IRISH MATHEMATICAL SOCIETY BULLETIN NUMBER 22 MARCH 1989

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# IRISH MATHEMATICAL SOCIETY BULLETIN

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, in March and December. The Bulletin is supplied free of charge to to members by Local Representatives, or by surface mail abroad. Libraries may subscribe to the Bulletin for IR£20 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files. Articles submitted in the form of typed manuscripts will be given the same consideration as articles in TEX.

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# LETTERS

## Textbook prices

Dear Editor,
I would like to draw the attention of your readers to the sometimes large difference between the US and European prices of mathematics textbooks. For example:

- "Modern Algebra, an Introduction" by John R. Durbin (Wiley) costs St£36.05 here, US$ 32.45 in the USA;
- "Methods of Mathematics Applied to Calculus, Probability and Statistics" by Richard W. Hamming (Prentice-Hall) costs St£59.90 here, US$ 56.00 in the USA;
- "Calculus with Analytic Geometry" by George F. Simmons (McGraw-Hill) costs St£42.95 here, US$ 48.95 in the USA.

The above are all hardback editions. The UK price is taken from the October 1988 edition of "British Books in Print", the US price from the November 1988 edition of "Books in Print". In each case the UK price exceeds the US price by a factor of at least 50%.

These are not isolated examples. The differential in prices usually seems to be a function not of the individual book but of the publisher. Most publishers have virtually identical US and European prices, but as you see there are exceptions.

Irish and British libraries are apparently aware of the problem and are trying to control it. We as individuals can help in this endeavour, and simultaneously get better value from our book budgets, by being wary in the choice of books ordered for our institutional libraries. This is particularly easily done when choosing textbooks for undergraduate reading, where there is generally a wide selection of titles.

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# IRISH MATHEMATICAL SOCIETY

## Annual General Meeting

## December 22, 1988

The Annual General Meeting of the Irish Mathematical Society was held at the Dublin Institute for Advanced Studies at 12.15 p.m. on Thursday December 12 1988.

Twenty-two members were present, and the President, Professor S. Dineen was in the Chair.

1. The minutes of the meeting of September 9 1988 were read, approved, and signed.

2. Arising from these, the President announced the decision of the Committee to expand the September Meeting to a two-day event, and to hold the 1989 meeting in Maynooth. The meeting will begin after lunch on Thursday, September 7, 1989, and conclude on Friday the 8th. Overnight accommodation will be available for those who require it. Suggestions for invited speakers and proposals for contributed talks should be sent to Professor O'Farrell, preferably before the end of February 1989.

   The Committee invites aplication by institutions willing to host the 1990 September meeting. Applications should be sent to the Secretary before the end of February 1989.

3. Correspondence: The two reports of the EUROMATH project were summarised for the meeting by the Secretary. These are:
   (1) State of the Art Report, July 1988.
   (2) The EUROMATH Surveys, August 1988.

   Members expressed a strong view that EUROMATH should support TEX. Members who wish to ontain copies of the reports could apply to John Carroll, head of the EUROMATH unit at NIHE Dublin, Glasnevin, Dublin 9.

4. Consideration of the Treasurer's report, which was delayed in the post, was postponed until the next meeting.



5. The Secretary presented his report, and was thanked for it.

6. Elections: The following were proposed, seconded, and elected unopposed:

   President: F. Gaines.

   Vice President: R.M. Timoney.

   Committee Members: P. Barry, B. Goldsmith, M. O'Reilly, M. Ó Searcóid, and R.O. Watson.

   Professor Dineen did not wish to serve on the Committee, but was appointed as the Society's representative to deal with EOLAS.

7. Maurice O'Reilly presented a short paper on 'Mathematics at Third Level: Questioning how we teach'. He raised interesting questions, on the subject of how teachers go about teaching, as opposed to the actual content of the courses. He provided two questionnaires, which teachers were invited to use for assessing themselves from this point of view. Copies of his material may be had by writing to him at Dundalk Regional Technical College. The paper and questionnaires provoked some spirited discussion. He was thanked by the President and the meeting closed.

*Anthony G. O'Farrell,*
*Secretary*

# IRISH MATHEMATICAL SOCIETY

## Secretary's Report 1987–88

The Society met three times during this year, and the Committee three times.

The main development this year was the first September Meeting of the Society. Other noteworthy items concerned the Olympiad, the Bulletin and Exchanges.

The September Meeting is intended to be an annual affair, devoted principally to scientific papers on Mathematics. The first was a one-day meeting, on the ninth of September, in Trinity College Dublin. The invited speakers were S. Gardiner, W.K. Hayman and T.W. Kõrner. Each gave a one-hour talk, and there was an hour of short contributed talks, and a panel discussion on the impact of computers on the curriculum in Mathematics. The meeting was well attended, and was generally considered a success. Encouraged by this, we propose in future to have two-day September meetings.

The Schools Mathematics Contest (sponsored by Eolas) continued to attract more entrants, and competition in the Contest and the Irish Mathematical Olympiad was very keen. The prize-giving ceremony was held at UCD on the second of December, and was attended by representatives of Eolas, the IMTA and the teachers of prizewinners.

This year, for the first time, an Irish team competed in the International Mathematical Olympiad in Australia. The team was coached and accompanied by Finbarr Holland and T.J. Laffey, who have put an enormous amount of work into the contest down through the years. The entry was coordinated by a joint committee of the Departments of Education and Foreign Affairs, and sponsored by the Government.

As the Society grows, its various functions become more time-consuming, and there is a danger that some of them might suffer as a result. It is the policy of the Committee to identify and circumscribe the various important functions, and find volunteers to manage each one on a long-term basis.

This year, the maintenance of the Membership List and the management of Institutional Members, formerly the responsibility of the Treasurer, was taken over by Bob Critchely. It has been agreed to rent the list to interested publishers.



The management of exchanges, formerly the function of the Bulletin Editor, was also separated, and is looked after by me. Existing exchanges have been regularised, and a campaign to create new ones launched. To date, six new exchanges have been agreed.

The Bulletin continues under the editorship of Ray Ryan, assisted by Ted Hurley. After some period of uncertainty, Seán Dineen successfully negotiated with Eolas to have the printing done gratis, indefinitely. It has been agreed to include advertisements in the Bulletin. Ray Ryan is launching a membership drive, using free copies of the Bulletin as an inducement.

The Society has become a member of the European Mathematical Trust, and has appointed Tony Seda as Chairman of the Euromath Coordinating Committee for Ireland.

The system of Local Representatives has proven very effective in ensuring the cohesion of the Society. This year, we added Local Representatives in the Queen's University of Belfast (D.W. Armitage) and in St. Patrick's College, Drumcondra (John Cosgrave).

The Society sponsored a meeting on Group Theory at UCG, a meeting on Operator Theory at UCC, a meeting on Matrix Theory at UCD, and a meeting on Differential Equations at NIHED.

*Anthony G. O'Farrell,*
*Secretary*

# NEWS

## Personal Items

- **Professor Les Foulds**, of Waikatu University, New Zealand, who works in Applied Graph Theory, is presently visiting UCD and TCD.
- **Professor Hansjorg Wacker**, of the Institut der Mathematik of the University of Linz, will be visiting the Mathematics Department of NIHE Limerick during the month of July. Professor Wacker works in Industrial Mathematics.
- **Professor Dan Luecking**, of the University of Arkansas at Fayetteville, will be visiting TCD during the Autumn term this year.
- **Ted Hurley** has been appointed to an Associate Professorship in Mathematics at UCG.
- **Donal O'Regan** has joined the Mathematics Department in Maynooth College.
- **David Spearman** will be visiting the University of Montpelier during the Autumn term.

## Second September Meeting of the IMS Maynooth College, September 7–8

Following the great success of the Society's First September Meeting in TCD last year, it was decided to expand this event to a two-day meeting. The Second September Meeting will take place in Maynooth on Thursday September 7th and Friday September 8th. The principal speakers will be **F. Almgren** (Princeton) who will speak on "Supercomputers and Minimal Surfaces", **S.K. Donaldson** (Oxford), who will speak on "Yang-Mills Theory and Four-Manifolds", and **J. Lewis** (DIAS). Contributed short talks are also invited. Overnight accomodation is available in Maynooth at a modest cost, and a dinner will be arranged for Thursday evening if sufficiently many participants are interested. Further details can be had from A. G. O' Farrell, Mathematics Department, Maynooth College, Co. Kildare.





## International Mathematical Olympiad

Following the excellent performance last year in Sydney by the first Irish team to participate in the International Mathematical Olympiad, preparations are now being made to send a team to this year's Olympiad, which will be held in Braunshweig.

Our readers might care to pit their wits against the 1988 questions. There were four and a half hours allowed for each paper. Five competitors obtained full marks.

### First Paper

1. Consider two concentric circles of radii $R$ and $r$ $(R > r)$ with centre $O$. Fix $P$ on the small circle and consider the variable chord $PA$ of the small circle. Points $B$ and $C$ lie on the large circle, $B\ P\ C$ are collinear and $BC$ is perpendicular to $AP$.
(i) For what value(s) of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?
(ii) What are the possible positions of the midpoint $U$ of $BA$ and $V$ of $AC$ as $\angle OPA$ varies?

2. Let $n$ be an even positive integer. Let $B$ be a set and let $A_1, A_2, \ldots A_{n+1}$ be subsets of $B$ such that
(i) each $A_i$ has $n$ elements,
(ii) each intersection $A_i \cap A_j$ $(i \neq j)$ has exactly one element,
(iii) every element of $B$ belongs to at least two of the $A_i$.
For which $n$ can one assign to every element of $B$ one of the numbers 0 and 1 in such a manner that each $A_i$ has exactly $n/2$ of its elements assigned the value 0?

3. A function $f$ defined on the positive integers (and taking positive integer values) is given by:
$$f(1) = 1, \quad f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n),$$
$$f(4n+3) = 3f(2n+1) - 2f(n),$$
for all positive integers $n$. Determine with proof the number of positive integers $n \leq 1988$ for which $f(n) = n$.

## Second Paper

4. Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

5. In a right-angled triangle $ABC$ let $AD$ be the altitude drawn to the hypotenuse and let the straight line joining the incentres of the triangles $ABD$, $ACD$ intersect the sides $AB$, $AC$ at the points $K$, $L$ respectively. If $E$ and $E_1$ denote the areas of the triangles $ABC$ and $AKL$ respectively, show that $E/E_1 \geq 2$.

6. Let $a$, $b$ be two positive integers such that $ab+1$ divides $a^2+b^2$. Prove that $(a^2+b^2)/(ab+1)$ is a perfect square.

Each participating country was invited to submit five questions for consideration for use in the contest. The jury then selected the six questions from this pool. Ireland was honoured to have one of its questions, number 4, composed by Finbarr Holland, selected for the contest.

We wish every success to the Irish team this year.

# James Callagy (1908–1988)

Jim Callagy was a distinguished member of the Irish mathematical community. He grew up in Galway City where he was a pupil at St. Joseph's Secondary School. His early mathematical career is closely associated to that of his life-long friend, Martin Newell, who later became Professor of Mathematics at University College, Galway and then President of the College. In 1930, Jim and Martin Newell graduated with honours degrees in Mathematics from University College, Galway, after highly distinguished undergraduate careers, during which they closely rivalled each other in academic achievement and participation in the student life of the College. The two of them produced a student magazine, and it will be interesting to those who remember Jim's light, neat figure that he was secretary of the College rugby football club. Another intriguing fact that survives from that period is the result of the first year Arts examination in Jim's first year in College. First overall, a young woman from Rosmuc, second James Callagy and third Martin Newell. It would be interesting to know the identity and subsequent career of the top scholar.

In 1930, Jim went to St. Muireadach's College in Ballina, where he taught until 1934. In 1934, he married and he and his wife Lilly went to live in Listowel, where Jim taught Mathematics at St. Michael's College until 1950. That year, Jim and Lilly returned to Galway and Jim joined the famous Preparatory College of St. Enda's.

St. Enda's was then at its height and, as teacher of honours Mathematics, Jim played a key role among a list of impressive and endearing figures, such as Aodh Mac Dhubhain, Tomás Ó Loideáin, Tomás Ó Sé and Micheál Mac Gabhna, all of them outstanding teachers and individuals who dedicated themselves with enthusiasm and imagination to excellence in education through the Irish language. He lived through the years of hope and excitement, and eventually knew the sorrow of what has since been recognised as a signal blow to the cause of the Irish language, when the country seemed to flinch from the prospect of a possible success of the revival of Irish. The Preparatory Colleges were abolished and Enda's was turned into a staid and conventional secondary A school, which quickly followed the fate of other A schools throughout the country. This transition was undoubtedly a traumatic one for Jim as it was for the other teachers who shared his ideals.

When Jim retired from St. Enda's in 1973, he had taught Mathematics at secondary level for 43 years. During all these years, he had been a meticulous and dedicated teacher and his success both on the academic and the personal level is demonstrated by the enormous number of past pupils of his who achieved success and distinction, and at the same time remained staunch friends of his in later years. His years of teaching however, had been, by his own account, arduous and often frustrating. Then, on retirement, life suddenly accorded him an opportunity which, had it arisen earlier, might perhaps have spared him much frustration and permitted greater personal fulfilment. Over the years, in spite of his dedication to his teaching and his devoted commitment to his wife and family, Jim had found time to pursue his passion for Mathematics and the history of Mathematics. It was this expertise that now afforded him the opportunity of a new career. In 1973, he joined the Mathematics Department at University College, Galway.

His years at U.C.G. were very successful and, again by his own account, very happy. He taught several existing courses in English and Irish and developed a new and highly popular course in the history of Mathematics which is on the curriculum ever since. He found the atmosphere in U.C.G. congenial and instantly developed the closest rapport with his colleagues within the Mathematics Department and outside it. He had always been a man of great charm and erudition. These qualities made him extremely popular among colleagues, many of whom were much younger than him. His meticulously and elegantly dressed figure was well loved in College and his company and conversation were always sought after.

His long and distinguished career in Irish education made Jim well known to a large number of people in this country, but he was also well-liked and respected abroad. For years, he was a regular and important participant in the Summer School of International Post University Courses in Belgium, and the President of the Post University Courses and Honorary Rector of the State Unversity of Ghent, Professor A. Cottenie, was a personal friend of his.

An Irish colleague of Jim's who accompanied him to one of the meetings of the Summer Schools was charmed at the warmth and regard with which Jim was greeted by the other participants. On that occasion, Jim was welcomed as the senior member of a very small number of participants who had attended the Summer School every year over a period of twenty years, and was called upon to deliver an appreciation of the occasion. Characteristically, he did so with accomplished elegance and humour.



The memory that Jim Callagy leaves behind him is a very fond one and a poignant one. His great knowledge of local people and history and their points of contact with world events and European culture were fascinating. His loss inevitably provokes thoughts on the nature and deficiences of the transmission of knowledge and cultural continuity.

Jim's long career was intimately linked to the Irish educational system, with all its contrasts and conflicting qualities. The constant demands on him by this system, over a period of half a century, could be challenging and exciting but also sometimes capricious and unimaginative. Throughout his life, Jim responded with impeccable professionalism, contributing handsomely to the best the system could offer and struggling bravely against its shortcomings and frustrations. Ultimately, a product of it himself, he was an example of the best that it is capable of.

*Tony Christofides*

# IMS MEMBERSHIP

## Ordinary Membership

Ordinary Membership of the Irish Mathematical Society is open to all persons interested in the activities of the Society. Application forms are available from the Treasurer and from Local Representatives. Special reciprocity rates apply to members of the Irish Mathematics Teachers Association and of the American Mathematical Society.

## Institutional Membership

Institutional Membership is a valuable support to the Society. Institutional members receive two copies of each issue of the Bulletin and may nominate up to five students for free membership.

## Subscriptions Rates

The rates are listed below. The membership year runs from 1st October to 30th September. Members should make payments by the end of January either direct to the Treasurer or through Local Representatives. Members whose subscriptions are more than eighteen months in arrears are deemed to have resigned from the Society.

| | |
|---|---|
| Ordinary Members | IR£5 |
| IMS–IMTA Combined | IR£6.50 |
| Reciprocity Members from IMTA | IR £1.50 |
| Reciprocity Members from AMS | US$6 |
| Institutional Members | IR£35 |

Note: Equivalent amounts in foreign currency will also be accepted.

# ARTICLES

## Group Presentations, Topology and Graphs

Timothy Porter

A few years ago John McDermott wrote a short article [1] for the Newsletter, as it was then called. This article takes up the relationship studied briefly in his article and looks at several other uses of simple graph-theoretic ideas in the study of group presentations. The level of graph theory involved is not much deeper than that used in his article The material is used quite successfully in both a three year course in Knot Theory and in an M.Sc. course in algebra at U.C.N.W., Bangor.

## 1 Group Presentations

As examples of group presentations, we will use a few very simple ones such as:
$C_6$, the cyclic group oforder 6, having an obvious presentation $(a : a^6)$, but also another slightly more subtle one, $(x, y : x^2, y^3, [x, y])$. $D_3$, the dihedral group of order 6, with a presentation $(a, b : a^3, b^2, (ab)^2)$.
In each case we specify a set of generators and some relations between them. To be slightly more precise, we recall:

$$X \subset G \text{ generates } G \text{ if } X \subset H \leq G \text{ implies } H = G$$

i.e., if there is no proper subgroup of $G$ containing $X$. In this case every $g \in G$ can be written nonuniquely as a word in elements from $X \cup X^{-1}$.

The relations in the presentation are there to handle the *problem of nonuniqueness of representative words.* This is simply illustrated by the following example.

In $C_6$, $X = \{a\}$, $a^8 = a.a.a.a.a.a.a.a$, and $a^2 = a.a$ representing the same element. This makes it awkward to talk about the relationships between different words; in some way we want to say that $a^8$ and $a^2$ are different words but at the same time they are equal as elements of $C3$. The solution to the conundrum is to form the free group on a set $Y \cong X$, satisfying $Y \cap G = \emptyset$ to so that no confusion of symbols can arise. More generally, if $Y$ is a set, we denote by $F(Y)$ the free group on $Y$, that is the group formed from all words in $y$'s and $y^{-1}$'s (with any occurences of $yy^{-1}$ etc. cancelled). If $Y$ has $n$ elements we say $F(Y)$ has rank $n$ and write

$$r_{F(Y)} = n.$$

Any $\omega \in F(Y)$ can be written *uniquely* in the form

$$\omega = y_{i_1}^{\alpha_1} \dots y_{i_n}^{\alpha_n}$$

where $y_{i_1}, \dots y_{i_n} \in Y$, $\alpha_1 \dots \alpha_n \in \{-1, 1\}$ and $i_j \neq i_{j+1}$ $(1 \leq j \leq n)$. Now pick $f\colon Y \to G$. This will induce $\varphi\colon F(Y) \to G$ defined recursively by

$$\varphi(y\omega) = f(y)\varphi(\omega)$$

*If $\varphi$ is onto then $f(Y)$ generates $G$*, so $\ker\varphi$ measures the *nonuniqueness* of representative words, i.e., the relations between the generators. To be able to study $\ker\varphi$ we pick $R \subset \ker\varphi$ so that $R \subset N \lhd F(Y) \Rightarrow N \geq \ker\varphi$, i.e., so that $\ker\varphi$ is the normal closure of $R$ in $F(Y)$.

**Example** Let $\mathcal{P}$ be the presentation $(x, y : x^3, y^2, (xy)^2)$ of $D_3$, $F(Y) = F(x, y) = F_2$, free of rank 2

$$f\colon Y \to D_3 \quad \text{is given by} \quad \begin{cases} f(x) = a, & \text{rotation} \\ f(y) = b, & \text{reflection} \end{cases}$$

Any relation between $a$ and $b$ is a consequence of $r = x^3$, $s = y^2$, and $t = (xy)^2$ i.e. is a product of conjugates of $r$, $s$ and $t$.

We note that $F = F(Y)$ acts on $N = N(R)$:

$$F \times N \to N, \quad (\omega, c) \mapsto {}^{\omega}c = \omega c \omega^{-1}$$

and any element, c, of N can be written in the form:

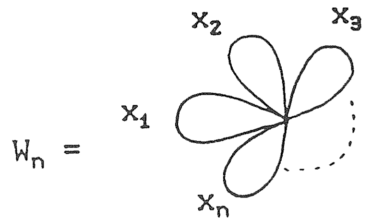
$$c = {}^{\omega_1}(r_1^{\varepsilon_1}) \dots {}^{\omega_n}(r_n^{\varepsilon_n})$$

with $r_i \in R$, $\varepsilon_i = \pm 1$, $\omega_i \in F$, $i = 1, 2, \dots n$.



## 2 Graphs

As explained in [1], a graph, $\Gamma$, consists of a set of vertices joined by edges; the edges may be directed, labelled (coloured) etc., as required. A path from a vertex, $x$, to a vertex, $y$, in a graph consists of a sequence, $x_0, x_1, \dots, x_k$, of vertices such that $x = x_0$, $y = x_k$ and each pair $x_{i-1}$, $x_i$ is joined by an edge; if there are several edges joining $x_{i-1}$ to $x_i$ we must specify which of the edges is being used. As an example consider the following graph having a single vertex:




A path in $W_n$ is exactly a word in the symbols $x_i$ and their "inverses" $x_i^{-1}$; however, unlike the elements of a free group, in the paths we can have occurences of $x_i x_i^{-1}$. To construct a group from these edge paths in a graph, $\Gamma$, one first proves that any path in $\Gamma$ determines a unique reduced path, i.e., a path in which such pairs do not occur; one then picks some vertex, $v$, and looks at the reduced paths that start and end at that vertex. Composition is given by putting two reduced paths next to each other and then forming the reduced path determined by them. The group one gets is called the fundamental group of the graph, and it is denoted $\pi_1(\Gamma, v)$. This group will in general depend on the choice of vertex, $v$, but if the graph is connected, i.e., if any two vertices can be joined by some path in $\Gamma$, then any two choices of base vertex give isomorphic groups, so if $\Gamma$ is connected we can write $\pi_1(\Gamma)$ without serious risk of ambiguity. By picking a maximal spanning tree as indicated in [1] one can prove:

*If $G$ is a connected finite graph with $\alpha_0$ vertices and $\alpha_1$ edges, then $\pi_1(\Gamma)$ is free of rank $\alpha_1 - \alpha_0 + 1$.*

In fact a basis for $\pi_1(\Gamma)$ can easily be found. Each element of the basis corresponds to an edge not in the spanning tree. The corresponding reduced path goes out in the tree to the start-vertex of the edge, crosses that edge in the given direction and returns to $v$ again in the spanning tree.

We can model a graph, $\Gamma$, by a topological space, $X$, say, by taking a set of points in $\mathbf{R}^n$ as the vertices and for each edge an arc in $\mathbf{R}^n$ joining the corresponding vertices. With this model $\pi_1(\Gamma, v)$ is of course isomorphic to the topologically defined fundamental group $\pi_1(X)$ of the space $X$ based at the vertex corresponding to $v$, (c.f. [2]).

## 3 Covering spaces and covering graphs

From topology we next take some results from the theory of covering spaces. These we really only need in the case that all the spaces concerned are graphs so the theory of "covering graphs" would suffice This can we found in the book by Stillwell, [2], which is an excellent source for much of this material.

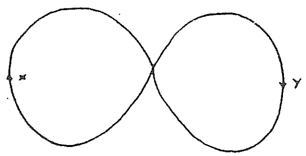
If $X$ is a space, and $H \leq \pi_1(X)$ then to $H$ there corresponds a covering space, $p\colon X_H \to X$ with

$$p_*\colon \pi_1(X_H) \to \pi_1(X)$$

a monomorphism with $\operatorname{Im} p_* = H$. If $X$ is a graph, so is $X_H$.

To illustrate the idea in the case we want, take $X$ to be $W_m$ so that $\pi_1(X) \cong F(x_1, \ldots, x_m) = F$, say, and take $H \leq F$. Then $X_H$ is the the Schreier diagram of $H$ in $F$ having the cosets of $H$ in $F$ as vertices and for each generator, $x_i$, and coset $xH$ an edge labelled $x_i$ from $xH$ to $x_i xH$.

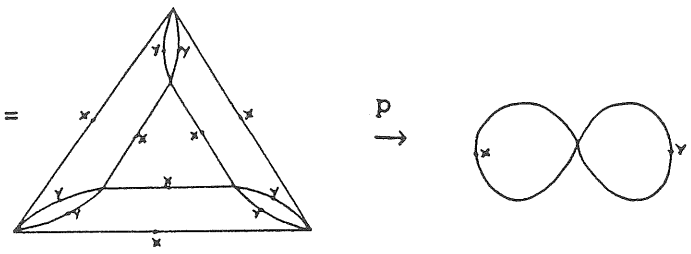
The map from $X_H$ to $X$ maps all the vertices to the one vertex of $X$ and maps edges according to their labels. e.g. $\mathcal{P} = (x, y, : x^3, y^2, (xy)^2)$, the presentation of $D_3$ given earlier. Then

$X =$ 


$$H = N(x^3, y^2, (xy)^2)$$



Then

$X_H =$ 


In this case the Schreier diagram is easily seen to be the Cayley graph of the group with respect to the given generators.

The above discussion provides the basis for a proof of the following famous theorem.

**Neilsen-Schreier:** *If $F$ free, and $H \leq F$ then $H$ is free.*

Together with our remark earlier on the rank of the fundamental group of a graph and basic, easily verified, facts about induced maps between fundamental groups we get:

**The Schreier Index Formula:** *If $r_F$ and $r_H$ are finite and $|F : H| = i < \infty$ then*

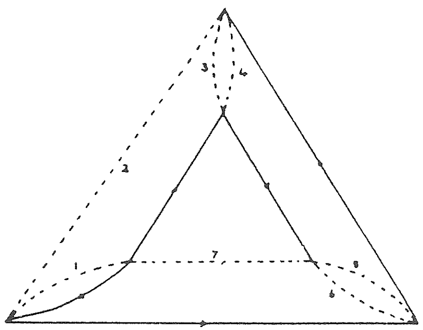
$$r_H = r_F i - i + 1$$

"Proof": $X_H$ has $i$ vertices and $r_F i$ edges.

## 4 Bases for $N(R)$

We saw that $H \cong \pi_1(X_H)$ is free. Can we find a basis? In other words, can we find elements freely generating $H$? Rather than looking at this in general we will consider our previous example in more detail. In that case $r_F = 2$, $|F : H| = |D_3| = 6$, so $r_H = 2 \times 6 - 6 + 1 = 7$, so we want a seven element basis for $H$.

We need to calculate $\pi_1(X_H)$ so we choose a maximal spanning tree $T$, in $X_H$; say

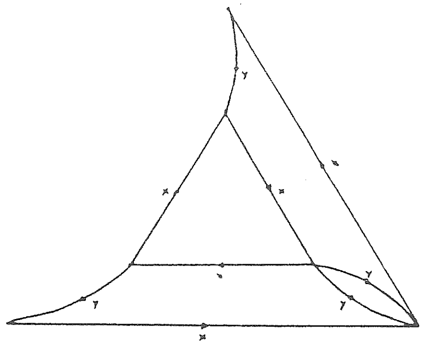


Generators of $\pi_1(X_H)$ correspond to the edges not in the tree as earlier. For our choice of tree they are:

| | | |
|---|---|---|
| 1. $\underline{y}y$ | 2. $xx\underline{x}$ | 3. $x x\underline{y}^{-1}x^{-1}y$ |
| 4. $xx\underline{y}x^{-1}y$ | 5. $y^{-1}xx\underline{y}x^{-1}$ | 6. $x\underline{y}x^{-1}x^{-1}y$ |
| | 7. $y^{-1}xx\underline{xy}$ | |

in each word the underlined element corresponds to the edge not in $T$.

Each of these is a consequence of the relations $r = x^3$, $s = y^3$ and $t = (xy)^2$. This is not only a result of the overall theory but that can be "visualised" in the following way:





Take for instance the fourth basis element in our list:

$$\begin{aligned} 4.\ xxyx^{-1}y &= (xy)^2(y^{-1}x^{-1}y^{-2}xy)(y^{-1}x^{-1}y(xy)^2y^{-1}xy)(y^{-1}x^{-3}y) \\ &= t.^{y^{-1}x^{-1}}(s^{-1}).^{y^{-1}x^{-1}y}t.^{y^{-1}}(r^{-1}) \end{aligned}$$

Thus by interpreting the Cayley graph as a covering graph of the graph $W_n$ one can find bases for the subgroup of relations and also one can express those basis elements as products of conjugates of the chosen relations. This method also enables one to identify certain identities among relations, but as that subject really needs another article to do it justice. I will not say more here.

## 5 Automorphisms of Graphs

From the theory of covering spaces one has the idea of a deck automorphism. This is as follows:

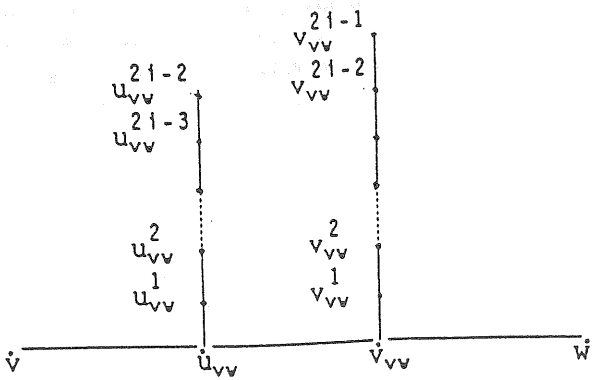
Suppose that $p: Y \to X$ is a covering graph, then a deck automorphism of $p$ is an automorphism $\varphi$ of the graph $Y$ such that $p\varphi = p$. Of course the deck automorphisms of a covering graph $(Y, p)$ form a group, $A(Y, p)$. In the case of the covering graph constructed from a presentation, this group is the subgroup of the automorphism group of the Cayley graph consisting of those automorphisms that preserve the labels. (We will look at this in our example shortly.) As the image of $\pi_1(X_H)$ in this case is normal, the covering graph is "regular" and the theory of covering spaces gives an isomorphism between $A(X_H, p)$ and $\pi_1(X_H)/\mathrm{Im}\{p^*: \pi_1(X_H) \to \pi_1(X)\}$. As this latter group is exactly the group that was being presented, we have the following result:

**Theorem: (Frucht,1938)** *If $G$ is a finite group, $X$ a generating set for $G$, $\Gamma$ the labelled graph of $G$ relative to $X$ then $G$ is isomorphic to the group of label preserving automorphisms of $\Gamma$.*

**Example** In our example in §3 with $G = D_3$, there are two obvious automorphisms of the graph that preserve labels. One rotates the graph through 120°; if the rotation is clockwise, this corresponds to the element $x$ in $D_3$, if anticlockwise to $x^{-1}$. The other exchanges the inner and outer triangles; this corresponds to $y$. These two generate the deck/label automorphism group of the graph and the relations are clearly those given in the presentation of $D_3$.

The fact that $\Gamma$ is considered as a labelled directed graph may spoil the pleasure of some people so it is worth while pointing out that there is a simple means to replace a labelled directed graph by an unlabelled undirected graph with the same automorphism group, as follows:

We suppose the graph is labelled by a finite set, $\{x_1, \ldots x_n\}$. If the edge from $v$ to $w$ is labelled with $x_i$ replace that edge by the subgraph:




Thus we replace the edge by a path $v, u_{vw}, v_{vw}, w$, at vertex $u_{vw}$, we attach a new path of length $2i-2$ and at $v_{vw}$ a new path of length $2i-1$. This labels the edge in a purely graph theoretic way.

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# Algebra and Superalgebra


Allan Solomon


*This article is based on a talk given by the author to the Dublin University Mathematical Society on April 30, 1987.*

On this 210th Anniversary of Gauss' birthday, I should like to start off by referring to the Fundamental Theorem of Algebra, for which Gauss gave four proofs. The assertion of this theorem—that every polynomial equation has a root—must be interpreted by extending the real numbers to the complex, i.e., every polynomial over the real numbers has a (complex) root. In fact, the theorem remains true if we extend to polynomials over the complex field; the complex numbers are algebraically closed. We may, however, extend the complex numbers in an elegant and non-trivial way to the quaternions. The system of quaternions $\mathcal{H}$ provides us with our first example of an algebra. An algebra $\mathcal{A}$ is a linear space over a field $\mathbf{F}$, on which a multiplication, having the usual distributive properties, is defined. Essentially, we have

$$\alpha x + \beta y \in \mathcal{A} \quad \text{and} \quad xy \in \mathcal{A} \qquad (x, y \in \mathcal{A}, \alpha, \beta \in \mathbf{F}),$$

with

$$(\alpha x)y = \alpha(xy),$$
$$(\alpha x + \beta y)z = \alpha xz + \beta yz,$$
$$z(\alpha x + \beta y) = \alpha zx + \beta zy$$

Here $\alpha, \beta$ are elements of the field $\mathbf{F}$ over which the linear space $\mathcal{A}$is defined; in the usual applications this will be the real field $\mathbf{R}$ on the complex field $\mathbf{C}$. An algebra may, or may not, have the associative property;

$$(xy)z = x(yz) \qquad \forall x, y, z \in \mathcal{A}.$$

If it does, it is called an ***associative algebra.*** In general, the algebras and superalgebras of my title are ***not*** associative algebras. However, most algebras that have applications may be represented by matrices; and since matrices multiply associatively, we may effectively ***embed*** our algebras in associative algebras.

However, our algebras will not be *commutative*; that is $xy \neq yx$ is general. The quaternions are a non-trivial extension of the complex numbers because they form a non-commutative system, perhaps the earliest such example. Let us look more closely at this example.

A quaternion $q$ is written $q = \alpha + \lambda i + \mu j + \nu k \quad (\alpha, \lambda, \mu, \nu \in \mathrm{R})$; so, as a vector space, $\mathcal{H}$ is 4-dimensional, with basis $\{1, i, j, k\}$. In order to define $\mathcal{H}$ as an algebrà, we must give a multiplication table for the basis elements; and these are the famous relations of Hamilton:

$$i^2 = j^2 = k^2 = -1; \quad ij = -j \cdot i = k, \quad j \cdot k = -k \cdot j = i, \quad h \cdot i = -i \cdot k = j.$$

So we see the non-commutativity in, for example $i \cdot j = -j \cdot i$.

How would such a non-commutativity, so non-intuitive from our experience with school alebra, arise naturally,? It arose naturally in a geometric context.

Consider the rotations of a sphere in $\mathrm{R}^3$ with its centre at the origin. First we have rotate the sphere by an angle $\pi/2$ about the (fixed in space) $K$-axis, followed by a $\pi/2$ rotation about the $J$-axis. Now we perform these two operations in reverse order; a $\pi/2$ roration about $J$ followed by a $\pi/2$ rotation about $K$. It is easy to see that the resulting position of the sphere is not he same in the two cases. These operations, of rotating the sphere, do not *commute*.

In fact, the operation of rotating the sphere through an angle $\theta$ about an axis through its centre may be represented by a quaternion

$$q = \alpha + \lambda i + \mu j + \nu k,$$

where $\alpha = \tan \theta/2$, and the axis has direction cosines $(\cos f, \cos g, \cos h)$ given by

$$\lambda = \tan \frac{\theta}{2} \cos f \quad \mu = \tan \frac{\theta}{2} \cos g \quad \nu = \tan \frac{\theta}{2} \cos h.$$

The rotation of the sphere is given by

$$xi + yj + zj \mapsto q(xi + yj + zk)q^{-1}$$

where $xi + yj + zk$ is a *unit* quaternion $(x^2 + y^2 + z^2 = 1)$ and so represents a point on the surface of the sphere.

For example, the first of the two rotations used above is given by $q_1 = 1 + k$ ($\theta = \pi/2$, and the axis contains $(0, 0, 1)$) The second is given by $q_2 = 1 + j$. If



we take the point $k$ on the sphere, we have:
(a) Applying first $q_1$ and then $q_2$: $k \mapsto q_1 k q_1^{-1} = (1 + k)k(1 - k)/2 = k \mapsto (1 + j)k(1 - j)/2 = i$
(b) Applying first $q_2$ and then $q_1$: $k \mapsto q_2 k q_2^{-1} = (i + j)k(1 - j)/2 = i \mapsto q_1 i q_1^{-1} = (1 + h)i(1 - h)/2 = j$.

This discovery of the connection between quaternions and rotations was made by Gauss in 1819, pre-dating Hamilton's work by almost a quarter of a century. However, Gauss did not publish it. The result was given in a paper of Olinde Rodrigues (1840) again pre-dating Hamilton, by three years. (I am indebted to my Open University colleague, Jeremy Gray for referring me to his article in the Archive for History of Exact Sciences where the details on this discovery of quaternions are laid out.)

Quaternions are not much used nowadays, mainly because they can be represented by the more familiar matrices with complex or real entries (see Diagram (a)). This is a pity. They do have their enthusiasts still, however.

A MATRIX REPRESENTATION OF THE QUATERNIONS

$$i = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$k = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$

$$ij = k, \quad jk = i, \quad ki = j$$

$$i^2 = j^2 k^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

These matrices are equivalent to the Pauli Spin Matrices

Diagram (a)

Symplectic groups—much used in dynamics and physics—have their most elegant expression in terms of quaternionic matrices—although again, these groups may be expressed otherwise. The fact that a quaternion is al element of a 4-dimensional space would seem to indicate a possible use in relativity;

however, the quaternion

$$q = \alpha + \lambda i + \mu j + \nu k$$

has a natural norm

$$||q||^2 = \alpha^2 + \lambda^2 + \mu^2 + \nu^2;$$

while the relativistic theory would require a form such as $-\alpha^2+\lambda^2+\mu^2+\nu^2$. For this reason, Professor John Synge introduced *Minkowski quaternions* [2] or *minquats* for short (called physical quaternions by Silberstein in 1912),

$$q = q_4 + q_1 i + q_2 j + q_3 k$$

where $q_4$ is a pure imaginary ($q_4 = \sqrt{-1}\,\alpha$, $\alpha$ real). Now it is easy to see that if we multiply two minquats $q$, $q'$ together, the scalar term $q_4 q_4'$ is no longer pure imaginary, and so the result is not a minquat. Thus minquats do not form an algebra. However, the general quaternions with complex coefficients *do* form an algebra, called biquaternions by Hamilton. The most amusing use of biquaternions I know is due to Louis Kaufman [3]: Define

$$H = H_1 i + H_2 j + H_3 k, \quad E = E_1 i + E_2 j + E_3 k,$$
$$J = J_1 i + J_2 j + J_3 k, \quad F = H + \sqrt{-1}E,$$

and define the operators $D$ and $\nabla$ by:

$$\nabla = \frac{\partial}{\partial x} + i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k,$$

$$D = \sqrt{-1}\frac{\partial}{\partial t} + \nabla$$

Then the six electrodynamics equations of Maxwell may be written as the single biquaternionic equation

$$DF = \sqrt{-1}\,\rho + J$$

The quaternions form a system very like ordinary numbers, in that we may add and subtract, multiply and divide. Technically, they form a *Division Ring*. But we have lost the commutativity of the real and of the complex numbers. If we relax associativity, we may define one final division ring, the *octonions*



of Cayley. I say a *final* division ring, for it may be shown that there are no others. The property which you cannot preserve is the absence of divisors of zero; that is when two non-zero elements are multiplied together, the result cannot be zero. Quaternions have this property, which is equivalent to saying that every non-zero element has an inverse: Thus, if

$$q \in \mathcal{H}, \quad q = \alpha + \lambda i + \mu j + \nu k \neq 0$$

then

$$q^{-1} = \frac{(\alpha - \lambda i - \mu j - \nu k)}{(\alpha^2 + \lambda^2 + \mu^2 + \nu^2)}.$$

Biquaternions to *not* have this property' that is, they *have* divisors of zero:

$$(\sqrt{-1} + k)(\sqrt{-1} - k) = 0.$$

A more important contribution of Cayley is the idea of a matrix. Sets of matries (over **R** or **C**, for examples) form Algebras; and the Algebras and Superalgebras I wish to consider in the sequel may all be represented by matrices.

## Clifford Algebras and Grassman Algebras

In passing, I should like to refer to two sorts of algebra which have many applications nowadays Clifford (1845–1879) Algebras and Grassmann (1809–1877) Algebras. These are both associative algebras.

First of all, *Clifford Algebras*. We take a basis $\{e_1, e_2, \ldots, e_k\}$ for a real $k$-dimensional vector space, and then define a multiplication of the basis vectors:

$$e_i e_j = -e_j e_i \quad (i \neq j)$$
$$e_i^2 = -1.$$

By this means we define an algebra—we *generate* an algebra, since we are allowed products. But due to the reduction we can make if two elements in a product are equal (e.g., $e_1e_2e_1e_3 = -e_1e_1e_2e_3 = e_2e_3$) we need only consider products in which all the basis vectors are unequal; so the algebra has for basis

| | | | | | |
|---|---|---|---|---|---|
| | 1 (no $e$'s), | $e_i$, | $e_ie_j$, | ... | $e_1e_2\ldots e_k$ |
| number of elements: | 1 | $k$ | ${}^kC_2$ | | 1 |

The total number of elements in the basis for the algebra is

$$1 + {}^kC_1 + {}^kC_2 + \ldots + {}^kC_k = (1+1)^k = 2^k$$

Let us consider some cases of Clifford Algebras:

1. $k = 1$: $2^1$ elements, basis $\{1, e_1, e_1^2 = -1\}$, $\mathcal{A} \cong \mathbf{C}$, the Complex Numbers
2. $k = 2$: $2^2$ elements, basis $\{1, e_1, e_2, e_1e_2\}$, $\mathcal{A} \cong \mathcal{H}$, the Quaternions.
3. $k = 3$: $2^3$ elements, basis $\{1, e_i, e_i, e_j, e_1e_2e_3\}$, $\mathcal{A} \cong \mathcal{H} \oplus \mathcal{H}$.

The 16-element case ($k = 4$) is related to an algebra introduced by Dirac (1902–85) to describe electromagnetism.

If instead of taking $e_1^2 = -1$, we assume $e_i^2 = 0$, we obtain the *Grassmann Algebras*, again of dimension $2^k$.

## Jordan Algebras and Lie Algebras

To introduce the remaining algebras, I wish to talk about, we turn to Quantum Mechanics. In one formulation, the basic laws of Quantum Mechanics are algebraic in character; this is the matrix mechanics of Heisenberg (1901–76). The dynamical quantities $Q$ and $P$ for position and momentum respectively are to be thought of as Hermitian matrices—since these correspond to *real* physical observables.

Hermitian conjugation is a complex conjugation which also reverses the order of matrices: thus

$$(AB)^\dagger = B^\dagger A^\dagger; \quad (\sqrt{-1}\,A)^\dagger = -\sqrt{-1}\,A^\dagger.$$

The operators representing real physical quantities, such as $P$ and $Q$, are Hermitian, that is

$$P = P^\dagger, \quad Q = Q^\iota.$$

It would therefore be very nice to form an *Algebra* of Hermitian matrices. Ordinary addition is no problem:

$$(A+B)^\dagger = A^\dagger + B^\dagger + A + B. \qquad (A = A^\dagger, B = B^\dagger)$$

But



$$(AB)^\dagger = B^\dagger A^\dagger = BA \neq AB \qquad (A = A^\dagger, B = B^\dagger).$$

in general. Nevertheless, it is not too difficult to devise multiplication rules which preserve hermiticity. These correspond to ***Jordan Algebras*** and ***Lie Algebras***.

Jordan Algebras:

$$A * B \equiv AB + BA$$

so that

$$(A*B)^\dagger = (AB+BA)^\dagger = B^\dagger A^\dagger + B^\dagger A^\dagger = BA + AB = A * B.$$

(Multiplication is always *commutative*.)

Lie(1842–1899) Algebras:

$$A * B = AB - BA$$

This multiplication actually preserves anti-hermiticity. If $A^\dagger = -A$ and $B^\dagger = -B$, then

$$(A*B)^\iota = (AB - BA)^\dagger = (B^\dagger A^\dagger - A^\dagger B^\dagger) = BA - AB = -A * B.$$

But if we consider our physical operator $P, Q$ etc., to be $\sqrt{-1}$ times an anti-hermitian operator, this amounts to preserving hermiticity. (Multiplication is always *alternating* or *anti-commutative*.)

Neither the Jordan nor the Lie Algebras are associative; but for the Lie Algebras associativity is replaced by the Jacobi identity:

$$(A*B)*C + (B*C)*A + (C*A)*B = 0$$

It is conventional to write the $*$ operation for Lie algebras as a bracket:

$$[A, B] = AB - BA$$

This implies the possibility of embedding the Lie Algebra in an associative algebra, where $(AB)C = A(B)$ — always possible for Lie Algebras (the Poincare-Birkhoff-Witt Theorem). However, not every abstract Jordan Algebra is thus obtainable.

Lie Algebras are the most frequently met in Physics, since the basic operator $Q$ and the momentum operator $P$, may be expressed as a Lie Algebra bracket:

$$[Q,P] \equiv QP - PQ = \sqrt{-1}\,\hbar\,1 \qquad (h = 10^{-27}\text{erg sec})$$

This relation gives rise to the famous Heisenberg uncertainty principle, which imposes limits on the simultaneous accuracy of measurement of the observables $Q$ and $P$. And the above Lie Algebra, consisting of $\{Q, P, 1\}$, is a very elementary and very famous Lie Algebra, sometimes called the Heisenberg algebra. This leads to the very physical *Boson and Fermion Algebras.*

If we define

$$b = Q + iP/\sqrt{2}, \quad b^\dagger = Q - iP/\sqrt{2},$$

then, taking units for which $\hbar = 1$,

$$[b, b^\dagger] \equiv bb^\dagger - b^\dagger b = 1$$

gives an even simpler form of this Lie algebra. It is found in applicatins that the operator $b$ is associated with a particle in Physics with zero spin (or an even number of spin units of $(1/2)\hbar$)). Such a particle is called a *boson.* Examples are mesons in nuclear physics and, most important of all, the photon in Quantum Optics.

If by analogy, we assume a similar Jordan algebra for a different operator $f$, we get the basic anti-commutation relation

$$\{f, f^\dagger\} \equiv ff^\dagger + f^\dagger f = 1 \quad (\hbar = 1)$$

Such a relation is satisfied by particles in physics which possess an odd number of spin-units. Examples are the particles of the nucleus, neutrons and protons, and, most importantly, the electron.

The property of a particle obeying either commutation or anti-commutation relations is called its "statistics", and can have a profound effect on the observed properties. For example, Helium Four consists of bosons, and becomes superfluid at about two degrees above absolute zero. The very similar isotope Helium Three, on the other hand, is a gas of Fermions and becomes superfluid, only at about under a thousandth of a degree above absolute zero.

If we wish to consider algebras in which both types of statistics are simultaneously present, we are led to superalgebras.



## Superalgebras

Superalgebras are simply mixtures of a Lie Algebra with a Jordan Algebra; or, an algebra which incorporates both the commutation relation of a Lie Algebra and the anti-commutation of a Jordan Algebra. Physically, we culd call them boson-fermion algebras. Thus the basic operation is neither commutation $[x, y] \equiv xy - yx$, or anti-commutation $\{x, y\} = xy + yx$, but an operation which can be either, depending on the elements $x$ and $y$.

Abstractly, we write our superalgebra $\mathcal{A}$ as a sum of algebras

$$\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}};$$

that is, every element $x$ in $\mathcal{A}$ belongs either to $\mathcal{A}_{\bar{0}}$ or $\mathcal{A}_{\bar{1}}$;, and

$$[x, y\} = -(-1)^{\alpha\beta}[y, x\}$$

where $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$; thus $\alpha, \beta = \bar{0}$: $[x, y\} = -[y, x\}$ (Lie type)
$\alpha, \beta = \bar{1}$: $[x, y\} = [y, x\}$ (Jordan type)
$\alpha = \bar{0}, \beta = \bar{1}$: $[x, y\} = -[y, x\}$ (Lie type)
and

$$[\mathcal{A}_\alpha, \mathcal{A}_\beta\} \subset \mathcal{A}_{\alpha+\beta}.$$

The supersymmetry associated with superalgebras provides a theoretical framework for some current theories of Particle Physics [4] (although I am informed it has not been observed to date experimentally) and this idea has been used in Nuclear Physics and, more recently, in Condensed Solid State Physics.

We give a simple example of a superalgebra in Diagram (b), representing the elements of the 4-dimensional algebra $\mathcal{A}$ by $2 \times 2$ matrices over $\mathbf{R}$. Note that although the example may be simple, the algebra $\mathcal{A}$ is not 'simple' in the technical sense, in that it possesses a (non-trivial) ideal; in fact $\{\alpha 1 : \alpha \in \mathbf{R}\} \subset \mathcal{A}$ is such an ideal. Just as a complete classification of all the simple Lie Algebras (finite dimensional over fields of characteristic zero) has been given by E. Cartan (1869–1951) and others, a similar classification has been made for superalgebras by Victor Kac of M.I.T. (1977).

An interesting confluence of the ideas of Clifford and Grassmann Algebras with those of Lie Algebras and Superalgebras arises when we consider representations of the latter by matrices [5]. We may reduce both types of bracket (Lie and Jordan) to a single type (Lie) by introducing a representation in terms of matrices over a Grassmann algebra instead of, say, the reals. We

AN EXAMPLE OF A SUPERALGEBRA

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x, y \in \mathcal{A}_{\bar{1}}, \quad h, 1 \in \mathcal{A}_{\bar{0}}$$

$\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ is generated by $x, y, h, 1$.

$$[x, y\} = xy + yx = 1$$

$$[h, x\} = hx - xh = x$$

$$[h, y\} = hy - yh = -y$$

In all other cases, $[a, b\} = 0$.

Diagram (b)



illustrate this using our small superalgebra $\mathcal{A}$ above. If every element of $\mathcal{A}_{\bar{1}}$ is assumed to be multiplied by an element (of the odd part) of a Grassmann algebra then, for example, since

$$[e_1 x, e_2 y] \equiv e_1 x e_2 y - e_2 y e_1 x = e_1 e_2 (xy + yx) = e_1 e_2 [x, y\}$$

we obtain closure by consideration only of the commutator (Lie) bracket. (We assumed in the above that the elements of the Grassmann algebra commuted with the elements of $\mathcal{A}$; we may alternatively assume that the $e_i$ anti-commute with $\mathcal{A}_{\bar{1}}$, commute with $\mathcal{A}_{\bar{0}}$. And since we only used the property $e_1 e_2 = -e_2 e_1$, a Clifford algebra would also provide a convenient representation for a superalgebra.)

We conclude this note by indicating how these algebras may arise when considering physical systems. The dynamics of such systems are governed by a hamiltonian $H$, an operator expressed in terms of other operators of the theory. The time evolution of an operator $A$ is given by

$$\sqrt{-1}\,\frac{d}{dt} A = [A, H] \equiv AH - HA$$

where we have a Lie Bracket on the right-hand side. This bracket is a natural operation when both $A$ and $H$ belong to a Lie Algebra, or a Superalgebra (with $H$ in the even part $\mathcal{A}_{\bar{0}}$). This would occur when, for example, the operators are linear or bilinear in boson or fermion operators $(b, f)$ described above. Otherwise, an approximation process ("linearization") may be used (called 'Mean Field Theory' in Many Body Physics). Suppose $H = AB$, where $A, B$ are some operators. We may write the identity

$$H = AB \equiv (A - \lambda)(B - \mu) + A\mu + \lambda B + \lambda\mu.$$

Typically, $\lambda, \mu$ are thought of as the expectation values of the operators $A, B$ respectively in some state $\omega$ of the system. In the event that we may neglect the $(A - \lambda)(B - \mu)$ term—rationalizing this by assuming we do not consider states for which operators $A, B$ stray far from the $\omega$ values—we may approximate:

$$H_{\mathbf{approx}} \sim A\mu + \lambda B - \lambda\mu.$$

This approximation is only consistent if
Case (i): $A, B$ commute; that is, $AB = BA$ and so the approximation for $BA$

leads to the same linearized value, here $\lambda, \mu$ are ordinary numbers.
Case (ii): $A, B$ anti-commute; that is, $AB = -BA$, which will be the case when $A$ and $B$ are fermion operators. In that case, consistency demands that $\lambda$ and $\mu$ anti-commute with one another, and also with the operators $A, B$, then $\lambda, \mu$ may be taken as Grassmann or Clifford numbers.

Thus a general hamiltonian, after linearization by this method, will look naturally like an element of a superalgebra, with $A_{\bar{1}}$-type elements multiplied by Grassmann (or Clifford) numbers, just as in the simple example above. This approach has recently been used to give a superlagebraic model of superconductivity [**6**].

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# Integrals of Subharmonic Functions


Stephen J. Gardiner


This article reviews a problem concerning potential theory in $\mathbf{R}^n$ which has its roots in classical complex analysis. One of the interesting features of the problem is the way in which the solution has gradually emerged, sometimes in a surprising fashion. The article is based on a lecture given at the First September Meeting of the Society, held at Trinity College, Dublin.

## 1 Background in C

Let $N(f,r)$ denote the maximum modulus of an analytic function $f$ on the circle $\{z \in \mathbf{C} : |z| = r\}$. The starting point for our discussion is provided by the following facts from elementary complex analysis.

**Hadamard's Three Circles Theorem.** *If $f$ is analytic on $\{|z| < R\}$ and $f \not\equiv 0$, then $\log N(f,r)$ is convex as a function of $\log r$.*

**Principle of Removable Singularities.** *If $f$ is analytic on $\{0 < |z| < R\}$ and $rN(f,r) \to 0$ as $r \to 0+$, then $f$ has an analytic continuation to $\{|z| < R\}$.*

The latter result is saying that either $N(f,r)$ behaves badly near 0 or else 0 is a removable singularity for $f$, in which case $N(f,r)$ is continuous at 0. The Three Circles Theorem has the following analogue for suprema over lines. (See [14, p.180] for an important application of this result in the proof of the M. Riesz convexity theorem.)

**Three Lines Theorem.** *Let $f$ be bounded and analytic on $\mathbf{R} \times (0,1)$, continuous on $\mathbf{R} \times [0,1]$, and let $f \not\equiv 0$. Then*

$$y \mapsto \sup\Big\{\log\big|f(x+iy)\big| : x \in \mathbf{R}\Big\}$$

*defines a convex function on $[0,1]$.*

We will be concerned with analogues of the above results for integrals of subharmonic functions. We recall that a function $s$ defined on a connected open subset $\omega$ of $\mathbf{R}^n$ $(n \geq 1)$ and taking values in $[-\infty, +\infty)$ is called *subharmonic* if $s \not\equiv -\infty$ and:

(i) $s$ is upper semicontinuous (u.s.c.), i.e. $\limsup_{Y \to X} s(Y) = s(X)$ for all $X \in \omega$;

(ii) the mean of $s$ over the boundary of any closed ball in $\omega$ is greater than or equal to its value at the centre.

**Notes.** (I) A function $h$ is harmonic (i.e. $h$ satisfies Laplace's equation) if and only if both $h$ and $-h$ are subharmonic.
(II) If $f$ is analytic on $\mathbf{C}$ and $f \not\equiv 0$, then $\log|f|$ is subharmonic. (Here we are identifying $\mathbf{C}$ with $\mathbf{R}^2$ in the usual way).
(III) Condition (ii) above can be replaced by (ii'): for any open set $W$ with compact closure in $\omega$, and for any continuous function $h$ on $\overline{W}$ which is harmonic on $W$ and satisfies $h \geq s$ on $\partial W$, we have $h \geq s$ on $W$.
(IV) Although it is usual to work with subharmonic functions on open subsets of $\mathbf{R}^n$, where $n \geq 2$, the definition also makes sense for $n = 1$. We discuss this further at the end of Section 3.

## 2 Convexity Theorems

If $s$ is a non-negative subharmonic function on $\mathbf{R}^{n-1} \times (0,1)$, put

$$M(x_n) = \int_{\mathbf{R}^{n-1}} s(x_1, \ldots, x_n)\, dx_1 \ldots dx_{n-1} \quad (0 < x_n < 1).$$

The following analogue of the Three Lines Theorem is essentially due to Hardy, Ingham and Pólya [8] in the case $n = 2$. (See also [13, 9]).

**Theorem 1** *If $M(\cdot)$ is locally bounded on $(0,1)$, then it is convex.*

**Proof** $(n = 2)$. Let $0 < \alpha < \beta < 1$, and choose $a, b$ such that $ay + b = M(y)$ for $y = \alpha, \beta$. Now define

$$h_\epsilon(x, y) = ay + b + \epsilon \cosh(\pi x)\sin(\pi y)$$



(a harmonic function), and

$$u_\ell(x, y) = \int_{-\ell}^{\ell} s(x + t, y)\, dt,$$

which is subharmonic because it is finite valued, u.s.c. (by Fatou's Lemma) and submeanvalued (by Tonelli's Theorem). Also $u_\ell \leq h_\epsilon$ on $\mathbf{R} \times \{\alpha, \beta\}$ and

$$u_\ell(x, y) - h_\epsilon(x, y) \to -\infty \quad (|x| \to \infty,\ \alpha \leq y \leq \beta),$$

so (cf. (ii') above) $u_\ell \leq h_\epsilon$ on $\mathbf{R} \times [\alpha, \beta]$. Letting $\epsilon \to 0+$ and $\ell \to \infty$, we get $M(y) \leq ay + b$ for $y \in [\alpha, \beta]$, proving convexity.

**Question.** Is local boundedness the "right" condition?

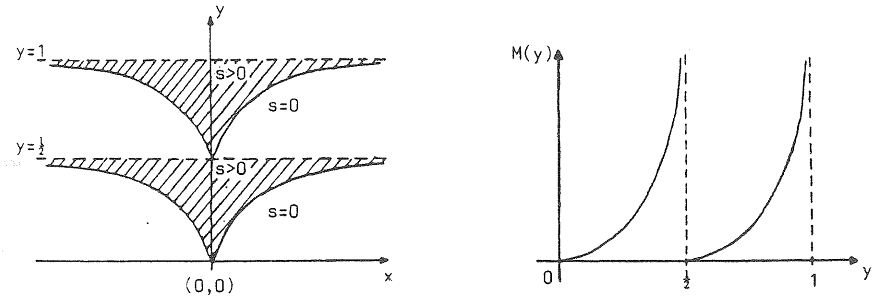
The hypothesis cannot be dispensed with entirely. To give some idea of possible behaviour we give below a few simple examples when $n = 2$.

Examples (i) $s(x, y) \equiv 1$; $\quad M(y) \equiv +\infty$.

(ii) $s(x, y) = e^{2\pi x}|\sin \pi y|$; $\quad M(y) = \begin{cases} 0 & \text{if } y \in \{0, \frac{1}{2}, 1\} \\ +\infty & \text{otherwise.} \end{cases}$

(iii) $s(x, y) = \dfrac{e^y}{x^2 + (y+1)^2}$; $\quad M(y) = \dfrac{\pi e^y}{y + 1}$.

(iv)



Thus $M(\cdot)$ may be everywhere infinite, or everywhere finite, or neither. Even if $M(\cdot)$ is always finite, it need not be convex.

**Theorem 2.** *If $M(\cdot)$ is locally integrable on $(0,1)$, then it is finite and convex.*

This result, due to Kuran [10], shows that convexity holds provided we restrict the type of discontinuity that is allowed to occur. It was substantially improved when Rippon [12] applied a result of Domar to obtain the following.

**Theorem 3.** *If $\log^+ M(\cdot)$ is locally integrable on $(0,1)$, then $M(\cdot)$ is finite and convex.*

It was also shown in [12] that the hypothesis here is best possible, so the convexity property of $M(\cdot)$ is now satisfactorily described. However, we will mention a recent generalization [7] which shows what happens when integration of $s$ is carried out with respect to fewer of the co-ordinates.

## 3 A Generalization

A subset $E$ of $\omega$ is called *polar* if there is a subharmonic function on $\omega$ which takes the value $-\infty$ on $E$. A function $s$ is said to be *quasi-subharmonic* if the function $\widehat{s}(X) = \limsup_{Y\to X} s(Y)$ is subharmonic, and $\widehat{s}$ equals $s$ except on a polar set.

Let $X = (X', X'') \in \mathbf{R}^{n-m} \times \mathbf{R}^m$, $(2 \le m \le n-1)$, and put

$$P(s, X'') = \int_{\mathbf{R}^{n-m}} s(X', X'')\, dX',$$

$$P_\infty(s, X'') = \sup\{s(X', X'') : X' \in \mathbf{R}^{n-m}\}.$$

**Theorem 4.** *Let $s$ be subharmonic on $\mathbf{R}^{n-m} \times (0,1)^m$.*

*(i) If $\{\log^+ P(s^+, \cdot)\}^{m+\epsilon}$ is locally integrable on $(0,1)^m$, then $P(s,\cdot)$ is either subharmonic on $(0,1)^m$ or identically valued $-\infty$.*

*(ii) If $\{\log^+ P_\infty(s^+, \cdot)\}^{m+\epsilon}$ is locally integrable on $(0,1)^m$, then $P_\infty(s,\cdot)$ is quasi-subharmonic on $(0,1)^m$.*

**Notes .** The hypotheses can be weakened slightly [7]. A version of (i) with stronger hypotheses was proved independently by Aikawa [1].



**Example.** To see that, in (ii), quasi-subharmonicity is the best that can be said, let $E$ be a polar subset of $(0,1)^m$ $(m \ge 2)$, and let $u$ be a negative subharmonic function taking the value $-\infty$ on $E$. Then the function $s(X) = -\{-u(X'')\,|X|^{2-n}\}^{1/2}$ can be shown to be subharmonic on $\mathbf{R}^{n-m} \times (0,1)^m$, and clearly

$$P_\infty(s, X'') = \begin{cases} -\infty & \text{if } u(X'') = -\infty \\ 0 & \text{elsewhere on } (0,1)^m. \end{cases}$$

Consider now the notions of harmonicity and subharmonicity for functions of one real variable. A "harmonic" function $h$ must satisfy $d^2h/dx^2 \equiv 0$, so $h(x) = ax + b$. From condition (ii′) of §1, if a "subharmonic" function $s$ satisfies $s(x) \le h(x)$ at $x = \alpha, \beta$, the same inequality holds for $x \in (\alpha, \beta)$, so $s$ is convex. Since it is impossible for a convex function to take the value $-\infty$, the only polar subset of $\mathbf{R}$ is the empty set. Hence, in $\mathbf{R}$, the terms "convex", "subharmonic" and "quasi-subharmonic" are synonymous. Thus Theorem 4 generalizes (in different ways) Theorems 1 – 3 and the Three Lines Theorem.

## 4 Growth Theorems

We now consider analogues for $M(\cdot)$ of the Principle of Removable Singularities. In what follows, we assume that $s$ is a non-negative subharmonic function on the half-space $\mathbf{R}^{n-1} \times (0,+\infty)$, and that $M(\cdot)$ is finite and convex on $(0,+\infty)$. We also note that, if $M(\cdot)$ is bounded on $(a,+\infty)$ for some $a > 0$, then $M(\cdot)$ is decreasing (wide sense).

The following is due to Flett [6].

**Theorem 5.** *If $M(y) = O(y^{n-1})$ as $y \to +\infty$, then $M(\cdot)$ is decreasing.*

**Proof** Let $B(X,r)$ denote the open ball of centre $X$ and radius $r$, and let $\nu$ denote the volume of $B(O,1)$. By hypothesis there exists $c > 0$ such that $M(y) \le cy^{n-1}$ for all $y \ge \frac{1}{2}$. If $x_n \ge 1$, then

$$\begin{aligned} s(X) &\le \frac{1}{\nu(x_n/2)^n} \int_{B(X,x_n/2)} s(Y)\,dY \quad (\text{cf §1, (ii)}) \\ &\le \frac{1}{\nu(x_n/2)^n} \int_{\mathbf{R}^{n-1}\times(x_n/2,3x_n/2)} s(Y)\,dY \\ &= \frac{1}{\nu(x_n/2)^n} \int_{x_n/2}^{3x_n/2} M(y)\,dy \end{aligned}$$

$$\leq \frac{c}{\nu(x_n/2)^n} \int_{x_n/2}^{3x_n/2} y^{n-1}\, dy = \text{constant}.$$

Thus $s$ is bounded on $\mathbf{R}^{n-1} \times [1, +\infty)$, and it follows that $s$ is majorized by its Poisson integral $I_s$ on $\mathbf{R}^{n-1} \times (1, \infty)$. Hence

$$M(x_n) = \int_{\mathbf{R}^{n-1}} s(X) dx_1 \ldots dx_{n-1} \leq \int_{\mathbf{R}^{n-1}} I_s(X) dx_1 \ldots dx_{n-1} = M(1) \qquad (x_n > 1)$$

by Tonelli's theorem, and so $M(\cdot)$ is bounded on $(1, +\infty)$.

In fact, Kuran [10] showed that the exponent in Theorem 5 can be increased.

**Theorem 6.** *If $M(y) = o(y^n)$, then $M(\cdot)$ is decreasing.*

**Example .** To see that the exponent cannot be further increased in the case $n = 2$, let $\alpha > 1$ and

$$s(re^{i\theta}) = \begin{cases} r^\alpha \cos \alpha(\theta - \frac{\pi}{2}) & (|\theta - \frac{\pi}{2}| < \frac{\pi}{2\alpha}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then $s$ is subharmonic on $\mathbf{R} \times (0, +\infty)$ and $M(y) = \text{const.}\, y^{\alpha+1}$ for $y > 0$. (For $n \geq 3$, a similar example is based on Legendre functions).

However, Nualtaranee [11] was able to refine Kuran's hypothesis.

**Theorem 7.** *If $M(y) = O(y^n)$, then $M(\cdot)$ is decreasing.*

The problem of finding the "correct" condition is now clearly down to a matter of "fine tuning". A contribution in this direction was obtained by Rippon [12] using a result of Dahlberg about minimally thin sets in half-spaces.

**Theorem 8.** *If $s$ has a harmonic majorant on $\mathbf{R}^{n-1} \times (0, +\infty)$ and*

$$\int_1^\infty \min\left[1, \{y/M(y)\}^{1/(n-1)}\right] dy = +\infty, \qquad (*)$$

*then $M(\cdot)$ is decreasing.*

Condition $(*)$ was also shown to be the best possible. Using the convexity of $M(\cdot)$ it can be seen that $(*)$ is implied by the condition $\liminf_{y\to+\infty} y^{-n} M(y) < +\infty$. It is now not difficult to obtain the following improvement of Theorem 7.



**Corollary.** *If $\liminf_{y\to+\infty} y^{-n} M(y) < +\infty$ then $M(\cdot)$ is decreasing.*

**Open Question .** Can the hypothesis about the harmonic majorant be removed from Theorem 8?

This question appears to be difficult. If the answer is "yes", then Rippon's condition $(*)$ is best possible [12].

# 5 An Extension

We mention now a recent result [3] which shows what can be said about the growth of $M(\cdot) = M(s, \cdot)$ when we drop the requirement that $s$ be non-negative. Again, $s$ denotes a subharmonic function on $\mathbf{R}^{n-1} \times (0, +\infty)$.

**Theorem 9.** *If $\log^+ M(s^+, y) = o(y)$ and*

$$\int_1^\infty y^{-n-1} M(s, y)\, dy < +\infty,$$

*then $M(s, \cdot)$ and $M(s^+, \cdot)$ are decreasing, and $M(s^-, y) = o(y)$.*

The proof of Theorem 9 begins by estimating the distributional Laplacian of $s$ on strips and using this to show that $s$ has a harmonic majorant on $\mathbf{R}^{n-1} \times (0, +\infty)$. With regard to the sharpness of the result we mention the following. (i) If $\log^+ M(s^+, y) = O(y)$, then all three conclusions fail. (ii) If we replace $y^{-n-1}$ by $y^{-n-1-\epsilon}$, the counterexample of §4 (involving Legendre functions) applies. (iii) The conclusion about $M(s^-, \cdot)$ is best possible in that, if $\phi(y)$ decreases to 0 as $y \to +\infty$, then there is a negative subharmonic function $s$ such that $M(s^-, y) \geq y\phi(y)$.

# 6 Other Results

A number of papers have dealt with $M(\Phi \circ s, \cdot)$, where $\Phi$ is an increasing, convex function (whence $\Phi \circ s$ is subharmonic). We mention here only the case $\Phi(x) = x^p$, where $p > 1$. The following is a refinement of a result of Brawn [4] in the light of Theorem 3.

**Theorem 10.** *If $s$ is non-negative and subharmonic on $\mathbf{R}^{n-1} \times (0, 1)$ and $\log^+ M(s^p, \cdot)$ is locally integrable on $(0, 1)$, then $\{M(s^p, \cdot)\}^{1/p}$ is finite and convex.*

The convexity property here is replaced by subharmonicity if we integrate only over $\mathbb{R}^{n-m}$ as in §3, (see [6]). With regard to growth theorems, we mention the following result of Armitage [2].

**Theorem 11.** *If $s$ is non-negative and subharmonic on $\mathbb{R}^{n-1} \times (0, +\infty)$ and $M(s^p, y) = O(y^{n+p-1})$ as $y \to +\infty$, then $M(s^p, y)$ decreases to 0 as $y \to +\infty$.*

Thus, with $s$ replaced by the "strongly subharmonic" function $s^p$, we can weaken the hypotheses of Theorem 7 and strengthen the conclusion.

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# Toeplitz Operators

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There are few classes of operators on a Hilbert space about which one has very detailed information, apart from the normal operators and the compact operators. An exceptional class about which much is known is the class of Toeplitz operators. This paper gives a brief survey of some aspects of their theory, from its origin near the begining of this century to the present day.

## 1 The basic results

The study of Toeplitz operators was initiated in a paper in 1911 (by Toeplitz [13]) in which the relationship of finite square matrices that are constant on diagonals to the corresponding infinite matrices was investigated. The class of Wiener-Hopf operators was studied in parallel (from 1931) until Rosenblum [11] observed in 1965 that that the two classes of operators are unitarily equivalent. There now exists a vast literature on this area. The theory is interesting in its own right, but also has applications to and connections with many other areas, for example, Function Theory, Prediction Theory, C*-algebras, other areas of Operator Theory, Probability, and Physics.

Having indicated why Toeplitz operators are studied, let us now define them and look at some of their properties.

Let T denote the circle group, $\mathrm{T} = \{z \in \mathrm{C} \mid |z| = 1\}$, and let $\lambda$ denote normalized Haar measure (= normalized arc length) on T. For $p \in [1, +\infty]$ let $L^p = L^p(\mathrm{T}, \lambda)$. If $\varphi \in L^\infty$ we get a bounded linear operator $M_\varphi$ on $L^2$ by setting

$$M_\varphi(f) = \varphi f \quad (f \in L^2).$$

$M_\varphi$ is called the *Laurent operator* with *symbol* $\varphi$. It is of course a normal operator, i.e. it commutes with its adjoint, and the map $\varphi \mapsto M_\varphi$ is an isometric $*$-homomorphism of $L^\infty$ into $B(L^2)$ (for any Hilbert space $H$, we let $B(H)$ denote the Banach algebra of all bounded linear operators on $H$). The matrices of these operators are very special when taken relative to the standard orthonormal basis $(e_n)_{n\in\mathrm{Z}}$ (where $e_n : z \mapsto z^n$) — they are constant





along diagonals. However our interest is not really in Laurent operators, whose theory is very easy, but in their compressions to a certain subspace $H^2$. Recall that for $p \in [1, +\infty]$ the *Hardy space* $H^p$ is the set of all $f \in L^p$ with "negative" Fourier coefficients $\langle f, e_n\rangle = \int f\bar{e}_n\, d\lambda$ $(n < 0)$ all equal to zero. Let $P$ denote the projection of $L^2$ onto $H^2$. If $\varphi \in L^\infty$ then $T_\varphi \in B(H^2)$ is defined by setting

$$T_\varphi(f) = P(\varphi f) \quad (f \in H^2).$$

$T_\varphi$ is called the *Toeplitz operator* with *symbol* $\varphi$. It might appear at first sight that the theory of Toeplitz operators should be like that of Laurent operators, since the difference between their definitions may appear trivial, but in fact the two theories are profoundly different. The theory of Toeplitz operators is deep, and hard.

Let us begin by noting some elementary facts. Every Toeplitz operator has matrix with constant diagonals relative to the standard orthonormal basis $(e_n)_{n=0}^\infty$, and conversely any $T \in B(H^2)$ with such a matrix relative to $(e_n)_{n=0}^\infty$ is a Toeplitz operator. One has $\|T_\varphi\| = r(T_\varphi)$ (the spectral radius) $= \|\varphi\|_\infty$. The spectral theory of $T_\varphi$ is complicated by the fact that invertibility of $T_\varphi$ is not equiva lent to invertibility of $\varphi$ (although the corresponding statement is true for $M_\varphi$). One does have implication in one direction however: if $T_\varphi$ is invertible then $\varphi$ is invertible. Hence $\sigma(\varphi) \subseteq \sigma(T_\varphi)$, a result due to Hartman and Wintner [6]. $\sigma(\ )$ denotes the spectrum — for $a$ an element of an algebra having a unit 1 the *spectrum* $\sigma(a)$ of $a$ is the set of all $z \in \mathrm{C}$ such that $z1 - a$ is not invertible. As indicated above one does not in general have $\sigma(T_\varphi) = \sigma(\varphi)$. An example is provided by the unilateral shift $U = T_{e_1}$ as its spectrum is the closed unit disc, but the spectrum of $e_1$ is the unit circle. However $\sigma(T_\varphi)$ is not too much bigger than $\sigma(\varphi)$, because $\sigma(T_\varphi)$ is contained in the closed convex hull of $\sigma(\varphi)$ (Brown-Halmos [ 1]).

The above results are relatively near the surface. In contrast is the beautiful and surprising theorem of Widom [15] which states that the spectrum of a Toeplitz operator is connected.

One of the reasons that Toeplitz operator theory is not easy is that the equation $T_{\varphi\psi} = T_\varphi T_\psi$ does not hold in general (e.g. take $\varphi = e_1$ and $\psi = \bar{e}_1$). But there is a subclass of Toeplitz operators for which the above equality does hold: We say $T_\psi$ is *analytic* if $\psi \in H^\infty$. In this case we have $T_\varphi T_\psi = T_{\varphi\psi}$ and $T_{\bar{\psi}} T_\varphi = T_{\bar{\psi}\varphi}$ for all $\varphi \in L^\infty$. Hence the map

$$H^\infty \to B(H^2), \quad \psi \mapsto T_\psi,$$

is an algebra isomorphism onto the closed subalgebra $A$ of all analytic Toeplitz operators. One can easily show that $A$ is the commutant of $U$, i.e. the set of all operators commuting with $U$, and so a maximal commutative subalgebra of $B(H^2)$. The open unit disc $\Delta$ can be embedded in the character space of the Banach algebra $H^\infty$ in a natural way, and if $\psi \in H^\infty$ then its Gelfand transform $\hat{\psi}$ restricted to $\Delta$ is a bounded analytic function. (By the way, the character space of $H^\infty$ is quite complicated. The famous Corona Theorem of Carleson says that $\Delta$ is dense in this space.) One can now state Wintner's theorem **[16]** : If $\psi \in H^\infty$ then $\sigma(T_\psi) = cl(\hat{\psi}(\Delta))$ where $cl$ denotes closure in **C**.

In the next section we shall indicate how one can use C*-algebras to get some other results of the classical theory of Toeplitz operators, but we end this section with a brief remark on Wiener-Hopf operators.

If $\varphi \in L^1(\mathbf{R})$ the *Wiener-Hopf* operator $W_\varphi \in B(L^2(\mathbf{R}^+))$ is defined by setting

$$(W_\varphi f)(x) = \int_0^\infty \varphi(x-t)f(t)\,dt \quad (f \in L^2(\mathbf{R}^+)).$$

If $z \in \mathbf{C} \setminus \{0\}$ then the Wiener-Hopf equation is

$$(z + W_\varphi)f = g$$

where $g$ is given and $f$ is the unknown function. The conformal map of the upper half-plane onto the unit disc sets up a unitary equivalence between a Wiener-Hopf operator and its corresponding Toeplitz operator.

## 2 C*-algebras and Toeplitz operators

For $T$ a bounded linear operator on a Hilbert space $H$, $N(T)$ denotes its nullspace $\{x \in H : Tx = 0\}$. Recall that $T$ is *Fredholm* if $T(H)$ is closed and $N(T), N(T^*)$ are finite-dimensional. In this case the Fredholm index is defined to be $index\,(T) = dim\,N(T) - dim\,N(T^*)$. The *essential spectrum* of $T$ is the set

$$\sigma_e(T) = \{z \in \mathbf{C} \mid z1 - T \text{ is not Fredholm}\}.$$

Obviously $\sigma_e(T) \subseteq \sigma(T)$. There is a very useful characterization of Fredholm operators due to Atkinson, but to state it we need to introduce a few more



concepts. An operator is *compact* if it is the norm limit of a sequence of finite-rank operators. The set $K(H)$ of these operators on $H$ forms a closed ideal, and the quotient algebra $B(H)/K(H)$ is called the *Calkin* algebra. If $\pi$ is the canonical map from $B(H)$ to $B(H)/K(H)$ then the Atkinson characterization is that $T$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra.

A key step in the spectral analysis of Toeplitz operators is the following lemma due to Coburn [2]. The proof is so short that we include it, as it also illustrates nicely the connections with Function Theory.

**Lemma** *If $\varphi$ is a function in $L^\infty$ not almost everywhere equal to zero then either $T_\varphi$ or $T_\varphi^*$ has zero nullspace.*

**Proof** Recall that the theorem of F. and M. Riesz says that a nonzero function $f$ in $H^2$ cannot vanish on any set of positive measure in **T**. Now suppose that $f \in N(T_\varphi)$ and that $g \in N(T_\varphi^*)$. Then $\overline{\varphi f}$ and $\varphi\bar{g}$ are in $H^2$, so $\varphi f \bar{g}$ and $\overline{g\varphi f}$ are in $H^1$. By the "analyticity" property of the Hardy spaces, we must have $\varphi f \bar{g}$ is constant a.e. But $\int \varphi f \bar{g}\,d\lambda = 0$, so $\varphi f \bar{g} = 0$ a.e. If neither $f$ or $g$ is zero, then by the F. and M. Riesz theorem we must have $\varphi = 0$ a.e., a contradiction. QED

It is immediate from this lemma that if $T_\varphi$ is Fredholm then $T_\varphi$ is invertible if and only if $index\,(T_\varphi) = 0$. From this it is not difficult to prove:

**Theorem (Krein-Widom-Devinatz)** *If $\varphi$ is a continuous function on* **T** *then the operator $T_\varphi$ is Fredholm if and only if $\varphi$ does not vanish anywhere, and in this case $index\,(T_\varphi)$ is equal to minus the winding number of $\varphi$ with respect to the origin.*

This beautiful result thus identifies an analytic index with a topological index, and is a simple prototype of the Atyiah-Singer Index Theorem. A direct consequence of it is the fact that the spectrum of $T_\varphi$ is connected if $\varphi$ is continuous, thus giving an easy proof of a special case of Widom's Theorem.

Many of the above results (and other results) are obtained by C*-algebraic techniques. The idea is this: Let $T(\mathbf{Z})$ denote the C*-algebra generated by the Toeplitz operators with continuous symbol (the reason for the appearance of the symbol **Z** will become clear presently). Then its commutator ideal (i.e. the smallest closed ideal $I$ for which the quotient algebra modulo $I$ is abelian) is $K(H^2)$. The map

$$C(\mathbf{T}) \to T(\mathbf{Z})/K(H^2), \ \varphi \mapsto T_\varphi + K(H^2),$$

is a $*$-isomorphism. To see how this gives connectedness of the essential spectrum note that if $\varphi \in C(\mathrm{T})$ then its spectrum is just its range, so it is connected as T is connected. It follows that the spectrum of $T_\varphi + K(H^2)$ is connected, and by the Atkinson characterization this is just the essential spectrum of $T_\varphi$. With a little more work a simple proof that $\sigma(T_\varphi)$ has connected spectrum can be given in this case.

One can also consider the C*-algebra generated by all Toeplitz operators, and use it to derive various results. For details see Douglas [4].

The above discussion indicates the usefulness of the algebra $T(\mathrm{Z})$ in Single Operator Theory, but it is also useful in C*-algebra theory. It is generated by a non-unitary isometry (viz $U$), and up to isomorphism it is the only such C*-algebra. Moreover one can use the short exact sequence

$$0 \to K(H^2) \to T(\mathrm{Z}) \to C(\mathrm{T}) \to 0$$

(or rather a reduced form of it) to give a relatively easy proof of the Bott periodicity theorem in K-theory (for locally compact spaces and for C*-algebras).

It is natural that one should try to extend these ideas and techniques to more general situations, and this has been done by many mathematicians including Douglas, Devinatz, Howe, Kaminker, Muhly and Singer and many others. We now discuss one of these extensions.

## 3 Extended theories of Toeplitz operators

An *ordered group* is a pair $(G, \leq)$ consisting of a (discrete) abelian group $G$ and a linear partial order $\leq$ on $G$ which is translation invariant (i.e. $x \leq y$ implies that $x+z \leq y+z$). Obvious examples are Z, R and all subgroups of R. Ordered groups exist in superabundance, for if $G$ is a discrete abelian group with Pontryagin dual group $\hat{G}$ then the following are equivalent conditions:
(1) There is a linear order $\leq$ on $G$ making $G$ an ordered group.
(2) $G$ is torsion-free
(3) $\hat{G}$ is connected.

Fix an ordered group $G$, and let $m$ denote normalized Haar measure on $\hat{G}$, and $L^p = L^p(\hat{G}, m)$, $1 \leq p \leq +\infty$. If $f \in L^1$ we say that $f$ is of *analytic type* if the Fourier transform $\hat{f}(x) = 0$ for all $x \in G$ for which $x < 0$. $H^p = H^p(G)$



denotes the norm-closed vector subspace of all $f \in L^p$ of analytic type. As is well known if $\varepsilon(x) : \hat{G} \to \mathrm{T}$, $\gamma \mapsto \gamma(x)$, $(x \in G)$ then $(\varepsilon(x))_{x \in G}$ form an orthonormal basis for $L^2$. It follows that if $G^+ = \{x \in G : 0 \leq x\}$ then $(\varepsilon(x))_{x \in G^+}$ is an orthonormal basis for $H^2$. One can extend Fourier Analysis and Function Theory to this context (see Rudin [12]). The generalized Hardy spaces $H^p$ display analytic behaviour, for if $f, \bar{f} \in H^p$ then $f =$ constant a.e. We can now define Toeplitz operators as before, and many of the elementary results extend easily. However we shall primarily be interested in certain C*-algebras generated by Toeplitz operators.

Let $T(G)$ be the C*-algebra generated by all $T_\varphi$ for which $\varphi \in C(\hat{G})$, and let $K(G)$ be its commutator ideal. Before stating some results in this area we need a few definitions.

A C*-algebra $A$ is *primitive* if it has a faithful irreducible representation (i.e. there is an injective $*$-homomorphism $\varphi : A \to B(H)$ where $H$ is some Hilbert space with no nontrivial subspace invariant for every $\varphi(a)$ $(a \in A)$). $A$ is *simple* if it has no closed ideals apart from 0 and $A$. Simple C*-algebras are primitive. In a loose sense the primitive and simple C*-algebras are thought of as the building blocks from which all C*-algebras are made, and for this and other reasons it is very important to have many examples of such algebras.

**Theorem** *Let G be an ordered group.*
*(1) $T(G)$ is primitive (and therefore $K(G)$ is primitive also).*
*(2) $K(G)$ is simple if and only if $G$ is (order isomorphic to) a subgroup of* R.

(1) and the forward implication in (2) are due to the author [9]. The backward implication in (2) is due to Douglas [5]. The study of the algebras $K(G)$ in the case of subgroups of R has become especially important recently with connections having been discovered with Connes' non- commutative Differential Geometry. There are are many more interesting things that can be said about these more general Toeplitz theories. For example there is the very rich spatial theory due to Muhly and others which has not even been touched on above. We finish up with a few remarks on the K-theory of the algebras $K(G)$. In [5] Douglas asked if subgroups $G_1, G_2$ of R were order isomorphic when $K(G_1)$ and $K(G_2)$ are isomorphic. This was answered affirmatively in [10] in a special case, and in general in [7]. The method of proof in both cases involved computing the K-groups of $K(G)$.

A good elementary introduction to the theory of Toeplitz operators is [4].

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# MATHEMATICAL EDUCATION

# Mathematics at Third Level
# Questioning How We Teach

Maurice O'Reilly

This essay comes in four parts: context, focussing questions, philosophies and questions for exploration.

## Contexts

What I have to say arises from three contexts: (a) my/our own personal experience, (b) the constraints of the educational system in which I/we operate, and (c) the last two meeings of the Society.

On the first, each of us has his/her own 'teaching CV'. I myself have been on the staff of Dundalk RTC since September 1981 teaching courses at National Certificate and National Diploma level in the Science, Business Studies and Engineering Schools. Latterly, my work has been with students of computing, science and marketing. The Leaving Certificate grades in mathematics of student entering Dundalk RTC may range from D in the lower course to a good honour (in the higher): an indication that students' formation, and perhaps their capabilities, in Mathematics can vary greatly.

The second context has to do with such issues as institutional goals, educational resources, physical space, class sizes, class contact hours and course objectives. All of these are characterized by elements of structure rather than experience.

The third context is one which has drawn attention to at least two important areas in the teaching of Mathematics: (i) the low numbers who choose to follow mathematics courses (perhaps indicative of a flaw in the popular





perception of the subject), and (ii) the challenge posed by the increasing use of computers in every domain of mathematics teaching.

It is my belief that when we talk about teaching mathematics, we need to go behond merely prescribing courses. We need to consider questions of *how* we teach whatever-it-is that we teach. It is perhaps remarkable that in, for example, Ralston & Young's interesting study [5] on the future of college mathematics, there is nothing to be found on this issue. For sure, the content of courses is important, but not to the absolute exclusion of considering how we spend our time in the classroom (I use 'classroom' to include lecture theatre!). And, after all the talk, then there's the doing ...

## Focussing Questions

To focus attention on the issues, I invite the reader to reply to the following questions:

1. What do I teach?
2. Where is my attention when I teach?
3. What different modes (methods) of teaching do I use as a mathematics educator?
4. How much time during a typical scheduled class am I silent?
5. Under what circumstances do I have my most fruitful pedagogic insights?
6. How do I cope in teaching when under (severe) time pressure?
7. To what extent are students actively engaged in my classes? Is this the same for all my classes?
8. To what extent do students know *in advance* what to expect in my class? Not only in terms of content, but also in terms of process?
9. What do students appreciate in a good lecture?
10. What do I consider important other than *content* in my teaching?

It is tempting to ask for answers to be shared, but when it comes to *how* we teach, our 'academic objectivity' tends to wear thin and unhelpful comparisons between approaches may ensue! Academics are sometimes identified as possessing a certain arrogance (if discreetly expressed!), or at least conceit, about the importance and validity of their work. Perhaps we mathematicians have developed this conceit to a fine art, since it is our practice, not only to proclaim our truths, but also to *prove* them! This is all very appropriate in a mathematical context, but what happens when it slips into our teaching methods too?

## Philosophies

Perhaps there are two areas of attention in learning mathematics: theory and practice. At first, students usually perceive theory as a body of knowledge to be taken down in their notes. It is important that these notes be coherent so that they can be consulted usefully at a later date. It is only after intelligent study, involving practice, review of theory, more practice, etc., that understanding grows.

In [7], Sheffield identified the most important aspect of lecturing as 'to stimulate students to become active learners in their own right'. This might be said of all teaching!

The key question an educator of mathematics (or indeed any subject) can ask is how can I provide the best range of opportunities for my students to learn?

There are various modes of teaching which provide different opportunities for students' learning. These may be characterised by the proportion of participation/control assumed by the lecturer as opposed to the student. The spectrum includes traditional lectures, facilitated group work, private study, etc.

It is often the case that the educator spends most of his/her teaching time operating in just one mode. Likewise, he student may remain in the rut of just one learning mode.

Much of the above question can be restated as: How can I ensure that I use the appropriate mode (and variety of modes) in a particular teaching circumstance?

In an environment where many (most?) weak students are struggling even to begin to grasp our subject, surely it is up to us to develop *process* in our mathematics teaching?



The 'traditional' process in mathematics teaching is linear with emphasis on content and is supported by an exclusively logical structure. To become mathematicians, we have thrived in such an environment, and now, as teachers, we perpetuate it. Yet we know from our *research* work that intuition plays a vital role in doing mathematics. Where is intuition in our teaching? Is Poincare's essay 'Mathematical Discovery' [4], better known among psychologists than among ourselves? Hadamard [2] has said that 'logic merely sanctions the conquests of the intuition'. (An update of Hadamard's work is found in Muir [3].) Why is mathematics so often presented in a state of over-rumination: chewed beyond flavour?

Our own research involves exploration: exploration which is rooted in experience. What is the analogy of this in our teaching? Let me put it this way: Suppose $E_{ij}$ is the experience of student $i$ in topic $j$. How can we benefit from $\bigcup_i E_{ij}$ rather than merely $\bigcap_i E_{ij}$?

Before offering some (more) open-ended questions, a few cautionary words. Our inertia in improving our teaching can operate in subtle ways. One such way is to allow oneself to be side-tracked into proclaiming the superiority of one's own methods over those of others. Another way is to dismiss a method glibly by insisting, often inaccurately, that one already applies a particular recommended method in one's teaching. It is hoped that the following questions may stimulate honest, self-critical and constructive exploration of how we teach methematics.

## Questions for Exploration

1. What is he best question I can ask to motivate topic, $T$?
2. How can I best spend time with my students in subject, $S$?
3. When is it desirable to alert students to my teaching mode?
4. How can I encourage students to engage more actively in my classes?
5. How can I encourage students to learn mathematics *intuitively*?
6. How can I encourage a healthy dynamic between theory and practice?
7. How can I encourage a spirit of mathematical confidence and independence in my students?

8. If I have new ideas about teaching mathematics, how can I be sure that I succeed in implementing, evaluating and extending them?
9. How can assessments be designed in order to complement good teaching so as to promote further opportunities for learning?
10. Are problem-solving and theorem-proving the only relevant elements in assessment?
11. What contribution could 'academic councils' make to the development of effective teaching?
12. What structures are necessary in my institution to support and promote effective teaching?

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# Linking Mathematics with Industrial Problems

P. F. Hodnett

There is a growing interest in establishing links between University Mathematics Departments and industrial and commercial organizations in order to identify industrial problems amenable to mathematical analysis. There is a variety of reasons for this including:

1. the desire of mathematics faculty to contribute to the solution of real life problems;
2. the desirability of involving graduate students with such problems;
3. offering industry the opportunity to view the useful mathematical expertise of graduates with possible resultant job offers;
4. the desire by industry to create links with mathematics faculties to avail of faculty expertise and to aid in student recruitment for the company;
5. the desire by industry to avail of technical expertise in areas of shortage of such expertise in the company.

As a result, a number of Universities in different parts of the world have established such links. The type of link varies somewhat from place to place. Probably the oldest continuing link scheme (running for more than fifteen years) is operated at the Mathematics Institute, University of Oxford, U.K. where a one week study group is held annually involving Oxford faculty members (augmented by invitees from other Universities) graduate students and industrial participants to discuss and hopefully outline solution paths to industrial problems. With initial help from Oxford faculty a similar one week study group is now held annually at both Rensselaer Polytechnic Institute in the USA and at C.S.I.R.O. in Australia. A different type of process (also running for more than fifteen years) is operated at Claremont Colleges, California, USA, where the postgraduate education of mathematics students is through involvement with industrial problems funded by industry and identified by faculty with an industrial partner. In return for funding, Claremont Colleges contracts to deliver material specified in a contract. The TechnoMathematics Group at Kaiserslautern University, Federal Republic of Germany and the Mathematics Institute, University of Linz, Austria also have well established links involving the education of mathematics students through the solution of industrial problems although at these Universities the procedure is less formalised than in the Oxford model. The TechnoMathematics Group at Kaiserslautern has particularly strong links with the German automobile manufacturing industry while Linz has particuarly strong links with the electricity supply companies and the chemical plants in Austria.

Similar initiatives have begun in Ireland and in the recent past a workshop in Applied Mathematics was held at NIHE, Limerick, in the first of what is planned to be a continuing series. The objective of the workshops is to involve the mathematics faculty and postgraduate students at NIHE, Limerick in real applications of mathematics in both the manufacturing industry and commercial organizations; the workshop participants offer help to industry in solving problems that appear to be amenable to mathematical modelling and analysis.

The task of identifying a set of suitable industrial problems for the workshop required substantial effort on the part of three mathematics faculty members, despite the fact that (1) NIHE, Limerick has well-established industrial links through its industrial placement program, which is an integral element of all degree programs; and (2) the range of potential problems was wide in that problems were regarded as acceptable if adequate relevant expertise resided in the mathematics faculty (augmented by faculty from other departments at NIHE, and, possibly, by support from faculty of other universities).

The three problems considered at the workshop were of widely different types: (1) wave-induced washout of submerged vegetation in Irish lakes; (2) creep behaviour of ultra-high-molecular-weight polyethylene under dynamic load, with potential application to the design of femoral prostheses; and (3) keg utilization.

## Workshop structure

For this first venture, if was decided to confine the activity to one day so as to facilitate participation by both the industrial representatives and faculty and students from other institutions. During the morning session, each of



the three problems was described in detail by the industrial proposer of the problem.

In the afternoon, separate groups with appropriate expertise and interests that had been identified for each problem participated in discussion sessions. Each discussion session began with an introductory presentation of material related to the problem under consideration by an NIHE, Limerick academic. This presentation was followed by an open discussion chaired by the introductory speaker.

The objectives of the afternoon session were (1) to identify potential solution paths for the problems (if possible) and (2) to identify groups of academics who would commit to work on a continuing basis with the industrial presenters of the problems towards solving the problem.

## Submerged vegetation washout

The problem on washout of submerged vegetation in Irish lakes was presented by the Central Fisheries Board, Dublin. The Irish Central Fisheries Board, which is responsible for monitoring and maintaining fish stocks in Irish lakes, has observed that vegetation growing on lake bottoms is sometimes washed away by the action of the wind on the lake surface. Since the vegetation is necessary for the health and survival of fish stocks, it is desirable to prevent vegetation washout; to do so, however, it is necessary to understand the mechanism through which washout can be predicted for a given wind speed, wind direction, and lake geometry. To achieve this, it is necessary to understand how the action of wind-driven waves on the surface of the lake is transmitted to the lake bottom to create stresses that cause washout of vegetation from the lake bottom.

For this water wave problem, mathematics faculty members from NIHE, Limerick with expertise and interest in fluid mechanics and wave problems were joined by fluid mechanics colleagues from mechanical engineering, NIHE, Limerick as well as experts in fluid mechanics and waves from University College, Cork and NIHE, Dublin. Joint work continues on this problem, and to date an initial model based on linear water wave theory has been developed.

# Improved Artificial Joint Design

The problem related to the design of femoral prostheses was presented by a large medical products manufacturing company that provides the medical profession with a wide range of artificial replacement joints and limbs for the human body. At present, the company produces an artificial hip joint head that fits into a receiver cup made of a different material. The company wishes to replace the material now used in the receiver cup by an ultra-high-molecular-weight polyethylene (UHMWPE) material. To do so, it is necessary (1) to establish the response of the UHMWPE material to anticipated static and dynamic load; and (2) to establish a model for predicting the creep penetration of the metal head into the UHMWPE after $N$ walking cycles and after various periods of use (days, months, years).

To consider these two linked materials and mechanical problems, a number of mathematics faculty members with expertise in numerical analysis (since numerical analysis is expected to play an important role in the modelling of these problems) were joined by materials and mechanical engineering colleagues from NIHE, Limerick and a materials expert from NIHE, Dublin. Work on the problem continues, aided in this case by the fact that a research professorship in mechanical engineering at NIHE, Limerick is sponsored by this industrial company and his work is substantially concerned with establishing the mechanical response of medical prostheses under static and dynamic load.

# Optimising Keg Use

The problem of keg utilization was presented by a large brewing company. The company wishes to optimize the use of its keg population (used to transport its wide range of brewing products) for known current demand and future anticipated demand. The problem as presented at the wokshop was somewhat diffuse and not clearly defined. The background is that this company holds a population of approximately 800,000 kegs, purchased during a 20-year period and of three types (i.e. 51.1 L aluminium, 50.0 L aluminium, 50.0 L stainless steel). The company's four production centers serve three main markets, i.e., Ireland, Europe and the U.S., with a dozen different brewing brands. Identification of kegs/markets/products is currently done by color banding. A proposal within the company is to change to a universal keg (50 L stainless steel). Subproblems related to the general optimization problem are (1) how



to estimate and control/reduce losses in the current keg population; (2) how to measure and improve utilization in a situation in which return times vary widely in different markets, from one week to 18 months; (3) with a universal keg, how to monitor intercompany transfers (thre are a number of separate companies within the group) and how to allocate control over their own keg populations to individual companies.

The consensus reached by the workshop participants was that to make progress in solving this problem, the company needs a range of reliable statistics (currently not available) on the keg population and that the company needs to invest resources to provide the necessary data. Work on this problem continues, aided by the close existing contacts betweeen the Mathematics Department at NIHE, Limerick and this company which on a regular basis, receives applied mathematics degree students from NIHE, Limerick for industrial placement.

# Conclusion

Review of the first workshop on applied mathematics in anticipation of planning for future workshops, clearly indicates that substantial expenditures of time, energy, and effort are required (1) to identify suitable industrial problems, (2) to organize an appropriate expert group to address a given problem, and (3) to ensure that collaboration between the industrialist and the academic group continues after the workshop, until an acceptable solution to the problem has been identified. it appears, therefore, that to continue the operation on an ongoing basis will require the expenditure of substantial human resources. However, there is considerable enthusiasm within the Mathematics Department at NIHE, Limerick for the organisation of further workshops.

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# NOTES

## Error Correcting Codes

John Hannah

### Introduction

In this article, I will describe an elementary approach to error-correcting codes which can be presented to second year (general level) students. In this approach, students can see simple abstract concepts being used to solve an easily described practical problem.

Although most texts give the impression that you need to know some finite field theory in order to learn coding theory, you can in fact get a good grasp of the basic ideas by knowing about little more than matrix multiplication and modulo - 2 arithmetic. Thus, for example, I include such codes as a brief topic in my second year linear algebra course (to help justify looking at abstract vector spaces rather than just spaces over the real numbers). Codes could also be discussed in introductory courses on abstract algebra or discrete mathematics.

From the student's point of view, the need for error-correcting codes is easily appreciated. Digital data occur in many parts of modern life. Information is stored as strings of binary digits (0 and 1) in such diverse areas as computers, satellites and record-players. It is important to be able to transfer such data reliably between different systems, whether it be between the memory and the processor of a computer, or between Earth and Voyager satellite near Uranus. Unfortunately, most communication channels are prone to noise of one sort or another, and errors can appear in the data. Coding theory tries to construct efficient ways of sending digital data while at the same time guarding against these errors.





### The Parity-check Code

Perhaps the simplest example of a binary code is the parity-check code. Here original data, a block of $n$ binary digits, has an extra "check digit" attached, and a block of $n+1$ digits is transmitted instead. The check digit is 0 if there are an even number of 1's in the original message, and 1 if there are an odd number. Hence the name "parity check" code. Using addition moduo 2 the encoding procedure can be expressed as:

$$\underset{\text{information digits}}{(a_1, a_2, \ldots, a_n)} \quad \rightarrow \quad \underset{\text{codeword}}{(a_1, a_2, \ldots, a_n, a_{n+1})}$$

where

$$a_{n+1} = a_1 + a_2 + \cdots + a_n.$$

For example, if we were using 4-digit blocks we would have the encodings

$$(1101) \rightarrow (11011)$$

and

$$(1010) \rightarrow (10100).$$

In this code a correctly received codeword must have an even number of 1's. Hence if there is exactly one error during transmission, the receiver will realize that an error has occurred. For example (10101) cannot come from a correctly sent message. So we can say that the parity check code *detects all single errors.* If there is no error detected then the receiver decodes the message simply by removing the redundant digit $a_{n+1}$.

There are two drawbacks to this simple code. Firstly, if only one error does occur, we still cannot tell which digit is wrong, and so we cannot correct the error. For example (10101) could have come from a single error in either (10100) or (11101). Secondly, double errors go undetected. For instance, errors in the second and fifth digits of (11011) result in a received word (10010) which looks correct since it satisfies the parity check $1+0+0+1=0$.

The idea behind coding theory is to look for more sophisticated "parity checks" which will improve the performance of the above code. By introducing extra redundant digits (like $a_{n+1}$ in the parity check code) we can home in on the incorrect digit and detect more errors. Of course, if there are too many redundant digits the code will become inefficient: the actual message will make up only a small part of the codeword and transmission will become time-consuming and expensive. So good codes have as few redundant digits as possible and detect or correct as many errors as possible.

## The ISBN Code

The ISBN numbers, which are used to classify books, are another example of a code. Again just one check digit is involved, but this time the arithmetic is done modulo 11. In a typical ISBN number

$$0 - 474 - 00130 - X$$

the first nine digits $a_1, a_2, \ldots, a_9$ are the information digits, and the check digit $a_{10}$ is calculated from the formula

$$a_{10} = \sum_{n=1}^{9} n a_n \pmod{11}.$$

If, as in the above example, this check digit is 10, it is written as $X$. It is easy to see that this code again detects all single errors. The coefficients $n$ are used (instead of the 1 used in the parity check code) so that the code will also detect all transposition errors, where two digits are accidentally interchanged. Transpositions are among the most common errors that occur when data are communicated by humans (rather than by electric or magnetic fields!)

I will discuss binary codes in the rest of this article, but obviously a discussion of why modulo-11 arithmetic is used here would fit well into an abstract algebra course.

## Constructing an error-correcting code

To illustrate the ideas involved, I shall show how to construct one of the family of Hamming codes. These codes can correct single errors or, alternatively, detect double errors.

Suppose we allow ourselves four check digits, instead of the single check digit in the above parity check code, and suppose that our data consists



of strings of $n$ information digits $(a_1, a_2, \ldots, a_n)$. Each of the check digits $a_{n+1}, \ldots, a_{n+4}$ will come from an equation of the form

$$a_{n+1} = \text{sum of some of } a_1, a_2, \ldots, a_n$$

and these same four equations will be used as check equations for the received message $(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+4})$. We can arrange these check equations in matrix form as

$$Ha = 0$$

where $a$ is the received codeword, 0 is the zero vector and $H$ is a matrix of the form

$$H = [Q \mid I_4]$$

where $I_4$ is the $4 \times 4$ identity matrix and $Q$ is a $4 \times n$ matrix of 0's and 1's. These equations determine whether $a$ is a correctly encoded string of $n + 4$ digits. Clearly if the received codeword gives $Ha \neq 0$, then an error has occurred. But what is the precise effect on $Ha$ of an error in the codeword $a$? We can represent the received codeword as $a + e$, where $a$ is the intended codeword, and the vector $e$ has a 1 in each position where an error occurred but has zero components otherwise. This is because each of the errors $0 \to 1$ and $1 \to 0$ can be obtained by adding 1 (mod 2) to the original entry. When we test the received codeword using the matrix $H$ we get

$$H(a + e) = Ha + He = 0 + He.$$

If there has been exactly one error, in the $i$th position say, then we have

$$He = i\text{th column of } H.$$

For example, an error in the first entry corresponds to having $e = (1, 0, \ldots, 0)^T$ and the calculation of $H(a + e)$ will yield the first column of $H$.

Thus to be able to *detect the existence* of one error, we just need to make sure that each column of $H$ is nonzero (that is, not every entry in the column is zero). If we also want to *locate the position* of such an error, then we just need to make each column of $H$ different. Notice that since the only possible entries are 0 and 1, locating the position of an error is the same as being able to correct that error.

In our example $H$ has four rows and so there are $2^4 = 16$ possible columns to choose from, all but one of them being nonzero. Thus if we want to be able to correct all single errors, then $H$ must have at most 15 columns. This means that 4 check digits can be used to protect strings of up to $11 (= 15 - 4)$ information digits. If we use 4 check digits to protect exactly 11 digits then there is esentially only one possible marix $H$ (since swapping the columns of $H$ amounts to relabelling the original digits):

$$H = \left[\begin{array}{ccccccccccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Once $H$ is found we can write down the equations for the check digits $a_{12}, a_{13}, a_{14}, a_{15}$ in terms of the information digits $a_1, a_2, \ldots, a_{11}$.

From the point of view of hand-calculations (which is all I expect my students to do), correcting single errors is a simple procedure: in the above notation, you search among the columns of $H$ for one that looks like $He$. Of course, this is not very satisfactory if you intend to use a computer. The searching part of the algorithm can be sidestepped though, if you are willing to rearrange all 15 columns of the above matrix $H$. The idea is to use for the $i$th column of $H$ the binary representation of the number $i$, so that when $He$ is calculated it tells you directly which entry was wrong. (See the article by Levinson [2]).

The same code can also be used to detect double errors, but this time correction is not possible. The same calculation as before shows that with two errors $H(a + e) = He$ is a sum of two columns from $H$, and since all these columns are different, their sum (modulo 2, of course) must be nonzero. So the error is detected. But with our matrix $H$ the sum of any two columns is another column, since $H$ contains all possible nonzero columns. So this double error would be indistinguishable from some other single error.

This is as far as I take my students. After all, it is a course on linear algebra. In an abstract course, you could go further: to correct two or more errors you really need to construct finite fields of order $2^m$. My colleague, Kevin O'Meara, offers such a course to third-year students. My main aim is to whet their appetities by showing what can be achieved using a few simple ideas, and by making them aware that there is still more to be achieved.



# References

[1] Clark, G.C. and Cain, J.B. *Error-correction coding for digital communications*, Plenum, 1981 pp.1–7 and 49–62. Their illustration (on pages 59–61), using the problem of identifying one counterfeit coin among a collection of coins, may appeal to you if you wanted to incorporate these ideas in a discrete mathematics course. Another interesting introduction to the subject is offered by

[2] N. Levinson, *Coding Theory: a counterexample to G.H. Hardy's conception of applied mathematics*, Amer. Math. Monthly 77 (1970), 249–258. For a more advanced, but still very readable treatment, you could try:

[3] V. Pless, *Introduction to the theory of error-coding codes*, Wiley, 1982.

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# Cayley-Hamilton for Eigenvalues

Robin Harte

The Cayley-Hamilton theorem says that a linear operator $T: X \to X$ on a finite dimensional space $X \cong \mathbf{C}^n$ satisfies its *characteristic equation*:

$$p_T(T) = 0 \tag{1}$$

where

$$p_T(T) = (z-\lambda_1)^{\nu_1}(z-\lambda_2)^{\nu_2}\dots(z-\lambda_k)^{\nu_k} \tag{2}$$

is the *characteristic polynomial* of $T$; thus $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct *eigenvalues* of $T$ and $\nu_1, \nu_2, \dots, \nu_k$ are their (algebraic) *multiplicities*. It is familiar that, if the inverse $T^{-1}$ exists, then it can be expressed as a polynomial in $T$ with the help of (2); dividing across by the non-vanishing constant term of $p_T$ and bringing it across the equality sign gives

$$p_T'(T)T = I = Tp_T'(T) \tag{3}$$

This note arises from the problem of calculating the *eigenvectors* associated with the eigenvalues $\lambda_j$. In the process we rediscover a well-known theorem (which was obviously not well enough known to the author!).

Begin with the observation that (1) may well be valid for polynomials $p_T$ other than the characteristic polynomial: it is possible for (1) to hold with integers $\nu_j$ in (2) smaller than the full algebraic multiplicities. If $p_T$ is the polynomial given by (2) we shall write

$$\hat{p}_T(T) = (z-\lambda_1)(z-\lambda_2)\dots(z-\lambda_k), \tag{4}$$

and call this the *reduced* polynomial of $T$; then it may or may not happen that

$$\hat{p}_T(T) = 0\,. \tag{5}$$

If (5) holds we shall call the operator $T$ *reduced.* The well-known theorem [4, Chapter IV Theorem 5] is simply stated:

**Theorem** *If $T: X \to X$ is a linear operator on a finite dimensional space then*

$$T \text{ reduced} \iff T \text{ diagonal.} \tag{6}$$





Proof *Diagonal* means of course that T is a direct sum of scalars:

$$X = \sum_{j=1}^{k}(T-\lambda_j I)^{-1}(0) = \bigoplus_{j=1}^{k}(T-\lambda_j I)^{-1}(0)\,; \tag{7}$$

if this happens then (5) follows at once by considering $\hat{p}_T(T)$ separately on each eigenspace. For forward implication in (6) we need the notion of "exactness" [2, 3, Chapter 10]: the pair $(R,S)$ of operators on $X$ is called *exact* if

$$R^{-1}(0) = S(X)\,. \tag{8}$$

Inclusion one way is just the condition

$$RS = 0\,; \tag{9}$$

for the opposite inclusion it is sufficient that there are operators $S'$ and $R'$ on $X$ for which

$$R'R + SS' = I\,. \tag{10}$$

We have this, in particular, when $R$ and $S$ are polynomials in $T$ without common divisor: if

$$S = q(T) \quad \text{and} \quad R = r(T) \quad \text{with} \quad \gcd(q,r) = 1 \tag{11}$$

then the Euclidean algorithm for polynomials gives polynomials $q'(z)$ and $r'(z)$ for which

$$q'(z)q(z) + r'(z)r(z) = 1\,,$$

giving (10) with $S' = q'(T)$ and $R' = r'(T)$. This happens many times over if $T$ is reduced: if (5) holds then we get (11) with

$$S = q(T) = T - \lambda_j I \quad R = r(T) = \prod_{i \neq j}(T - \lambda_i I) \tag{12}$$

Further, in this case, everuthing in (9) and (10) commutes, so that also

$$S^{-1}(0) = R(X), \quad R^{-1}(0) \cap S^{-1}(0) = 0 \quad \text{and} \quad S(X) + R(X) = X\,; \tag{13}$$

thus

$$X = S^{-1}(0) \oplus R^{-1}(0)\,.$$

Forward implication in (6) is now induction on the number $k$ of distinct eigenvalues $\lambda_j$; for if $T: X \to X$ is diagonal on each of its invariant subspaces $S^{-1}(0)$ and $R^{-1}(0)$ then it is diagonal on their direct sum $S^{-1}(0) \oplus R^{-1}(0)$. On $S^{-1}(0)$ the operator $T$ coincides with the scalar $\lambda_j I$; on $R^{-1}(0)$ T has only $k-1$ eigenvalues.

This theorem is not new, and can be found for example in Jacobson [4]. We believe our direct deduction from the Euclidean algorithm has some charm; the same argument gives, with no assumptions about $T$, the "primary decomposition"

$$X = \sum_{j=1}^{k} (T - \lambda_j I)^{-\nu_j}(0) = \bigoplus_{j=1}^{k} (T - \lambda_j I)^{-\nu_j}(0).$$

An alternative version of the argument, passing through the medium of "Taylor invertibility", is given by Gonzalez [1].

When an operator $T: X \to X$ is "reduced" in the sense of (5) then its eigenvectors can all be obtained without solving any more equations: with $S = T - \lambda_j I$ and $R$ as in (12), the first part of (13) says that the eigenspace corresponding to $\lambda_j$ is the range or "column space" of the matrix $R$ built out of the remaining eigenvalues. Of course in practice it will usually be easier and pleasanter to solve the equations $Sx = 0$ than to compute the matrix $R$.

## References

[1] M. Gonzalez, *Null spaces and ranges of polynomials of operators*, Pub. Math. U. Barcelona **32** (1988), 167–170.

[2] R.E. Harte, *Almost exactness in normed spaces*, Proc. Amer. Math. Soc. **100** (1987), 257–265.

[3] R.E. Harte, *Invertibility and singularity*, Marcel Dekker, New York, 1988.

[4] N. Jacobson, *Lectures in abstract algebra II*, Van Nostrand, New York, 1953.

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# BOOK REVIEWS

METRIC SPACES: ITERATION AND APPLICATION
by Victor Bryant, Cambridge University Press (1985), STG£5.95 (paperback).

METRIC SPACES
by E.T. Copson, Cambridge Tracts in Mathematics number 57, Cambridge University Press (1968), STG£22.50 (hardback), STG£7.95 (paperback).

INTRODUCTION TO METRIC AND TOPOLOGICAL SPACES
by W.A. Sutherland, Oxford University Press (1981), STG£10.95 (paperback).

Of the three books, I like the one by Bryant the least. It claims, with some justification, to make the subject interesting. But the result is a book which might be more appropriately described as an introduction to iteration and fixed point theory that includes a little on metric spaces. To be somewhat objective, the book does touch on many of the basic concepts (limits of sequences; closed, complete, compact and connected sets). The applications include the existence and uniqueness of solutions for ordinary differential equations. But my basic objection is the second class treatment given to open sets and the less than enthusiastic treatment of continuity. On page 35, having introduced closed sets via limits of sequences, we are told that open sets are not really necessary because "all theorems about open sets can be stated in terms of closed sets". While this is undeniable, most textbooks do not take such an upside down view, and I do not consider that one can be said to have learned 'metric spaces' without being comfortable with the notion of open set. Who would like to volunteer to rewrite a standard text on multivariable analysis (never mind ones about complex analysis, functional analysis or elementary manifolds) mentioning only closed sets? The last chapter (marked optional) of Bryant's short book does make some amends by looking into continuity (even uniform continuity and the fact that the continuous image of a compact set is compact) and defining open sets.

This brings us to the question of what the book sets out to achieve. It claims to be intended for courses for engineering or 'combined honours' students, or really for those who have taken but not grasped a single variable analysis course. Perhaps it could be used as the introductory part of a course on numerical methods for the more mathematically mature engineers, but I think it would give a poor foundation for further study in analysis, differentiable manifolds or topology.

The other two books under review are much more serious books from the point of view of the honours mathematics program. The excuse for this review is that Copson is now available in paperback, but I find the book alarmingly old-fashioned in its approach — so much so that it must have been old-fashioned even when it first appeared in 1968. The book is written for those whose education was based on the classic 'Pure Mathematics' by Hardy and the first 20 of its 143 pages are devoted to background information including sets, set notation, equivalence relations and functions. Most of this introductory section (except possibly for some material on the axioms for the real numbers and sequences) is inappropriate now. The definition of a function is introduced gradually by recalling the notion of conformal mapping! Worse than that we are subjected to a further section on *Functions defined on an abstract set* over half way through the book.

By contrast I find Sutherland's first chapter *Review of some real analysis* to be written in the lively style which persists throughout the book, even though the chapter does really just rehash things that belong in a prerequisite course on analysis. Sutherland's style is more relaxed than Copson's throughout. When Copson gets around to the definition of a metric space, there is a surprising feature. He gives the 'wrong' definition! Well, of course it is not actually wrong, but decidedly unusual. Left out are the requirements that $\rho(x,y) = \rho(y,x)$ and $\rho(x,y) \geq 0$ — these are deduced from a slightly contorted version of the triangle inequality. I dislike also Copson's approach to examples. He leaves the examples till a few sections after the definition and starts with the discrete metric. Perhaps this was due to the effect of Bourbaki (who might have started with the empty metric space?).

Both Copson and Sutherland treat the examples of $\ell^p$ spaces early on, but I think they are misguided in never really treating them as normed spaces. In fact both of these books assume quite a degree of maturity on the part of the reader and reach more or less equal sophistication — the Baire category theorem, solutions of differential equations via the contraction mapping principle and the Arzelá-Ascoli theorem are treated. An early section on normed spaces would fit in well.

Copson is slightly more complete in some respects, but the main difference in content is that Sutherland launches into general topological spaces more



or less immediately after the definition of a metric space and the examples. Sutherland does use open sets and continuity as a springboard for general topology with the result that functions appear earlier than in Copson where completeness, connectedness and compactness are studied before functions appear. On balance, this is a reasonable place to contemplate topological spaces if one wants to do so in the course, but it may slightly affect those who only have time to cover basic metric space concepts.

What's lacking in these books? There is probably scope for more pictures. Copson has none at all, Sutherland has a few and Bryant has the most. None of the books considers algebraic topological ideas (like the fundamental group) although Sutherland shows us a trefoil knot as an example of a homeomorphic embedding of the circle in space and also deals with Möbius bands as quotient spaces of rectangles.

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AN INTRODUCTION TO HILBERT SPACE
by N. Young, Cambridge University Press.

In the very interesting "afterword" to "An introduction to Hilbert space", Nicholas Young quotes the following passage from G.H. Hardy's "A Mathematician's Apology":

> "I have never done anything useful. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."

This statement, which, as we shall see, is contradicted in this book, is interesting not so much for what it says about Hardy's attitude to mathematics; I am quoting it here at third hand and out of context. What is important about it, and other statements like it, is that they were interpreted in a particular way and had a profound influence on the teaching of mathematics in these islands. One of the consequences has been the traditional undergraduate textbook in pure mathematics which I call a grammar. The grammar launches without warning into a description of a given mathematical idiom, duly listing rules, declensions, exceptions and irregularities. It ends as abruptly as it has begun, leaving the unfortunate student with a strong feeling of kinship with the Scholars of Minoan Linear B who, having brilliantly deciphered that impenetrable script, found almost nothing to read in it.

The present book is in a very different tradition and is inspired by a totally different conception of the nature and role of mathematics. This is no grammar, but a literary work with a strong narrative line, inspired by two unifying themes. By the time one has reached the rather impressive conclusion, one is in no doubt that the subject has substance, that one is not dealing with an empty formal shell of theorems and corollaries but rather with a fascinating aspect of reality.

The book succeeds in giving a concise and lucid account of the elementary theory of Hilbert spaces. This is done very economically (the whole book is less than 240 pages long) and with modest technical equipment. For instance, Lebesgue integration is not assumed and, although it is mentioned from time to time, it is not essential to the understanding of the text. But there is much more. As already mentioned, there are two important unifying themes, that give the book a sense of purpose. Early on, at the end of a first chaper of only ten pages, we meet one of these, in the form of an inner product space of complex valued rational functions, analytic on the unit circle. We are shown an elegant connection between the inner product of two such functions and their poles in the unit disc.

For a while these functions drop out of the picture and the second theme is developed, in conjunction with the familiar material on orthonormal sets, Fourier series, functionals, duality and linear operators. This second theme is Sturm-Liouville systems and linear partial differential equations. Their description is interwoven with the general properties of Hilbert spaces and interest is kept high by the many examples, problems and exercises. Finally a very satisfying synthesis is achieved between orthonormal systems and solutions of second order partial differential equations.

The focus then returns to the first theme and the early example is expanded into the theory of Hardy spaces. In spite of Hardy's own passionate profession of uselessness, an interesting discursive chapter, an interlude in the author's own words, describes an application of these spaces to engineering problems of automatic control.

Multiplication operators are introduced and described clearly and economically, so that, within a matter of pages, one becomes familiar with Toeplitz and Hankel operators. The book closes with a series of very fine approximation theorems with a strong geometric flavour. We learn about best possible $L^\infty$-approximation of complex valued functions bounded on the unit circle by fuctions analytic in the unit disc (Nehari's problem); and of rational complex valued functions bounded on the unit circle by functions meromorphic in the unit disc (Adamyan-Arov-Krein theorem).



I hope that this book will set a trend in mathematical text books. One of the abiding difficulties of introducing students to advanced mathematical topics is to find an exposition which is complete and at the same time does justice to the subject. This book is an excellent example of how this can be done.

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# PROBLEM PAGE

## Editor: Phil Rippon

My first problem this time arose from a computer experiment in iteration theory. Over the last few years, one of the commomest purposes to which computers have been put has been the plotting of the Mandelbrot set, defined as follows.

First put

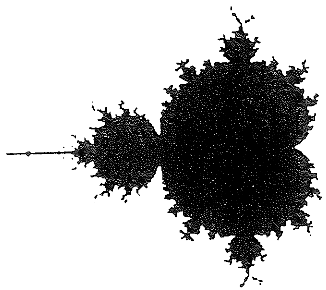
$$z_{n+1}(c) = z_n(c)^2 + c, \quad n = 0, 1, \ldots$$

where $c$ is a complex number and $z_0 = 0$. Thus $z_1(c) = c$, $z_2(c) = c^2 + c$, and so on. The Mandelbrot set is

$$M = \{ c : z_n(c) \not\to \infty \},$$

or equivalently, by an elementary argument,

$$M = \{ c : |z_n(c)| \leq 2, \quad n = 1, 2, \ldots \}.$$



M





To plot $M$ therefore, we calculate for each value of $c$ corresponding to a screen pixel, the sequence $z_n(c)$, $n = 1, 2, \ldots, N$ (where $N = 30$, say) and we plot the pixel if $|z_n(c)| \leq 2$, for $n = 1, 2, \ldots, N$. The familiar set $M$ appears below. Increasing the value of $N$ should in theory give a better approximation to $M$, but in practice there is an optimal $N$ depending on the screen resolution.

Recently, my colleagues David Grave, Robert Hassan and Peter Strain-Clark were using a transputer system to plot $M$ when they came upon an interesting relation of $M$, obtained by using the iteration formula

$$z_{n+1}(c) = \overline{z_n(c)}^2 + c, \quad n = 1, 2, \ldots \tag{1}$$

where $z_0(c) = 0$. This relation of the Mandelbrot set has a rather unexpected property.

**22.1** *Let $z_n(c)$ be defined by (1). Prove that the Mandelbar set*

$$M_{\text{bar}} = \{ c : |z_n(c)| \leq 2, \quad n = 1, 2, \ldots \}$$

*has rotational symmetry.*

The set $M_{\text{bar}}$ has many other intriguing properties; for example, it seems to contain many small copies of itself as well as small copies of the Mandelbrot set! Anyone who has a program for plotting the Mandelbrot set should be able to plot $M_{\text{bar}}$ by inserting a minus sign in the appropriate place.

Just one other problem this time. I've forgotten where I heard this, and would appreciate any reference to it.

**22.2** *Prove that it is impossible to tile the plane with triangles in such a way that at most 5 triangles meet at each vertex.*

Now to earlier problems. First a remark about problem **11.2** (in the new notation), which concerned sequences of the form

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots$$

If $a_0$, $a_1$, are real, then $a_n$ is periodic with period 9. It has now been proved, by Dov Aharanov and Uri Elias, that if $a_0$, $a_1$ are complex, then such sequences are always bounded (the proof looks very complicated and uses a theorem of Moser concerning 'twist mappings').

The first problem of Issue 20 was as follows:

**20.1** Find a formula whose value is 64, which uses the integer 4 twice, and no operations other than: $+$, $-$, $\times$, $/$, $\uparrow$, $\sqrt{\ }$ and !.

The answer is

$$64 = \left(\sqrt{\sqrt{\sqrt{4}}}\right)^{4!}$$

This problem is reminiscent of the "four fours" problem: how many of the integers can you express using the integer 4 four times, and the above operations? I remember wasting idle moments in my youth expressing all the integers from 1 to 100 in this way, but one of my OU students last year showed me a notebook he had completed some forty years ago which contained expressions for all integers up to 1000 and many beyond!

**20.2** Given positive integers $a_k$, $k = 1, 2, \ldots N$ (not necessarily distinct), prove that some sum of the form

$$a_{k_1} + a_{k_2} + \ldots + a_{k_m}, \quad 1 \le k_1 < k_2 < \ldots < k_m \le n$$

is equal to 0 mod $n$.

My colleague John Mason, who showed me this problem, calls it the "some sum" problem. There is a very neat solution, which shows that some sum of the form

$$a_p + a_{p+1} + \ldots + a_q, \quad 1 \le p \le q \le n \qquad (2)$$

is equal to 0 mod $n$. Indeed, if the $n$ integers

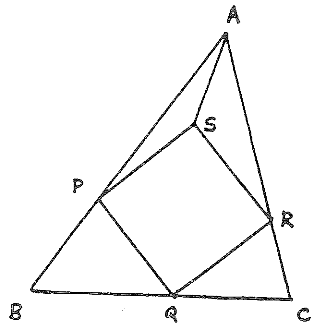
$$a_1 + a_2 + \ldots + a_m, \quad m = 1, 2, \ldots n$$

are distinct mod $n$, then one of them is equal to 0 mod $n$. Otherwise, two of them are equal mod $n$ and so their difference is equal to 0 mod $n$. Either way, we get a sum of the form (2).

**20.3** Show that if a square lies within a triangle, then its area is at most half the area of the triangle.

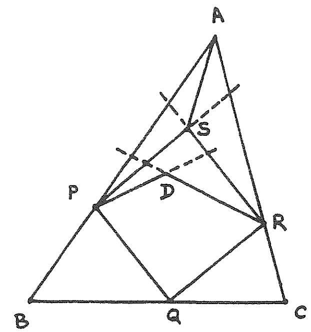
Here is a proof by paperfolding! We may assume, by scaling the triangle while keeping the square fixed, that at least three vertices of the square lie on the triangle.






We now claim that the reflections of the four triangles $APS$, $ARS$, $BPQ$, $CQR$, across their respective sides of the square $PQRS$ combine to completely cover the square.

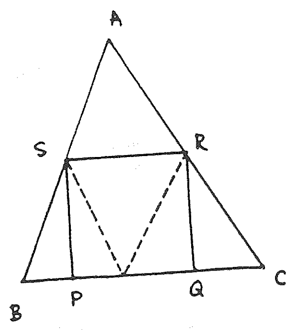
To see this, note that the angles $ASP$, $ASR$ are non-acute, so that the reflections of the triangles $ASP$ and $ASR$ together cover the quadrilateral $PSRD$.




It is then clear that the reflections of triangles $BPQ$, $CRQ$ across their respective sides of the square together cover the quadrilateral $PQRD$, since the reflection of $PB$ lies along $PD$, the reflection of $RC$ lies along $RD$, and the reflections of $QB$, $QC$ lie along a common line through $Q$.

In fact this proof assumes that angles $APS$, $ARS$ are both less than 45°. If one of them exceeds 45° (and at most one of them can), then a similar argument applies, but the point $D$ lies on the edge of the square.

Further consideration of this paperfolding approach shows that the case of equality occurs precisely when all four vertices lie on sides of the triangle and the side of the triangle which contains two vertices of the square is twice as long as the side of the square.

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