Planks' Constants

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One of the hazards to be faced by the student of general topology is tl proof of existence of spaces which are $T_{3\frac{1}{2}}$ (i.e. completely regular, or T chonoff) but not T_4 (i.e. normal). The standard example, the Tychono Plank (see [3]), has perhaps an unnecessarily austere public image since i usual presentation requires familiarity with ordinals which many undergra uates have not acquired. We here call attention to an alternative exampl due essentially to Thomas [4], which has no such prerequisite. Assuming a elementary understanding of cardinal numbers we go on to show how to exe the construction to produce a family of non-normal Tychonoff spaces, and discuss some questions which this extension raises.

Example 1 (The Thomas Plank (see [3]).) Let X and Y be infinite di crete spaces, where X is uncountable. Form their Alexandroff ("one-point' compactifications $A(X) = X \cup \{\infty\}$ and $A(Y) = \cup\{\infty\}$, their product spa $A(X) \times A(Y)$, and its subspace

$$P = (A(X) \times A(Y)) \setminus \{(\infty, \infty)\}$$

(If desired, $A(X)$ may be defined as carrying the Fort topology $\gamma \cup \epsilon(\infty$ where γ denotes the cofinite topology, and $\epsilon(\infty)$ the excuded point topolog in which the non-universal open sets are those to which ∞ does not belon — see [3].) Since $A(X)$ and $A(Y)$ are compact and T_2, as may be seen eith from the local compactness of X and Y or directly from the definition, so i their product which is thus T_4 and $T_{3\frac{1}{2}}$ also. Now the subsets

$$T = X \times \{\infty\}, \quad R = \{\infty\} \times Y$$

are closed in P. If, however, it were possible to find dispoint open subset G,H of P with $T \subseteq G$ and $R \subseteq H$, choose a countably infinite subset Y' of and note that

(i) H would have to contain all but finitely many points on each horizont cross-section $X \times \{y'\}$ of $X \times Y'$, from which it follows that $(X \times Y') \setminus$ is at most countable, whereas

(ii) G must contain at least one point (indeed, infinitely many points) on each vertical cross-section $\{x\} \times Y'$ of $X \times Y'$, and so $(X \times Y') \cap G$ is uncountable.

These observations cannot be reconciled with the disjointness of G and H, and the contradiction establishes that P is not T_4.

Note that this example could be simplified by taking Y to be countable, thus rendering the selection of Y' unnecessary. (Indeed, even further siimplification can be achieved by abstraction. Begin with an uncountably infinite set X, let $z \in X$ and let Y be a countably infinite subset of $S \setminus \{z\}$. Consider X with the Fort topology $\gamma \cup \varepsilon(z)$, the product space $X \times X$, and its subspace $P = (X \times X) \setminus \{(z, z)\}$; then $T = (X \setminus \{z\}) \times \{z\}$, $R = \{z\} \times Y$ are each closed in P, and a routine modification of the previous argument will suffice.)

Remarks The source of the contradiction here is the existence of a cardinal number, in this case $\aleph_0$, which is less than that of X but exceeds that of the complement of a "neighbourhood of infinity". It is easily seen that we can obtain other examples of non-T_4 spaces just by replacing $\aleph_0$ by another infinite cardinal; further, it will be convenient to allow different cardinals to be associated with X and with Y. More thought, however, is needed to ensure that we not not lose the $T_{3\frac{1}{2}}$ property in the process, since the demonstration of this depended on three results which could be described as "cardinality-sensitive", namely

(a) $A(X)$ is compact,

(b) the product of two compact spaces is compact,

(c) compact plus T_2 implies T_4.

This is what will occupy most of our attention for the remainder of the present note.

Definitions Let α denote an infinite cardinal number. A topological space X is called α-compact (see [1] or, for a more recent reference, [2]) if every open cover of X has a subcover consisting of fewer than α sets. Thus, for example, $\aleph_0$-compactness is just (classical) compactness, and $\aleph_1$-compactness is the Lindelöf property. Given any space X, choose an object ∞ which does not belong to X and denote by $A_\alpha(X)$ the topological space defined on $X \cup \{\infty\}$ by declaring open

(i) the open subsets of the space X,

(ii) the complements in $X \cup \{\infty\}$ of the α-compact closed subsets of X, and

(iii) $X \cup \{\infty\}$ itself.

The obvious modifications of the Alexandroff argument will sow that $A_\alpha(X)$ is a α-compact and contains X as a subspace, and that X is dense in $A_\alpha(X)$ precisely when X is not α-compact.

Lemma 1 *Suppose that X is a discrete space. Then $A_\alpha(X)$ is $T_{3\frac{1}{2}}$ for any finite cardinal α.*

Proof It is certainly T_1 since singletons are α-compact. Now if F is a given closed subsety of $A_\alpha(X)$ and $p \notin F$, we consider two cases:

(a) $p = \infty$. Define $f : A_\alpha(X) \to [0,1]$ by $f(y) = 1$ if $y \notin F$, $f(y) = 0$ if $y \in F$.

(b) $p \neq \infty$. Define $f : A_\alpha(X) \to [0,1]$ by $f(p) = 1$, $f(y) = 0$ for all $y \neq p$.

In either case f is constant on a neighbourhood of ∞ and thus continuous there. Every other point of $A_\alpha(X)$ is isolated, so continuity elsewhere is automatic.

Example 2 Choose any two infinite cardinal numbers α and β. Denote by $\bar{\alpha}$ the supremum of all cardinals less than α, so that if α has an immediate predecessor than $\bar{\alpha}$ is the predecessor, while if not we have $\bar{\alpha} = \alpha$. Choose sets X and Y whose cardinalities satisfy

$$\mathrm{card}(X) > \bar{\alpha}\,, \quad \mathrm{card}(X) > \beta\,, \quad \mathrm{card}(Y) \geq \beta\,.$$

Give X and Y their discrete topologies. By the lemma, the subspace

$$P = (A_\alpha(X) \times A_\beta(Y)) \setminus (\infty, \infty)$$

of the product space $A_{alpha}(X) \times A_\beta(Y)$ is $T_{3\frac{1}{2}}$. Now if T,R,G and H are as in Example 1, choose a subset Y' of Y having cardinality β and observe that

(i) the relative complement of H in each horizontal cross-section of $X \times Y'$ has cardinality at most $\bar{\alpha}$, and so the cardinality of $(X \times Y') \setminus H$ cannot exceed $\beta.\bar{\alpha}$ which is less than $\mathrm{card}(X)$, whereas

(ii) G must contain at least one point on each vertical cross-section of $X \times Y'$, so the cardinality of $(X \times Y') \cap G$ is at least $\mathrm{card}(X)$.

Thus the same contradiction as before has arisen, and P cannot be T_4.

Remarks Since $\bar{\aleph}_0 = \aleph_0$, the special case $\alpha = \beta = \aleph_0$ coincides with Example 1. If instead we choose $\alpha = \aleph_1$ (noting that $\bar{\aleph}_1 = \aleph_0$) and $\beta = \aleph_0$, we obtain a construct whose behaviour closely resembles that of the Tychonoff Plank. What we have obtained, then, is a family of "planks", parameterized so to speak by the two cardinals α and β which we regard as the "constants" describing a particular plank. The authors would at this point like to apologise for the excruciating pun in the title of this paper.

It is interesting to note what happens when one attempts to establish the $T_{3\frac{1}{2}}$ property (for Example 2) not directly, as in the lemma but by re-examining the points (a), (b) and (c) in the remarks following Example 1. Now $A_\alpha(X)$ is α-compact, but it is not in general true that a product of α-compact spaces is α-compact (see [3] for a simple example — Sorgenfrey's half-open square topology on a real plane—of a Lindelöf space X such that $X \times X$ is not Lindelöf) nor that an α-compact T_2 space is T_4 (for instance, see [3] again for the relatively prime integer topology on the positive integers). There are, however, special circumstances in which this line of argument recovers its validity, as we shall now see.

Definitions (i) An infinite cardinal number α is called *additively inaccessible* if it cannot be expressed as the sum of a lesser number of smaller cardinals: that is, if it is impossible to obtain a set of cardinality α by forming the union of a family of subsets, where each subset *and* the index set of the family have cardinality less than α. It is easily seen that a cardinal which has an immediate predecessor is additively inaccessible, but the problem of existence of other examples would lead us too deeply into axiomatic set theory to be appropriately discussed in this note.

(ii) A topological space in which each intersection of fewer than α open sets is open is called *α-saturated* (see [1] again). Thus every space is $\aleph_0$-saturated, a discrete space is α-saturated for every cardinal number α, and it is readily checked that, for discrete X, $A_\alpha(X)$ is α-saturated provided that α is additively inaccessible; indeed, we can as readily obtain a more general result:

Proposition 1 *Let α be an additively inaccessible cardinal number; then*

(i) *the union of fewer than α subsets of a space X, each of which is α-compact, is α-compact;*

(ii) *if X is α-saturated then so is $A_\alpha(X)$.*

Proof (i) If $\{C_i : i \in I\}$, where $\text{card}(I) < \alpha$, is a family of α-compact sets whose union is contained in that of a family $\{G_j : j \in J\}$ of open sets, then for each i in I there is a subset $J(i)$ of J such that $C_i \subseteq \bigcup\{G_j : j \in J(i)\}$ and $\text{card}(J(i)) < \alpha$. So $\bigcup\{C_i : i \in I\} \subseteq \bigcup\{G_j : j \in J'\}$ where the set $J' = \bigcup\{J(i) : i \in I\}$ has cardinality less than α.

(ii) Consider $x \in G = \bigcap\{G_i : i \in I\}$ where $\text{card}(I) < \alpha$ and each G_i is open in $A_\alpha(X), X$ being α-saturated. If $x \in X$ then $\bigcap\{G_i \cap X : i \in I\}$ is an open neighbourhood (in X) of x and is contained in G. If $x = \infty$ then $X \setminus G$ is α-compact by (i), and closed in X because X is α-saturated. Thus G is a neighbourhood of each of its elements, and must be open.

Proposition 2 *Let X and Y be α-saturated and α-compact, where α is additively inaccessible. Then $X \times Y$ is α-compact.*

Proof Given an open covering $\{G_\beta : \beta \in B\}$ of $X \times Y$, let y be any element of Y. For each x in X we can choose $\beta(x,y)$ in B, open $H(x,y) \subseteq X$ and open $J(x,y) \subseteq Y$ such that

$$(x,y) \in H(x,y) \times J(x,y) \subseteq G_{\beta(x,y)} \,.$$

Now fewer than α of the sets $H(x,y)$ will suffice to cover X; and if J_y denotes the (open) intersection of the $J(x,y)$ which correspond to these, we see that $X \times J_y$ is covered by a subfamily $\{G_\beta : \beta \in B_y\}$ of the given cover for which $\text{card}(B_y) < \alpha$.

Now the sets J_y, for y in Y, cover Y; so there is a subset Y' of Y such that $\text{card}(Y') < \alpha$ and $Y \subseteq \bigcup\{J_y : y \in Y'\}$. Then

$$X \times Y = \bigcup\{X \times J_y : y \in Y'\} \subseteq \bigcup\{G_\beta : \beta \in \bigcup\{B_y : y \in Y'\}\}$$

where $\bigcup\{B_y : y \in Y'\}$ has cardinality less than α, as required.

Proposition 3 *If X and Y are α-saturated topological spaces, then so is $X \times Y$.*

The proof is elementary.

Proposition 4 *An α-compact, α-saturated, T_2 topological space is T_4.*

The proof is the obvious modification of that of the classical case $\alpha = \aleph_0$.

Remarks These four propositions constitute an alternative proof that the space $A_\alpha(X) \times A_\beta(Y)$ in Example 2 is T_4 (and therefore that P is $T_{3\frac{1}{2}}$), but only in the case where α and β are additively inaccessible and equal. Thus they add nothing to our understanding of Example 2, and are included here partly for their intrinsic interest and partly to point out how a relatively innocuous-looking topological question can quickly lead to areas of set theory in which Zermelo-Fraenkel will not suffice.

References

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