is a simple means to replace a labelled directed graph by an unlabelled undirected graph with the same automorphism group, as follows:

We suppose the graph is labelled by a finite set, $\{x_1, \ldots x_n\}$. If the edge from v to w is labelled with x_i replace that edge by the subgraph:



Thus we replace the edge by a path v, u_{vw}, v_{vw}, w, at vertex u_{vw}, we attach a new path of length $2i-2$ and at v_{vw} a new path of length $2i-1$. This labels the edge in a purely graph theoretic way.

References

[1] J. McDermott, *Groups and trees*, Irish Mathematical Society Newsletter **10** (1984), 46–52.

[2] J. Stillwell, *Classical and Combinatorial Group Theory*, Graduate Texts in Maths., No. 72, Springer-Verlag, 1980.

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Algebra and Superalgebra

Allan Solomon

This article is based on a talk given by the author to the Dublin University Mathematical Society on April 30, 1987.

On this 210th Anniversary of Gauss' birthday, I should like to start off by referring to the Fundamental Theorem of Algebra, for which Gauss gave four proofs. The assertion of this theorem—that every polynomial equation has a root—must be interpreted by extending the real numbers to the complex, i.e., every polynomial over the real numbers has a (complex) root. In fact, the theorem remains true if we extend to polynomials over the complex field; the complex numbers are algebraically closed. We may, however, extend the complex numbers in an elegant and non-trivial way to the quaternions. The system of quaternions $\mathcal{H}$ provides us with our first example of an algebra. An algebra $\mathcal{A}$ is a linear space over a field $\mathbf{F}$, on which a multiplication, having the usual distributive properties, is defined. Essentially, we have

$$\alpha x + \beta y \in \mathcal{A} \quad \text{and} \quad xy \in \mathcal{A} \qquad (x, y \in \mathcal{A}, \alpha, \beta \in \mathbf{F}),$$

with

$$(\alpha x)y = \alpha(xy),$$
$$(\alpha x + \beta y)z = \alpha xz + \beta yz,$$
$$z(\alpha x + \beta y) = \alpha zx + \beta zy$$

Here α, β are elements of the field $\mathbf{F}$ over which the linear space $\mathcal{A}$is defined; in the usual applications this will be the real field $\mathbf{R}$ on the complex field $\mathbf{C}$. An algebra may, or may not, have the associative property;

$$(xy)z = x(yz) \qquad \forall x, y, z \in \mathcal{A}.$$

If it does, it is called an ***associative algebra***. In general, the algebras and superalgebras of my title are ***not*** associative algebras. However, most algebras that have applications may be represented by matrices; and since matrices multiply associatively, we may effectively ***embed*** our algebras in associative algebras.