

Finally, if M is singular, no conclusions can be drawn concerning the nature of the stationary point a .

We conclude with some examples.

Example 1 A sufficient condition for $f(x, y)$ to have a minimum at a stationary point (a, b) subject to a side condition parametrised by $x = \varphi_1(t)$, $y = \varphi_2(t)$ is

$$\begin{pmatrix} \varphi'_1 & \varphi'_2 \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix} + \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} \varphi''_1 \\ \varphi''_2 \end{pmatrix} > 0$$

Example 2 Let $f(x, y) = 1 - 2xy$. Then f has a maximum at $(0, 0)$ subject to $y - x^3 = 0$, but has neither a maximum nor a minimum at $(0, 0)$ subject to $y - x^2 = 0$. In both cases we have $M = 0$.

Example 3 The function $f(x, y, z) = 1 - 2xy - 2xz - 2yz$ has a stationary point at $(0, 0, 0)$. Parametrising the side condition $y = z$ by $\varphi(r, s) = (r, s, s)$, we find that

$$M = \begin{pmatrix} 0 & -2 \\ -2 & -2 \end{pmatrix}$$

which is indefinite. $f(x, y, z)$ has a maximum at $(0, 0, 0)$ subject to $x = y = z$, but has a minimum at $(0, 0, 0)$ subject to $-x = y = z$. The point $(0, 0, 0)$ is a saddle point subject to $y = z$.

Department of Mathematics
University College Galway

A Note on Integrating Composed Functions

Paul Barry

This note groups together several concepts that are met at different places in a first course on real analysis in a way that allows graphical representation. It provides a generalisation of the formula (see [1]):

$$\int_{f(a)}^{f(b)} f^{-1}(y) dy + \int_a^b f(x) dx = bf(b) - af(a) \quad (1)$$

which has a certain pedigree—see [2], [3] and particularly [4], where a proof is given in the case where f and f^{-1} are assumed only to be integrable.

We shall use the (Riemann-)Stieltjes integral as given, for instance, in [7]. We deal only with definite integrals.

We begin by recalling the formula for integration by parts for the Stieltjes integral. Let $u, v : [c, d] \rightarrow \mathbb{R}$, and assume the integral $\int_c^d u dv$ exists. Then

$$\int_c^d u dv + \int_c^d v du = u(d)v(d) - u(c)v(c) \quad (2)$$

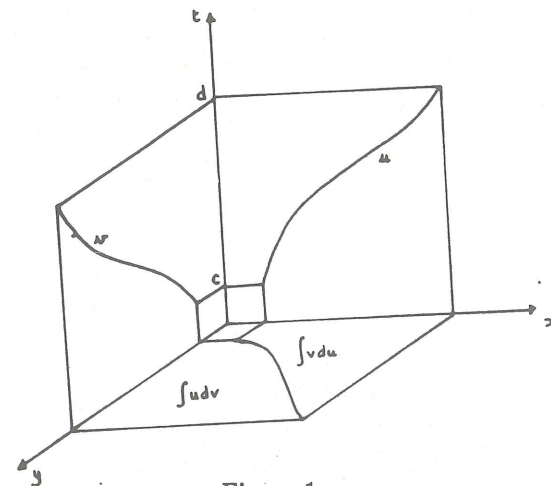


Figure 1

A particular case of this is illustrated in figure 1 where u and v are increasing. In this special case, note that $y = v \circ u^{-1}(x)$ represents the curve in the (x, y) -plane which also has parametric representation $t \mapsto (u(t), v(t))$. If moreover u and v are differentiable, we note that

$$\begin{aligned}\frac{dy}{dx} &= v'(u^{-1}(x)) (u^{-1})'(x) \\ &= v'(t) \cdot \frac{1}{u'(u^{-1}(x))} = \frac{v'(t)}{u'(t)}\end{aligned}$$

shows the link between the chain rule and the derivative of a parametrised curve.

We now apply these results to the following situation, where for simplicity we shall assume that $f: [a, b] \rightarrow [c, d]$ is strictly increasing, with $f(a) = c$ and $f(b) = d$. Let $g: [c, d] \rightarrow \mathbb{R}$ be integrable with respect to f^{-1} . Then from (2) we obtain

$$\begin{aligned}\int_c^d f^{-1} dg + \int_c^d g df^{-1} &= f^{-1}(d)g(d) - f^{-1}(c)g(c) \\ &= f^{-1}(f(b))g(f(b)) - f^{-1}(f(a))g(f(a))\end{aligned}$$

Letting $F = g \circ f$, we get

$$\int_c^d f^{-1} dg + \int_c^d g df^{-1} = bF(b) - aF(a) \quad (3)$$

By imposing stronger conditions on f and/or on g , we can find more manageable forms of (3). For example, if we assume that f is continuous, then the formula for change of variable in a Stieltjes integral yields the following:

$$\int_c^d f^{-1}(y) dg(y) = \int_{f^{-1}(c)}^{f^{-1}(d)} f^{-1}(f(x)) dg(f(x)) = \int_a^b x dF(x)$$

and

$$\int_c^d g(y) df^{-1}(y) = \int_{f^{-1}(c)}^{f^{-1}(d)} g(f(x)) df^{-1}(f(x)) = \int_a^b F(x) dx$$

Hence we obtain (not surprisingly!)

$$\int_a^b x dF(x) + \int_a^b F(x) dx = bF(b) - aF(a) \quad (4)$$

Finally, if we assume that g^{-1} exists and is continuous, then another appeal to the formula for substitution in a Stieltjes integral yields

$$\int_c^d f^{-1}(y) dg(y) = \int_{g(c)}^{g(d)} f^{-1}(g^{-1}(z)) dg(g^{-1}(z)) = \int_{F(a)}^{F(b)} F^{-1}(z) dz$$

Hence we get the following formula relating the integral of a composed function and its inverse:

$$\int_{F(a)}^{F(b)} F^{-1}(z) dz + \int_a^b F(x) dx = bF(b) - aF(a) \quad (5)$$

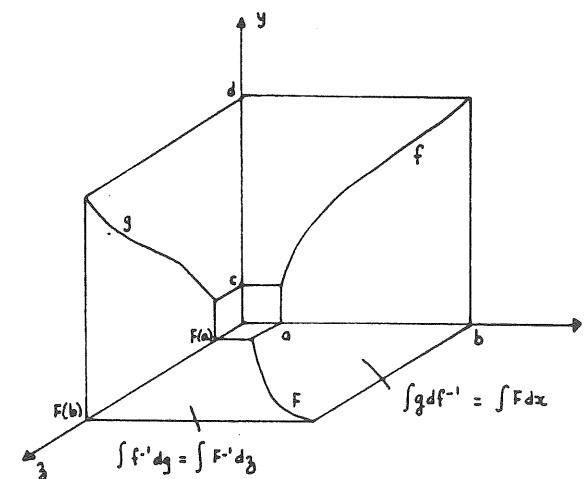


Figure 2

The special case $g(y) = y$ yields (1). Formulas (3)–(5) are illustrated in a special case in figure 2. Notice that in this case the curve $z = F(x)$ has parametrisation $y \mapsto (f^{-1}(y), g(y))$. In the case that f and g are differentiable, the $F'(x) = g'(y)/(f^{-1})'(y) = g'(y)f'(x)$ as we would expect.

As an example in this latter case, we take $f(x) = \tan x$ on $[0, \pi/4]$, and $g(y) = y^2$ on $[0, 1]$. Then (3) and (5) yield

$$\int_0^1 \arctan \sqrt{z} dz + \int_0^{\pi/4} \tan^2 x dx = 2 \int_0^1 y \arctan y dy + \int_0^1 \frac{y^2}{1+y^2} dy = \frac{\pi}{4}$$

In particular, we find (see figure 3):

$$\int_0^1 \arctan \sqrt{z} dz = \frac{\pi}{2} - 1$$

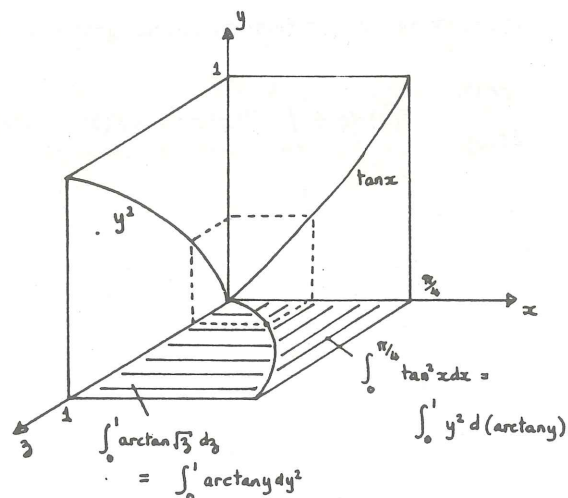


Figure 3

A less elementary but perhaps more instructive example is obtained by taking $f(x) = e^x$ and $g(y) = [y]$. Using (3), we get

$$\int_1^n \ln y d[y] + \int_0^{\ln(n)} [e^x] dx = n \ln(n) \quad (6)$$

The first integral here is $\sum_2^n \ln(j) = \ln(n!)$. Hence we obtain

$$\begin{aligned} \ln(n!) &= n \ln(n) - \int_0^{\ln(n)} [e^x] dx, \quad \text{or} \\ n! &= \frac{n^n}{e^{\int_0^{\ln(n)} [e^x] dx}} \end{aligned}$$

By estimating the integral appearing here, we can obtain some simple bounds

on $n!$. For example

$$\int_0^{\ln(n)} [e^x] dx < \int_0^{\ln(n)} e^x dx = n - 1$$

and hence

$$n! > e \left(\frac{n}{e}\right)^n$$

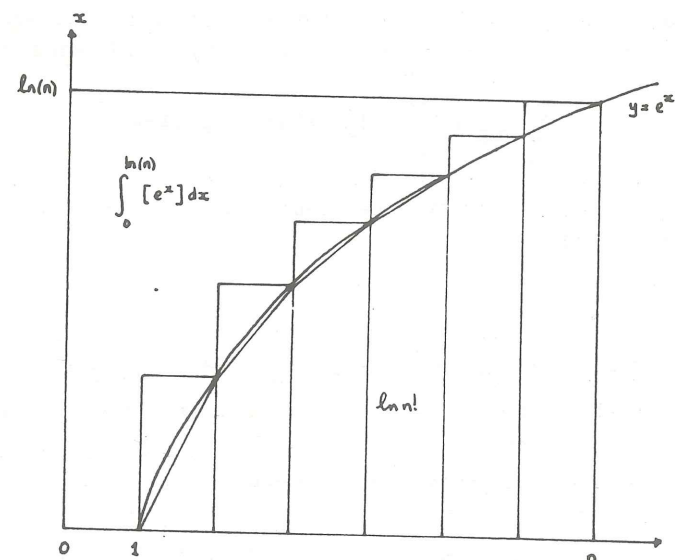


Figure 4

On the other hand, subtracting the areas of the triangles in figure 4,

$$\begin{aligned} \int_0^{\ln(n)} [e^x] dx &> \int_0^{\ln(n)} e^x dx - \sum_{j=2}^n \frac{1}{2} (\ln(j) - \ln(j-1)) \\ &= n - 1 - \frac{1}{2} \ln(n) \end{aligned}$$

which yields $n! < e\sqrt{n}(n/e)^n$. Hence at very little expense (6) yields the following well known bounds:

$$e \left(\frac{n}{e}\right)^n < n! < e\sqrt{n} \left(\frac{n}{e}\right)^n$$

Reverting to the general case, where f is arbitrary and $g(y) = [y]$, we obtain the formula

$$\sum_{f(a) < n \leq f(b)} f^{-1}(n) + \int_a^b [f(x)] dx = b[f(b)] - a[f(a)] \quad (7)$$

For example, $\sum_1^{N^2} \sqrt{n} = N^3 - \int_0^N [x^2] dx$.

So far we have only considered increasing functions. The reader may be interested in deriving and interpreting graphically the following equation:

$$\sum_1^N \frac{1}{n^s} + \int_1^{N^{-s}} [x^{-(1/s)}] dx = N^{1-s} - 1$$

I would like to record my indebtedness to my colleague Tom Power for references [4] and [8].

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Department of Physical and Quantitative Sciences
Regional Technical College, Waterford

CONFERENCES

Operator Theory and Operator Algebras University College Cork

The Third Annual Conference on Operator Theory and Operator Algebras will be held in University College Cork, from Wednesday 29th June to Saturday 2nd July, 1988. The principal speakers will include

I.D. Berg (Illinois)
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Further information can be obtained from the organizers:

G.J. Murphy & R.E. Harte
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Conference on Functional Analysis El Escorial, Spain

A Conference on Functional Analysis will be held in El Escorial, Madrid, from June 13 to 18, 1988. The Organising Committee consists of J. Ansemil, F. Bombal and J.G. Llavona.

It is expected that the main speakers will include R. Aron, K.D. Bierstedt, J. Diestel, J.M. Isidro and M. Valdivia.

Further information can be obtained from

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense
28040 Madrid, Spain

EARN:W116@EMDUCM11