Lagrange Multipliers

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It is not uncommon to hear a person say "I don't really understand Lagrange multipliers". The object of this note is to offer some explanation of what they are.

We recall that a necessary condition for the real-valued function $f(\mathbf{x})$, ($\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$) to have a stationary point at $\mathbf{a} \in \mathbf{R}^n$, subject to the "side conditions"

$$g_1(\mathbf{x}) = \ldots = g_k(\mathbf{x}) = 0 \tag{1}$$

is the existence of suitable Lagrange Multipliers, i.e. real numbers $\lambda_1, \ldots, \lambda_k$, such that

$$f'(\mathbf{a}) + \lambda_1 g_1'(\mathbf{a}) + \cdots + \lambda_k g_k'(\mathbf{a}) = 0 \tag{2}$$

Here, of course, $f', g_1', \ldots, g_k'$ are the derivatives of the relevant functions, so that f', for instance, is the vector

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

We shall assume throughout that we are dealing with functions which possess the required degrees of differentiability. Condition (2), together with the equations

$$g_1(\mathbf{a}) = \ldots = g_k(\mathbf{a}) = 0$$

usually enable one to determine the points $\mathbf{a}$.

Now think of the points satisfying the side conditions (1) as a variety V in $\mathbf{R}^n$. A point $\mathbf{a}$ is a stationary point of f subject to (1) if the directional derivative of f at $\mathbf{a}$ "in any direction contained in V" vanishes. More precisely, $\mathbf{a}$ is such that the directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{u}$ is zero for every unit vector $\mathbf{u}$ tangent to V at $\mathbf{a}$.

This directional derivative is the scalar product $\langle f'(\mathbf{a}), \mathbf{u}\rangle$. Thus $f'(\mathbf{a})$ is in the orthogonal complement of the tangent space to V at $\mathbf{a}$. Let us denote this tangent space by $T_{\mathbf{a}}V$. Assuming that the side conditions (1) are not redundant, $g_1'(\mathbf{a}), \ldots, g_k'(\mathbf{a})$ are linearly independent and span the

normal space to V at $\mathbf{a}$. Thus these vectors form a basis for the orthogonal complement of $T_{\mathbf{a}}V$, and therefore

$$f'(\mathbf{a}) + \lambda_1 g_1'(\mathbf{a}) + \cdots + \lambda_k g_k'(\mathbf{a}) = 0$$

for some $\lambda_1, \ldots, \lambda_k$

More analytically now, let $f : A \to \mathbf{R}$, and $g_i : A \to \mathbf{R}$, for $i = 1, \ldots, k$ be sufficiently smooth functions—say with continuous second order derivatives—on an open subset A of $\mathbf{R}^n$. Suppose we have a "parametrisation" or "local coordinate system" for V at $\mathbf{a}$. Thus, we have an open subset B of $\mathbf{R}^k$ and a homeomorphism $\varphi : B \to \mathbf{R}^n$ which maps B onto an open set in V containing $\mathbf{a}$. We assume that φ is as smooth as the other functions considered. The existence of such a function is guaranteed by the implicit function theorem.

The problem of finding stationary points of f subject to (1) can now be reduced to that of finding ordinary stationary points, with no side conditions, for the function $f \circ \varphi$.

Letting $\varphi^{-1}(\mathbf{a}) = \mathbf{t}_0$, we apply the chain rule to the equation

$$(f \circ \varphi)'(\mathbf{t}_0) = 0 ,$$

which is a necessary condition for $\mathbf{t}_0$ to be a stationary point for $f \circ \varphi$. This gives

$$(f \circ \varphi)'(\mathbf{t}_0) = f'(\mathbf{a})\varphi'(\mathbf{t}_0) = 0$$

Hence $f'(\mathbf{a})$ is orthogonal to each of the columns of the matrix $\varphi'(\mathbf{t}_0)$, and it is well known that these columns span $T_{\mathbf{a}}V$.

In order to determine the nature of the stationary point $\mathbf{a}$, one must look at the quadratic part of $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})$ for values of $\mathbf{h}$ for which $\mathbf{a}+\mathbf{h}$ lies on V, i.e. those $\mathbf{h}$ such that $\mathbf{a}+\mathbf{h} = \varphi(\mathbf{t}_0 + \mathbf{s})$, $\mathbf{s} \in \mathbf{R}^k$. Then

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = Q(s) + \eta(\mathbf{s})$$

where

$$Q(\mathbf{s}) = \frac{1}{2}\left(f''(\mathbf{a})\left(\varphi'(\mathbf{t}_0)\mathbf{s}\right)^2 + f'(\mathbf{a})\varphi''(\mathbf{t}_0)(\mathbf{s})^2\right)$$

$|\eta(\mathbf{s})|$ being of the order of $||\mathbf{s}||^3$. Bear in mind that $f''(\mathbf{a})$ is a scalar valued bilinear mapping, while $\varphi''(\mathbf{t}_0)$ is a bilinear mapping with values in $\mathbf{R}^n$.

Let M be the matrix associated with the bilinear form Q. If M is non-singular and definite then $\mathbf{a}$ is an extreme point of f subject to (1). If M is non-singular and indefinite then $\mathbf{a}$ will be a conditional saddle point of f.

Finally, if M is singular, no conclusions can be drawn concerning the nature of the stationary point $\mathbf{a}$.

We conclude with some examples.

Example 1 A sufficient condition for $f(x,y)$ to have a minimum at a stationary point (a,b) subject to a side condition parametrised by $x = \varphi_1(t)$, $y = \varphi_2(t)$ is

$$\begin{pmatrix} \varphi_1' & \varphi_2' \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} + \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} \varphi_1'' \\ \varphi_2'' \end{pmatrix} > 0$$

Example 2 Let $f(x,y) = 1 - 2xy$. Then f has a maximum at $(0,0)$ subject to $y - x^3 = 0$, but has neither a maximun nor a minimum at $(0,0)$ subject to $y - x^2 = 0$. In both cases we have $M = 0$.

Example 3 The function $f(x,y,z) = 1 - 2xy - 2xz - 2yz$ has a stationary point at $(0,0,0)$. Parametrising the side condition $y = z$ by $\varphi(r,s) = (r,s,s)$, we find that

$$M = \begin{pmatrix} 0 & -2 \\ -2 & -2 \end{pmatrix}$$

which is indefinite. $f(x,y,z)$ has a maximum at $(0,0,0)$ subject to $x = y = z$, but has a minimum at $(0,0,0)$ subject to $-x = y = z$. The point $(0,0,0)$ is a saddle point subject to $y = z$.

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A Note on Integrating Composed Functions

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This note groups together several concepts that are met at different places in a first course on real analysis in a way that allows graphical representation. It provides a generalisation of the formula (see [1]):

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy + \int_a^b f(x)\,dx = bf(b) - af(a) \tag{1}$$

which has a certain pedigree—see [2], [3] and particularly [4], where a proof is given in the case where f and f^{-1} are assumed only to be integrable.

We shall use the (Riemann-)Stieltjes integral as given, for instance, in [7]. We deal only with definite integrals.

We begin by recalling the formula for integration by parts for the Stieltjes integral. Let $u, v : [c,d] \to \mathbf{R}$, and assume the integral $\int_c^d u\,dv$ exists. Then

$$\int_c^d u\,dv + \int_c^d v\,du = u(d)v(d) - u(c)v(c) \tag{2}$$

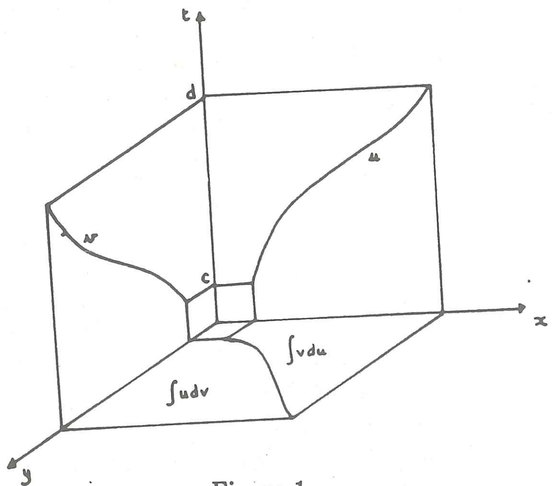

Figure 1