We argue as follows. Let x, y, r be elements of R with $xy \neq 0$. Then $(yx-xy)^n = yx-xy$ implies that $(yx)^n = yx = 0$. Similarly, $(x(ry)-(ry)x)^n = x(ry)-(ry)x$ implies that $xry = 0$. A simple induction argument now shows that all nilpotent elements are central. Thus R is commutative.

Of course, R need not be a field, as the example $(\mathbb{Z}_4, \oplus, \otimes)$ shows.

Finally, we are indebted to Professor T.J. Laffey who has supplied the following ingenious alternative proof of Theorem 1.

Let R be a finite ring with unity 1, let $T = T(R)$ be its group of units and suppose that $T \neq R \setminus \{0\}$. Let $0 \neq x \in R \setminus T$ and let $T_0 = \{ t \in T \mid xt = x \}$. We note that T_0 is a subgroup of T and that $V = \{ xv \mid v \in T \}$ is a subset of $R \setminus (T \cup \{0\})$, with $|V| = |T|/|T_0|$. Let $W = \{ t-1 \mid t \in T_0 \}$. We note that $|W| = |T_0|$ and that $W \subset R \setminus T$, since $t-1 \in T$ and $xt = x$ implies $x = 0$. Hence $|R| \geq |T| + |V| + 1 = |T| + |T|/|T_0| + 1$ and also $|R| \geq |T| + |T_0|$. Hence we deduce that $|R| - |T| \geq \max(|T_0|, |T|/|T_0| + 1$. So $|R| - |T| \geq \sqrt{|R|} + 1$.

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Periodic Functions

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This article arose out of correspondence between the author and Mark Heneghan regarding certain inconsistencies in the treatment of periodic functions in our secondary school texts. A complete and rigorous treatment of this topic requires the introduction of such concepts as convergent sequence, continuity, greatest lower bound, induction and linear independence. We have tried to minimize the impact of these concepts and at the same time to clarify the situation regarding the sum of periodic functions.

Definition 1 A function $f : \mathbb{R} \mapsto \mathbb{R}$ is *periodic* if there exists $a \neq 0$ such that

$$f(x+a) = f(x) \qquad \text{for all } x \in \mathbb{R}. \tag{1}$$

Any real number a satisfying (1) is called a *period* of f.

Remarks (1) If a is a period of f then so is $-a$, since $f(x) = f(x-a+a) = f(x-a)$.

(2) If a is a period of f and n is an integer then na is also a period of f. This follows from the identity

$$f(x+na) = f(x+(n-1)a+a) = f(x+(n-1)a),$$

using induction and our first remark.

(3) If a and b are periods of f then $a+b$ is also a period of f, since $f(x+a+b) = f(x+a) = f(x)$.

Example 1 Let f be given by $f(x) = \sin x$. Then f is periodic since $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$.

Example 2 Let f be given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

If a and x are rational and y irrational then $a+x$ is rational and $a+y$ is irrational, and hence $f(x+a) = 0 = f(x)$ and $f(y+a) = 1 = f(y)$. Thus every rational number is a period of f. If b is irrational and x is rational then $b+x$ is irrational and $f(x+b) = 1 \neq f(x)$ and hence b is not a period of f.

Examples 1 and 2 are typical of the only cases that can occur, as the following proposition demonstrates.

Proposition 1 *If f is a periodic function then exactly one of the following holds:*

(a) there exists a sequence $(a_n)_n$ of positive periods of f which converges to zero;

(b) there exists a positive number a such that na, $n = 0, \pm 1, \pm 2, \ldots$ form all the periods of f.

Proof If (a) does not hold then there exists a positive number δ such that the interval $[0, \delta)$ does not contain a positive period of f. We claim that no interval of the form $[a, a+\delta)$ contains two distinct periods of f. Suppose otherwise, so that there exist periods b and c with $a \leq b < c \leq a+\delta$ for some a. By Remarks (1) and (3) $c-b$ is also a period of f, but since $0 < c-b < \delta$ this is a contradiction. Hence our claim is proven.

Now consider the intervals $I_1 = [0, \delta]$, $I_2 = [\delta, 2\delta], \ldots, I_n = [(n-1)\delta, n\delta], \ldots$ Since f is periodic at least one of these intervals contains a period of f. Let n_1 be the least positive integer such that I_{n_1} contains a period of f. We have seen that I_{n_1} can contain only one period. This is then the smallest positive period of f. We denote it by a. By Remark (2), na, $n = 0, \pm 1, \pm 2, \ldots$ are periods of f. Suppose b is a further period. Then there exists an integer n such that $na < b < (n+1)a$. By Remarks (1) and (3) $(n+1)a - b$ is also a period of f. Since $0 < (n+1)a - b < a$ this is a contradiction, and so no such b exists. This completes the proof.

Remarks (4) If case (a) of Proposition 1 applies then, using our earlier remarks, it is not difficult to show that the periods of f form a dense subset of $\mathbb{R}$.

(5) If f is continuous and $(a_n)_n$ is a sequence of periods of f which converges to a, it is easy to see that a is also a period of f. Hence, using (4), we can conclude that if a continuous function f has a sequence of periods which converges to 0 then f is a constant function.

(6) If one is willing to use the concept of greatest lower bound then the proof of Proposition 1 can be shortened.

Definition 2 If case (b) of Proposition 1 applies to f then the smallest positive period of f is called *the period* of f.

Thus we have singled out a special period of f. The statement "f has period a" should be read as "f is a periodic function and *the period* of f is a".

Combining Proposition 1 and Remark 5 we see that if f is a non-constant, continuous periodic function and a is a period of f then there exists a positive integer n such that the period of f is a/n. To determine n one must investigate further the function f.

Example 3 Let $f(x) = \sin x$. By Example 1, f is periodic and the period of f is $2\pi/n$ for some positive integer n.

Now $f(0) = 0 = \sin(2\pi/n)$. If $n > 2$ then $2\pi/n < \pi$ and $\sin(2\pi/n) > 0$. Hence $n \leq 2$. We now check $n = 2$. Since $f(\pi/2 + 2\pi/2) = f(3\pi/2) = -1$ and $f(\pi/2) = 1$ it follows that 2 is not the correct value for n. Hence $n = 1$ and the period of f is 2π.

This result can, of course be obtained from a graph; while this suffices in practise, it is not a full proof.

We now consider the sum $f+g$ of two periodic functions (the case $f-g$ is handled in the same fashion).

Lemma 1 *If f and g are periodic and k is a common period of f and g then k is also a period of $f+g$.*

The proof is obvious.

Remark (7) If k is the period of both f and g, this does not give us precise information on the period of $f+g$ as the following example shows.

Example 4 Let $f(x) = \sin x + \cos(x/2)$, and let $g(x) = \sin x - \cos(x/2)$. It is easily seen that f and g are periodic and that 4π is the period of both functions (see Example 6). $(f+g)(x) = 2\sin x$ and so the period of $f+g$ is 2π.

Example 5 Let $f(x) = \sin ax + \cos bx$ where a and b are non-zero real numbers. We shall now show that f is periodic if and only if a/b is a rational number.

We confine ourselves to the case where a and b are both positive; the other cases are handled similarly.

Suppose first that a/b is rational. Let $a/b = p/q$ where p and q are positive integers. Then $2\pi p/a = 2\pi q/b = k$, say.

Let $g(x) = \sin ax$ and $h(x) = \cos bx$. Since

$$g\left(x + \frac{2\pi}{a}\right) = \sin\left[a\left(x + \frac{2\pi}{a}\right)\right] = \sin(ax + 2\pi) = \sin ax = g(x)$$

and

$$h\left(x + \frac{2\pi}{b}\right) = \cos\left[b\left(x + \frac{2\pi}{b}\right)\right] = \cos(bx + 2\pi) = \cos bx = h(x)$$

we have that $2\pi/a$ is a period of g and $2\pi/b$ is a period of h. By Remark (2) $k = p\frac{2\pi}{a} = q\frac{2\pi}{b}$ is a common period of g and h. Hence by Lemma 1 k is a period of $f = g + h$ and so f is a periodic function

Conversely, suppose that $f = g + h$ is periodic. Let k be a non-zero period of f. Then $f(0) = f(k) = f(-k) = 1$. Hence

$$\begin{aligned} \sin(ak) + \cos(bk) &= 1 \\ \sin(-ak) + \cos(bk) &= 1 \end{aligned}$$

and this implies $\cos bk = 1$ and $\sin ak = 0$. Therefore $bk = 2n\pi$ and $ak = m\pi$ for some integers n, m and so $\frac{a}{b} = \frac{m}{2n}$ is a rational number.

The example above shows how to construct non-periodic functions which are sums of periodic functions; $\sin x + \cos(\sqrt{2}x)$, for instance, is not periodic.

In our next example we show how to find the period of $\sin ax + \cos bx$.

Example 6 Let $a = \frac{p}{q}b$ where p and q are positive integers which have no common factors. By Example 5 $f(x) = \sin(ax) + \cos(bx)$ is periodic, and we wish to find its period.

We introduce an auxiliary function $g(x) = \sin(px) + \cos(qx)$. Then g is also periodic, and f and g are related as follows:

$$\begin{aligned} g\left(\frac{b}{q}x\right) &= \sin\left(p\frac{b}{q}x\right) + \cos\left(q\frac{b}{q}x\right) \\ &= \sin(ax) + \cos(bx) + f(x) \\ \text{and}\quad f\left(\frac{q}{b}x\right) &= g\left(\frac{b}{q}\frac{q}{b}x\right) = g(x)\,. \end{aligned}$$

We now find a relationship between the periods of f and g. If l is a period of f then

$$g\left(x + \frac{b}{q}l\right) = g\left(\frac{b}{q}\left(\frac{q}{b}x + l\right)\right) = f\left(\frac{q}{b}x + l\right) = f\left(\frac{q}{b}x\right) = g(x)$$

Hence $\frac{b}{q}l$ is a period of g. Since the period of a continuous periodic function is the smallest positive period, it follows that

$$(\text{the period of } g) \leq \frac{b}{q}(\text{the period of } f)$$

Similarly, if k is a period of g then $\frac{q}{b}k$ is a period of f. Hence

$$(\text{the period of } f) \leq \frac{q}{b}(\text{the period of } g)$$

Therefore, we have

$$(\text{the period of } f) = \frac{q}{b}(\text{the period of } g)$$

We now proceed to show that the period of g is 2π. Since $g(x + 2\pi) = \sin p(x+2\pi) + \cos q(x+2\pi) = \sin(px+2\pi) + \cos(qx+2\pi) = \sin px + \cos qx = g(x)$, it follows that the period of g is $2\pi/\alpha$ for some positive integer α. We must show that $\alpha = 1$. Now

$$\sin px + \cos qx = \sin\left[p\left(x + \frac{2\pi}{\alpha}\right)\right] + \cos\left[q\left(x + \frac{2\pi}{\alpha}\right)\right]$$

Hence

$$\sin px - \sin\left[p\left(x + \frac{2\pi}{\alpha}\right)\right] = \cos\left[q\left(x + \frac{2\pi}{\alpha}\right)\right] - \cos qx\,,$$

and

$$2\cos\left(px + \frac{p\pi}{\alpha}\right)\sin\left(-\frac{p\pi}{\alpha}\right) = -2\sin\left(qx + \frac{q\pi}{\alpha}\right)\sin\left(\frac{q\pi}{\alpha}\right)\,,$$

so that

$$\cos\left(px + \frac{p\pi}{\alpha}\right)\sin\left(\frac{p\pi}{\alpha}\right) = \sin\left(qx + \frac{q\pi}{\alpha}\right)\sin\left(\frac{q\pi}{\alpha}\right) \qquad (2)$$

Now suppose $\alpha \neq 1$; we shall show that this leads to a contradiction. Since p and q have no common factor, at least one of p/α and q/α is not an integer. Suppose that p/α is not an integer. Then the left hand side of (2) is not zero, and hence the same is true of the right hand side, which implies that q/α is also not an integer. Similarly, if we assume that q/α is not an integer, then it follows that p/α is not an integer. Therefore we have

$$\sin\frac{p\pi}{\alpha} \neq 0 \qquad \text{and} \qquad \sin\frac{q\pi}{\alpha} \neq 0\,.$$

If we differentiate (2) $4n$ times we get

$$p^{4n}\cos\left(px+\frac{p\pi}{\alpha}\right)\sin\left(\frac{p\pi}{\alpha}\right) = q^{4n}\sin\left(qx+\frac{q\pi}{\alpha}\right)\sin\left(\frac{q\pi}{\alpha}\right)$$

Letting $x = 0$ we get

$$\left(\frac{q}{p}\right)^{4n} = \frac{\cos\left(\frac{q\pi}{\alpha}\right)\sin\left(\frac{p\pi}{\alpha}\right)}{\sin^2\left(\frac{q\pi}{\alpha}\right)} \neq 0 \tag{3}$$

Now if $p \neq q$ then when $n \to \infty$ the left hand side of (3) tends to either 0 or ∞. However, the right hand side of (3) is a non-zero constant. Hence $p = q$. Since p and q have no common factors, this can occur only when $p = q = 1$. In this case, (2) becomes

$$\cos\left(x+\frac{\pi}{\alpha}\right) = \sin\left(x+\frac{\pi}{\alpha}\right)$$

and hence $\tan(x+\pi/\alpha) = 1$ for all x. If we let $x = -\pi/\alpha$ we obtain a contradiction, and hence we must have $\alpha = 1$.

To summarise the above, we have shown the following: the period of $\sin(px)+\cos(qx)$ is 2π, and the period of $\sin((p/q)bx)+\cos(bx)$ is $2\pi q/b$ when p and q are positive integers with no common factor. We can use this to find the period of $\sin nx + \sin nx$, where m and n are arbitrary positive integers, i.e., making no assumption about common factors. Let $d = \gcd(m,n)$. Then there are positive integers m', n' such that $m = m'd$, $n = n'd$ and $\gcd(m',n') = 1$. Letting $a = m$ and $b = n$ we see that the period of $\sin nx + \cos mx$ is

$$2\pi\frac{m'}{m} = \frac{2\pi}{\gcd(m,n)}$$

Similarly, the period of $\sin(x/n) + \cos(x/m)$ is

$$\frac{2\pi nm}{\gcd(m,n)}$$

Periodic functions of the form $\sin ax \pm \sin bx$ and $\cos ax \pm \cos bx$ are treated in the same way, and simple functions such as $\sin ax \cos bx$ can be reduced to the cases discussed above by the use of appropriate trigonometric identities.

At this point, the reader may well ask the following questions:

(a) Is there any criterion for deciding if the sum of periodic functions is periodic?

(b) Are there any general methods of determining the period of a sum from the periods of the component functions?

(c) How does one determine the period of a general trigonometric polynomial i.e., a linear combination of powers of the functions $\sin x$ and $\cos x$?

(d) How large is the class of functions consisting of trigonometric polynomials?

A special case of question (a) is answered in Example 5. The same method can be used to obtain the following general result:

Proposition 2 *Let f and g be continuous periodic functions such that $f+g$ is non-constant. Let a and b be non-zero periods of f and g respectively. Then $f+g$ is periodic if and only if a/b is rational; furthermore, every period of $f+g$ has the form na/m for some integers n and m.*

This result is not true in general without the assumption of continuity.

As regards (d), the Stone-Weierstrass theorem shows that every continuous periodic function can be approximated uniformly by trigonometric polynomials, and the theory of Fourier Series shows that every continuous periodic function is the (pointwise) infinite sum of sines and cosines. Thus, by considering sums of sines and cosines, one is led to a very large class of functions, and there is no general simple method for calculating the period of a trigonometric polynomial.

There are, however, a number of techniques which can be used for arbitrary periodic functions, and which may help to locate the period. We briefly discuss these in our final example.

Example 7 The function $f(x) = 3\sin x - 4\sin^3 x$ has period $2\pi/3$. The easiest way to see this is to note that $f(x) = \sin 3x$. In the general case, however, we may not have such a nice formula for f, or we may be considering something like $\sin 16x$ expanded in sines and cosines and may not recognise the simple form of the function. Hence, we illustrate some techniques for finding the period without using the fact that $f(x) = \sin 3x$.

First, one checks easily that f is not constant. Now, since $\sin x$ has period 2π, we know that the period of f is $2\pi/n$ for some positive integer n. If $2\pi/n$ is the period of f then, since $f(0) = 0$ and $f(x) = \sin x\,(3 - 4\sin^2 x)$, we must have either $\sin(2\pi/n) = 0$ or $3 - 4\sin^2(2\pi/n) = 0$. Now $\sin(2\pi/n) = 0$ implies $n \leq 2$ and $3 - 4\sin^2(2\pi/n) = 0$ implies $\sin(2\pi/n) = \pm\sqrt{3}/2$. Since $f(x) > 0$ for small positive values of x it follows that the first positive zero of f is at least $\pi/3$. Hence $2\pi/n \geq \pi/3$, i.e. $n \leq 6$. It remains therefore to check the cases $n = 1, \ldots, 6$.

Now $f \geq 0$ on $[0, \pi/3]$, and the factorisation $\sin x\,(3 - 4\sin^2 x)$ shows that $f \leq 0$ on $[\pi/3, 2\pi/3]$. Hence $2\pi/n \geq 2\pi/3$, giving $n \leq 3$. Since 2π is a period of f, it suffices to check the cases $n = 2$ and $n = 3$.

$n = 2$: Since $f(\pi/2) \neq f(3\pi/2)$ we cannot have $n = 2$.

$n = 3$: Checking some values of x such as $\pi/6$ and $\pi/2$, one finds that $n = 3$ is not ruled out. Hence we check to see if $2\pi/3$ is a period of f.

Now
$$\sin\left(x + \frac{2\pi}{3}\right) = -\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$
and
$$\sin^2\left(x + \frac{2\pi}{3}\right) = \frac{1}{4}\sin^2 x - \frac{\sqrt{3}}{2}\sin x\cos x + \frac{3}{4}\cos^2 x\,.$$
Hence
$$\begin{aligned} f\left(x + \frac{2\pi}{3}\right) &= \frac{1}{2}\left(-\sin x + \sqrt{3}\cos x\right)\left(3 - \sin^2 x + 2\sqrt{3}\sin x\cos x - 3\cos^2 x\right) \\ &= \sin x\,\left(-\sin x + \sqrt{3}\cos x\right)\left(\sin x + \sqrt{3}\cos x\right) \\ &= \sin x\,\left(3\cos^2 x - \sin^2 x\right) = \sin x\,\left(3 - 4\sin^2 x\right) \\ &= f(x) \end{aligned}$$

Hence $2\pi/3$ is a period of f, and therefore it is *the* period of f.

To summarise the methods used in this example:

(1) By inspection, find one period of the function (the smaller the better).

(2) Locate some zeros of f. The period is at least equal to the maximum distance between *adjacent* zeros. If it is not possible to find any zeros, try to locate points at which f takes the same value and proceed as above.

(3) The first step rules out all but a finite number of possible values. Using (2), check these values at a number of points. This will generally rule out most values.

(4) Finally, check which of the remaining values are periods of f.

At some stage one should also check that the function is non-constant. If one begins to get a constant value for f while carrying out the above steps, one should try to prove that f is constant.

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