includes a nice treatment of the equivalence of norms on finite dimensional spaces.

Chapter 7 is "Operator Algebra and Commutativity". This contains a lot of basic functional analysis, for example the "Stone-Weierstrass Theorem". Chapter 8 is "Inner Products and Orthogonality". The title says it all— again basic functional analysis. Chapter 9 is "Liouville's Theorem and Spectral Theory". Another good basic chapter, including the beginnings of the theory of $C^*$-algebras and their representations.

Chapter 10 is "Comparison of Operators and Exactness". The introduction to this chapter states "The various kinds of invertibility have "relative" analogues, in which one operator is compared to another. If we mix both left and right comparisons and then specialize we come down to concepts of "exactness"". Enough said.

Chapter 11 is "Multiparameter Spectral Theory". This contains the Taylor spectrum, an idea toward which much of the book seems aimed. There is also useful material on the Silov boundary and Tensor Products.

A final section is a collection of "Notes, Comments, and Exercises" for each chapter. It is clear from these that the author has researched his subject with diligence and thoroughness, and this section adds greatly to the value of the book as a compendium of results. In fact one is tempted to suggest that the book might have been called "everything you ever wanted to know about Spectral Theory, but were afraid to ask—in case you were told". The one thing that could be found missing is some mention of the many areas in which Spectral Theory finds its applications, and which provide it's Raison d'Etre.

Much of the above comment may seem negative, so let me hasten to add that this is a book that I am glad to have on my shelves. It has appeal on three levels. Firstly, the standard introductory results of functional analysis and operator theory are all there. Secondly, it collects many of the more esoteric notions of Spectral Theory, and finally it contains the authors own ruminations on completeness and the lack of it.

*Donal O'Donovan*
*School of Mathematics*
*Trinity College Dublin.*

---

# PROBLEM PAGE

## Editor: Phil Rippon

Here are a couple of attractive problems which fall into the category of 'geometric doodling'. The first one appears in Coxeter's book 'An Introduction to Geometry', but I heard it first from my school maths teacher.

21.1 *What is the minimum number of (strictly) acute angled triangles into which a square can be partitioned?*

The next problem was asked recently by an OU maths student at summer school. It has a very neat solution and I'd be interested to hear of any references to it.

21.2 *Find a configuration of finitely many points in the plane such that the perpendicular bisector of each pair of the points passes through at least two of the points.*

Finally, a wonderful sequence problem due to John Conway, who offered a prize of \$1000 in July this year for a solution (his audience thought he had offered \$10,000!).

21.3 *Let $a(n)$, $n = 1, 2, \ldots$, be defined as follows: $a(1) = 1$, $a(2) = 1$, and*

$$a(n+1) = a(a(n)) + a(n+1-a(n)), \qquad n = 2, 3, 4, \ldots$$

*Thus the sequence begins:*

$$1, 1, 2, 2, 3, 4, 4, 4, \cdots.$$

*The problem is to determine an integer $N$ such that*

$$\left|\frac{a(n)}{n} - \frac{1}{2}\right| < 0.05 \qquad \text{for } n > N.$$

A solution was given three weeks later by Colin Mallow, a mathematician working at Bell Labs. In September, a British newspaper offered a magnum of champagne for a solution and awarded two prizes a week later, one of which went to a pupil from St. Paul's School, London. The Problem Page has no prizes to offer, but anyone who takes a closer look at this sequence will be amazed by its properties!

Next, here are the solutions to the problems which appeared in December 1987.

**19.1** *To each vertex of a regular pentagon, an integer is assigned in such a way that the sum of all five integers is positive. If three consecutive vertices are assigned the numbers $x$, $y$, $z$ respectively and $y < 0$, then the following operation is allowed: the numbers $x$, $y$, $z$ are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five integers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.*

I am grateful to Tom Laffey for sending me the background to this problem, which appeared in the International Mathematical Olympiad at Warsaw in 1986, including a booklet on the Olympiad published by the Australian Mathematics Competition organisation. The U.S. student Joseph Keane was awarded a special prize for his solution, which begins as follows:
"As with so many problems of this type the key to a solution is the discovery of a function whose value decreases when the given operation is performed but which is always a whole number."

Keane goes on to show that if the five integers are $v$, $w$, $x$, $y$, $z$, then the expression:

$$\begin{aligned}&|v|+|w|+|x|+|y|+|z|+\\&|v+w|+|w+x|+|x+y|+|y+z|+|z+v|+\\&|v+w+x|+|w+x+y|+|x+y+z|+|y+z+v|+|z+v+w|+\\&|v+w+x+y|+|w+x+y+z|+|x+y+z+v|+\\&|y+z+v+w|+|z+v+w+x|\end{aligned}$$

is decreased by $|s-y|-|s+y|$, where $s=v+w+x+y+z$, when the operation is applied to the triple $x$,$y$,$z$. Since $s>0$ and $y<0$, this expression has the required property, showing that the operation can be performed only a finite number of times.

A similar solution, provided by the proposer of the problem, uses the quadratic



expression

$$\frac{1}{2}\left[(v-x)^2+(w-y)^2+(x-z)^2+(y-v)^2+(z-w)^2\right],$$

which decreases by $-sy$ when the operation is applied to $x$,$y$,$z$. My own solution uses the expression

$$\begin{aligned}&v^2+w^2+x^2+y^2+z^2+\\&(v+w)^2+(w+x)^2+(x+y)^2+(y+z)^2+(z+w)^2+\\&(v+w+x)^2+(w+x+y)^2+(x+y+z)^2+(y+z+v)^2+(z+v+w)^2\end{aligned}$$

which decreases by $-sy$ also.

This approach to the problem generalises to $n$ integers placed at each vertex of a regular n–gon, as does a remarkable alternative solution due to J.M. Campbell of Canberra, which proves in addition that the final configuration is independent of the order in which the operations are performed.

To explain this solution, we let $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ denote the integers in order around the pentagon, and $\sigma_j$ denote the operation which reverses the sign of $a_j$ and adds $a_j$ to its pentagon neighbours. The operation $\sigma_j$ has a very simple effect on the sequence of progressive sums:

$$\ldots,-a_4-a_5,-a_5,0,a_1,a_1+a_2,a_1+a_2+a_3,\ldots$$

defined by $s_0=0$, and $s_i=s_{i-1}+a_i$, $i\in\mathbb{Z}$, where $a_i$ is defined by taking the subscript mod 5. Indeed, one easily checks that $\sigma_j$ swaps any two consecutive sums of the form $s_{j-1}$, $s_j$ (taking $j$ mod 5) and leaves all other $s_i$ unchanged. Also, the operation $\sigma_j$ is allowed, that is $a_j<0$, if and only if $s_j>s_{j+1}$, and so the operations $\sigma_j$ can be performed until the sequence $s_i$ is sorted into increasing order.

The fact that $s_i$ is in increasing order after only finitely many operations follows from the observation that

$$s_{i+5}=s_i+s,\qquad i\in\mathbb{Z},$$

where $s=a_1+a_2+a_3+a_4+a_5>0$. This implies that the number $N_i$ of terms less than $s_i$, which lie to the right of $s_i$, is finite. Note that $N_{i+5}=N_i$, $i\in\mathbb{Z}$, and also that, in sorting the sequence, the term $s_i$ moves precisely

$N_i$ places to the right. Hence the total number of operations required is $N_1 + N_2 + N_3 + N_4 + N_5$ and the final configuration is indeed independent of the order in which the operations $\sigma_j$ are performed. Moreover, Campbell is able to give an explicit formula for the $N_i$ and for the final numbers around the pentagon.

The next problem is known in Maths Education circles as the Krutetskii Problem (see page 150 of 'The Psychology of Mathematical Abilities in Schoolchildren', by V.A. Krutetskii, The University of Chicago Press) and is attributed to Lovász by Ross Honsberger in an article in 'The Mathematical Gardner' (a volume dedicated to Martin Gardner). It is followed by an intriguing variant proposed by my colleague John Mason.

19.2 *A finite number of petrol dumps are arranged around a racetrack. The dumps are not necessarily equally spaced and nor do they necessarily contain equal volumes of petrol. However, the total volume of petrol is sufficient for a car to make one circuit of the track. Show that the car can be placed, with an empty tank, at some dump so that, by picking up petrol as it goes, it can complete one full circuit.*

19.3 *The petrol dumps are arranged as in* **19.2**, *but this time the total volume of petrol is sufficient for two circuits of the track. Can two cars be placed with empty tanks at the same dump so that, by picking up petrol as they go, they can each complete one full circuit in opposite directions? (The cars may cooperate in sharing petrol from the dumps.)*

There are various ways to solve **19.2**, but the following approach has the advantage that it can be used to solve **19.3**. Indeed, I came across it while working on **19.3**.

Suppose that $p_k$, $k = 1, 2, \ldots, n$, denote the volumes of petrol at the dumps $D_k$, $k = 1, 2, \ldots, n$, in order anticlockwise around the circuit, and that $d_k$, $k = 1, 2, \ldots, n$, denote the distances from $D_k$ to $D_{k+1}$. If $p_k$ and $d_k$ are measured in comparable units, say gallons, then the assumption is that:

$$p_1 + p_2 + \ldots + p_n \geq d_1 + d_2 + \ldots + d_n. \qquad (1)$$

Let us take as inductive hypothesis the statement that a solution is always possible with $n$ dumps. Certainly, this is true for $n = 1$. Now consider



$n + 1$ dumps $D_1, \ldots, D_{n+1}$ with associated petrol $p_1, \ldots, p_{n+1}$ and distances $d_1, \ldots, d_{n+1}$, such that

$$p_1 + p_2 + \ldots + p_{n+1} \geq d_1 + d_2 + \ldots + d_{n+1}.$$

If $p_i \geq d_i$, for each $i$, then the car can start from any dump and complete the circuit. Otherwise $p_i < d_i$, for some $i$, and we consider a new configuration with $n$ dumps, in which $D_i$ is removed and the petrol $p_i$ is added to $D_{i-1}$. Since (1) holds for this configuration, a solution is certainly possible, by the inductive hypothesis.

Following this solution, the car leaves $D_{i-1}$ with an unknown volume $P$ of petrol such that

$$P \geq d_{i-1} + d_i.$$

Since $p_i < d_i$, we deduce that $P - p_i > d_{i-1}$. It follows that the car could have completed the original $n+1$ dump circuit, with the same starting point, by picking up $p_i$ gallons of petrol at $D_i$ instead of at $D_{i-1}$. Thus we have a proof by induction.

The set up is similar in **19.3** with dumps $D_1$, $D_2$, ..., $D_n$, distances $d_1$, $d_2$, ..., $d_n$ and petrol $p_1$, $p_2$, ..., $p_n$, but now we assume that

$$p_1 + p_2 + \ldots + p_n \geq 2(d_1 + d_2 + \ldots + d_n). \qquad (2)$$

Once again the inductive hypothesis is that a solution is always possible with $n$ dumps, and this clearly holds for $n = 1$. Now consider $n + 1$ dumps $D_1, \cdots, D_{n+1}$ with associated petrol $p_1, \ldots, p_{n+1}$ and distances $d_1, \ldots, d_{n+1}$, such that

$$p_1 + p_2 + \ldots + p_{n+1} \geq 2(d_1 + d_2 + \ldots + d_{n+1}).$$

If $p_i \geq d_{i-1} + d_i$, for each $i$, then the two cars can begin at any dump and complete opposite circuits by picking up at each dump exactly enough petrol to reach the next dump. Otherwise $p_i < d_{i-1} + d_i$, for some $i$, so that

$$p_i = q_{i-1} + q_i, \qquad \text{where } 0 < q_{i-1} < d_{i-1}, \quad 0 < q_i < d_i,$$

and we consider a new configuration with $n$ dumps, in which $D_i$ is removed and the petrol $q_{i-1}$, $q_i$ is added to $D_{i+1}$, $D_{i-1}$, respectively. Since (2) holds for this configuration, a solution is certainly possible, by the inductive hypothesis.

Following this solution, the 'anticlockwise' car leaves $D_{i-1}$ with an unknown volume $P$ of petrol such that

$$P \geq d_{i-1} + d_i.$$

Since $q_i < d_i$, we deduce that $P - q_i > d_{i-1}$. It follows that the 'anticlockwise' car could have completed the original $n+1$ dump circuit, with the same starting point and without altering the petrol rations of the 'clockwise' car, by picking up $q_i$ gallons of petrol at $D_i$ instead of at $D_{i-1}$. Since a similar argument applies to the clockwise car, we again have a proof by induction.

**Remark** Problems **19.2** and **19.3** are in fact special cases of a more general problem in which the two cars have different rates of petrol consumption. The above argument needs only slight modification to deal with this more general problem.

*Phil Rippon*
*Faculty of Mathematics*
*Open University*
*Milton Keynes MK7 6AA, UK*

## Second Dublin Differential Equations Meeting

The Second Dublin Differential Equations Meeting will be held in NIHE Dublin on May 22–25, 1989. Invited speakers include S.S. Antman (Maryland), J. Carr (Heriot-Watt), W.N. Everitt (Birmingham), J.K. Hale (Georgia Tech.), R.E. O'Malley (Rennselaer) and V. Moncrief (Yale). The programme will include sessions of contributed talks and workshops on both theory and applications. Possible subjects for workshops include bifurcation theory, singular perturbations and gelation. Financial support has been received from the Irish and London Mathematical Societies. Those interested in participating are invited to write to Dr. D.W. Reynolds, School of Mathematical Sciences, N.I.H.E., Dublin 9.

# INSTRUCTIONS TO AUTHORS

Authors may submit articles to the Bulletin either as TEX input files, or as typewritten manuscripts. Handwritten manuscripts are not acceptable.

## Manuscripts prepared with TEX

The Bulletin is typeset with TEX, and authors who have access to TEX are encouraged to submit articles in the form of TEX input files. Plain TEX, AMS-TEX and LATEX are equally acceptable. The TEX file should be accompanied by any non-standard style files which have been used.

The input files can be transmitted to the Editor either on IBM or Macintosh diskette, or by electronic mail to the following Bitnet or EARN address:

MATRYAN@CS8700.UCG.IE

Two printed copies of the article should also be sent to the Editor.

Authors who prepare their articles with word processors other than TEX can expedite the typesetting of their articles by submitting the input file in the same way, along with the printed copies of the article. The file should be sent as an ASCII file.

## Typed Manuscripts

Typed manuscripts should be double-spaced, with wide margins, on numbered pages. Commencement of paragraphs should be clearly indicated. Handwritten symbols should be clear and unambiguous. Ilustrations should be carefully prepared on separate sheets in black ink. Two copies of each illustration should be submitted: one with lettering added, and the other without lettering. Two copies of the manuscript should be sent to the Editor.