data type methods of the first sections with the machine based methods of the latter ones. In situations when architectural features of the system are important, these can be incorporated into the X-machine by defining the set X suitably, perhaps including models of registers etc.

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Crossed Modules

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Crossed modules were invented almost 40 years ago by J.H.C.Whitehead in his work on combinatorial homotopy theory [W]. They have since found important roles in many areas of mathematics (including homotopy theory, homology and cohomology of groups, algebraic K-theory, cyclic homology, combinatorial group theory, and differential geometry). Possibly crossed modules should now be considered one of the fundamental algebraic structures. In this article we give an account of some of the main occurrences and uses of crossed modules and we describe some recent developments in their theory.

Before presenting the definition of a crossed module, we shall consider several motivating examples. Throughout G denotes an arbitrary group.

Example 1 Let N be a normal subgroup of G. The inclusion homomorphism $N \to G$ together with the action ${}^g n = gng^{-1}$ of G on N is a crossed module.

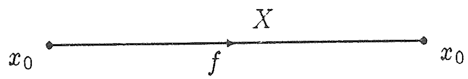
Example 2 If M is a $\mathbb{Z}G$-module then the trivial homomorphism $M \to G$ which maps everything to the identity is a crossed module.

Example 3 Let $\partial\colon H \to G$ be a surjective group homomorphism whose kernel lies in the centre of H. There is an action ${}^g h = \tilde{g}h\tilde{g}^{-1}$ of G on H where $\tilde{g}$ denotes any element in $\partial^{-1}(g)$. The homomorphism ∂ together with this action is a crossed module.

Example 4 Suppose that G is the group $Aut(K)$ of automorphisms of some group K. Then the homomorphism $K \to G$ which sends an element $x \in K$ to the inner automorphism $K \to K$, $k \mapsto xkx^{-1}$ is a crossed module.

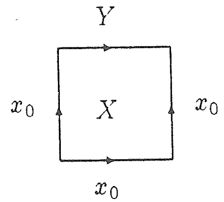
Each of these examples consists of a group homomorphism with an action of the target group on the source group. Before stating the precise algebraic properties needed by such a homomorphism for it to be a crossed module, let us consider some more substantial examples.

Example 5 Let X be a topological space in which a point x_0 has been chosen. Recall that the fundamental group $\pi_1(X, x_0)$ consists of homotopy classes of continuous maps $f : [0,1] \rightarrow X$ with $f(0) = f(1) = x_0$. (Two such maps are *homotopic* if one can be continuously deformed into the other in such a way that the image of 0 and 1 remains x_0 throughout the deformation.) We think of these maps as paths in X beginning and ending at x_0; the appropriate picture is

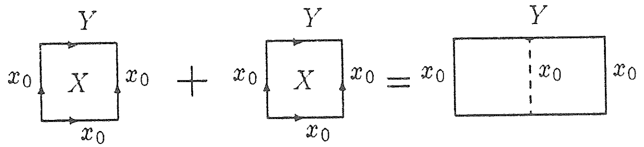


Composition of paths yields a (not necessarily abelian) group structure on $\pi_1(X, x_0)$.

Now if Y is a subspace of X containing the point x_0 then we can consider the second relative homotopy group $\pi_2(X, Y, x_0)$. This group consists of homotopy classes of continuous maps $g : [0,1] \times [0,1] \rightarrow X$ from the unit square into X which map three edges of the square onto the point x_0 and the fourth edge into Y. The appropriate picture of such a map g is



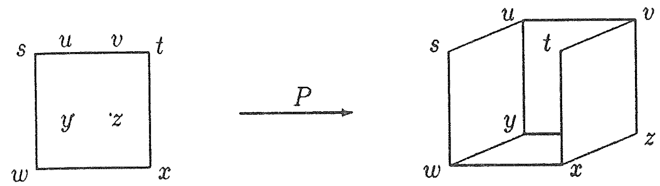
Juxtaposition of squares



yields a (not necessarily abelian) group structure on $\pi_2(X, Y, x_0)$.

By restricting to the fourth edge of the unit square we obtain a boundary homomorphism $\partial : \pi_2(X, Y, x_0) \rightarrow \pi_1(Y, x_0)$. Moreover there is an action of

$\pi_1(Y, x_0)$ on $\pi_2(X, Y, x_0)$ which makes the boundary homomorphism into a crossed module. To understand this action note that there is a continuous map



from the unit square onto four faces of the unit cube which sends s to s, t to t and so on. Now given a path $f : [0,1] \rightarrow Y$ representing an element of of $\pi_1(Y, x_0)$, and a square $g : [0,1] \times [0,1] \rightarrow X$ representing an element of $\pi_2(X, Y, x_0)$, we can construct a continuous map $^f g$ from the four faces of the unit cube to the space X by using g to map the face $uvyz$, and mapping each horizontal line in the remaining three faces by f. On composing $^f g$ with p we get a map which represents an element of $\pi_2(X, Y, x_0)$. It can be checked that the assignment $(f, g) \mapsto {}^f g \circ p$ induces an action of $\pi_1(Y, x_0)$ on $\pi_2(X, Y, x_0)$.

Example 6 Let M and N be normal subgroups of G. A non-abelian tensor product $M \otimes N$ has been introduced by R. Brown and J.-L. Loday [B-L]; it is the group generated by the symbols $m \otimes n$ ($m \in M$ and $n \in N$) subject to the relations

$$\begin{aligned} mm' \otimes n &= (mm'm^{-1} \otimes mn'm^{-1})(m \otimes n) \\ m \otimes nn' &= (m \otimes n)(nmn^{-1} \otimes nn'n^{-1}) \end{aligned}$$

for all $m, m' \in M$ and $n, n' \in N$. In general $M \otimes N$ is a non-abelian group. If however conjugation in G by an element of M (resp. N) leaves all the elements of N (resp. M) fixed then $M \otimes N$ is precisely the usual abelian tensor product of abelianised groups $M/M' \otimes N/N'$. For any normal subgroups M and N there is a homomorphism $\partial : M \otimes N \rightarrow G$ defined on generators by $\partial(m \otimes n) = mnm^{-1}n^{-1}$. There is also an action of G on $M \otimes N$ defined on generators by $^g(m \otimes n) = (gmg^{-1} \otimes gng^{-1})$. This homomorphism and action is a crossed module.

Example 7 Let Λ be an associative ring with identity, let $GL(\Lambda)$ be the general linear group, and let $E(\Lambda)$ be the subgroup of $GL(\Lambda)$ generated by the elementary matrices $e_{ij}(\lambda)$ with $i \neq j$ and $\lambda \in \Lambda$ (recall that $e_{ij}(\lambda)$ has 1's on the diagonal, λ in the (i,j) position, and 0 elsewhere). The group $E(\Lambda)$ is a normal subgroup of $GL(\Lambda)$, and the non-abelian tensor square $E(\Lambda) \otimes E(\Lambda)$ is known as the Steinberg group and denoted $St(\Lambda)$. (This definition of the Steinberg group is equivalent to the usual definition [B-L].) As a special case of Example 6 we have a crossed module $\partial : St(\Lambda) \to GL(\Lambda)$. It can be shown that $\partial(St(\Lambda)) = E(\Lambda)$. The groups $K_1(\Lambda) = \mathrm{Coker}(\partial)$ and $K_2(\Lambda) = \mathrm{Ker}(\partial)$ are known as the first and second algebraic K-theory groups of Λ.

The essential features of these examples are captured in the following definition.

Definition A *crossed module* consists of a group homomorphism $\partial : C \to G$ together with an action of G on C such that

(i) $\partial({}^g c) = g(\partial c)g^{-1}$,

(ii) ${}^{\partial c}c' = cc'c^{-1}$,

for all $c, c' \in C$ and $g \in G$.

If $\partial : C \to G$ and $\partial' : C' \to G'$ are crossed modules, then we say that a pair of homomorphisms $\varphi : C \to C$, $\psi : G \to G$ is a *morphism* of crossed modules if $\psi(\partial(c)) = \partial(\varphi(c))$ and $\varphi({}^g c) = {}^{\psi(g)}\varphi(c)$ for all $c \in C$ and $g \in G$.

An easy consequence of this definition is that for any crossed module $\partial : C \to G$ the group $\partial(C)$ is a normal subgroup of G; the quotient $G/\partial(C)$ is denoted by $\pi_1(\partial)$. Also it is easily checked that the action of G on C induces an action of $\pi_1(\partial)$ on $\mathrm{Ker}(\partial)$, and that $\mathrm{Ker}(\partial)$ is abelian; we denote the $\pi_1(\partial)$-module $\mathrm{Ker}(\partial)$ by $\pi_2(\partial)$.

In all algebraic theories the notion of a free object is important. For a crossed module $\partial : C \to G$ the notion of "freeness" is made precise by saying that ∂ is *free* on a function $\eta : W \to G$ from some set W into G if:

(i) W is a subset of C;

(ii) η is the restriction of ∂ ;

(iii) for any crossed module $\partial' : C \to G$, if $\nu : W \to C'$ is a function satisfying $\partial'\nu = \partial$, then ν induces a unique morphism $\varphi : C \to C'$, $\psi : G \to G$ of crossed modules with ψ the identity homomorphism.

Bearing Example 2 in mind, it is readily seen that free $\mathbb{Z}G$-modules are one instance of free crossed modules.

Another instance of free crossed modules arises from Example 5. For suppose that the space X can be constructed by choosing a point x_0 in X, then attaching copies of the unit interval $[0,1]$ to x_0 by gluing the end points 0 and 1 of each copy to x_0, and then finally attaching copies of the unit square $[0,1] \times [0,1]$ by gluing the edges of each copy along the various copies of the unit interval in some fashion. In other words, suppose that X is a ***reduced 2-dimensional CW-space***. The copies of the unit interval in X are called ***1-cells***, and the copies of the unit square are called ***2-cells***. Let Y be the subspace of X consisting of the 1-cells; in the jargon, Y is the ***1-skeleton*** of X. It was shown by J.H.C.Whitehead [W] that in this situation the boundary homomorphism $\partial : \pi_2(X, Y, x_0) \to \pi_1(Y, x_0)$ is a free crossed module. It is free on the function

$$\{\text{2-cells of } X\} \longrightarrow \pi_1(Y, x_0)$$

which sends each 2-cell to the element of $\pi_1(Y, x_0)$ represented by the boundary of the 2-cell.

To illustrate the above, suppose that X is the torus. Now the torus can be constructed by gluing together two 1-cells and one 2-cell. In this case we take Y to be the union of the two circles. Thus $\pi_1(Y) = F(a,b)$ is the free group on two elements a, b. The crossed module $\partial : \pi_2(X, Y, x_0) \to \pi_1(Y, x_0)$ is free on the function $\{w\} \to F(a,b)$, $w \mapsto aba^{-1}b^{-1}$.

Whitehead showed that the homotopy theoretic information contained in 2- dimensional reduced CW-spaces is completely captured in the algebra of free crossed modules. More precisely he showed that if X and X' are 2-dimensional reduced CW-spaces with Y and Y' their respective 1- skeleta, then the set of homotopy classes of continuous maps from X to X' is bijective with the set of (appropriately defined) homotopy classes of crossed module morphisms from $\pi_2(X, Y, x_0) \to \pi_1(Y, x_0)$ to $\pi_2(X', Y', x_0) \to \pi_1(Y', x_0)$. Using this bijection certain homotopy theoretic problems (such as the enumeration of the homotopy classes of maps from a compact connected closed surface to the projective plane) can be solved purely algebraically (cf. [E]).

The idea of studying 2-dimensional CW-spaces by means of their associated free crossed modules has applications to combinatorial group theory. Any presentation $< V : R >$ of the group G gives rise to a reduced CW-space with one 1-cell for each generator $v \in V$ and one 2-cell for each relator $r \in R$. The associated crossed module $\partial : C \to F(V)$ is free on the inclusion function

$R \hookrightarrow F(V)$ where $F(V)$ is the free group on V. Clearly $\pi_1(\partial)$ is isomorphic to G. The G-module $\pi_2(\partial)$ is known as *the module of identities*, and is a measure of the "non-trivial identities among the relations." A good introduction to this area can be found in [B-Hu].

A rather more algebraic use of free crossed modules is to do with the homology of groups. For suppose that $\partial : C \to G$ is a free crossed module, and let H denote the image of ∂ in G. It can be shown [E-P] that the commutator subgroup $[C, C]$ of C depends only on H. (In fact $[C, C]$ is isomorphic to the quotient of the non-abelian tensor product $H \otimes H$ by the subgroup generated by the elements $h \otimes h$ $(h \in H)$.) Moreover the intersection $[C, C] \cap \mathrm{Ker}(\partial)$ is isomorphic to $\mathrm{H}_2(H, \mathbb{Z})$, the second integral homology (or Schur multiplier) of H.

Crossed modules also have a role in the cohomology of groups. It has long been known that the second cohomology group $\mathrm{H}^2(G, A)$ of G with coefficients in a G-module A is bijective with the set of isomorphism classes of extensions of G by A. (Recall that a pair of group homomorphisms

$$A \xrightarrow{i} E \xrightarrow{p} G$$

is an *extension* of G by A if p is surjective, i is injective, $\mathrm{Im}(i) = \mathrm{Ker}(p)$, and the module action of $g \in G$ on $a \in A$ corresponds to conjugating by some $\tilde{g} \in p^{-1}(g)$.) In the mid 1970's various people (see [ML] for an incomplete list of references) discovered an analogous interpretation of the third cohomology group $\mathrm{H}^3(G, A)$ in terms of crossed extensions of G by A: a sequence of homomorphisms

$$A \xrightarrow{i} C \xrightarrow{\partial} N \xrightarrow{p} G$$

is a *crossed extension* of G by A if p is surjective, i is injective, $\mathrm{Im}(i) = \mathrm{Ker}(\partial)$, $\mathrm{Im}(\partial) = \mathrm{Ker}(p)$, and ∂ is a crossed module such that the resulting action of $N/\partial(C)$ on A corresponds to the module action of G on A. This interpretation has been used by Huebschmann [Hu] to obtain some new exact sequences in the cohomology of groups. In his recent book K.MacKenzie [MK] notes that the interpretation of $\mathrm{H}^3(G, A)$ carries over to the case of Lie groups and smooth morphisms. This leads him (via a more general result about Lie groupoids) to a reformulation of the Čech classification of principal bundles which works entirely in terms of abelian Čech cohomology.

One of the most fruitful areas in the theory of crossed modules stems from work by R. Brown and P.J. Higgins [B-Hi] on generalising to higher dimensions Van Kampen's famous theorem about the fundamental group of a space. This theorem states that if a space X is the union of pathwise connected open subspaces U and V such that the intersection $U \cap V$ is pathwise connected and contains a point x_0 , then the fundamental group $\pi_1(X, x_0)$ is isomorphic to the amalgamated sum

$$\pi_1(U, x_0) \underset{\pi_1(U \cap V, x_0)}{*} \pi_1(V, x_0) \; ;$$

in other words the fundamental group construction preserves certain amalgamated sums. It has been shown by Brown and Higgins [B-Hi] that the crossed module construction on pairs of spaces given in Example 5 also preserves certain amalgamated sums. This new "2-dimensional Van Kampen theorem" is a useful tool in algebraic topology, and has led to several new results. Perhaps more importantly it has led to a successful search for an algebraic structure which will model n-dimensional homotopy theoretic phenomena, and which will satisfy some sort of Van Kampen theorem.

It has long been known that a crossed module $\partial : C \to G$ is equivalent to a set Q which possesses both a group structure and the structure of a category, the group multiplication being compatible with the category composition $\circ$ in the sense that $(x \circ y)(x' \circ y') = xx' \circ yy'$ for all $x, x', y, y' \in Q$ such that the left hand side of the equation is defined. As a group, Q is the semi-direct product $C \times G$. The category composition on Q is defined for those pairs of elements (c, g) and (c', g') satisfying $g' = \partial(c)g$, and is given by $(c', g') \circ (c, g) = (c'c, g)$. In [L] J.-L. Loday used this description of a crossed module to show that crossed modules are equivalent "up to homotopy" to connected CW-spaces X whose homotopy groups $\pi_i(X, x_0)$ are trivial for $i > 2$. He went further and showed that groups possessing n compatible category structures, which we now call *cat^n-groups*, are equivalent "up to homotopy" to connected CW-spaces X with $\pi_i(X, x_0) = 0$ for $i > n + 1$. His method was to assign to each space X a space W containing n subspaces $U_1, \ldots, U_n \subseteq W$, and then to construct from the $(n+1)$-tuple $(W, U_1, \ldots, U_n)$ a cat^n-group.

It has since been shown [B-L] that this construction of a cat^n-group from an $(n+1)$-tuple of spaces satisfies a Van Kampen type theorem (that is, it preserves certain amalgamated sums). The technicalities involved in using this n-dimensional Van Kampen theorem have lead to some interesting algebraic problems, such as the computation of amalgamated products of cat^n-groups. In [GW-L] it was shown that algebraic problems about cat^2-groups are often better reformulated using a non-trivial equivalence between cat^2-groups and algebraic structures known as *crossed squares*. (Intuitively a crossed square is a crossed module in the category of crossed modules. Thus it consists of a morphism of crossed modules together with an "action" of the target crossed module on the source crossed module, and certain algebraic conditions are satisfied.) More generally in [E-S] the notion of a *crossed n-cube* was introduced and shown to be equivalent to a cat^n-group. Since the publication of [L] in 1982 over 55 articles have been published on the subject of cat*n*-groups; a fairly comprehensive bibliography can be found in [B].

Finally we should mention that by imitating in other algebraic settings the equivalence between cat^1-groups and crossed modules, one arrives at the notion of a *crossed module* in these settings. Crossed modules of Lie algebras turn out to be useful in studying the cyclic homology of an associative algebra [K-L]. Crossed modules of commutative rings are useful in studying the Koszul Complex [P]. And many of the (topologically motivated) results on crossed modules of groups, such as the description of group cohomology, carry over to these other settings.

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