

# REALIZING RINGS AS ENDOMORPHISM RINGS - THE IMPACT OF LOGIC

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## INTRODUCTION

The object of this paper is to give a brief survey of an area of Abelian group theory in which remarkable progress has been made in recent years. A second objective is to indicate how results and techniques from logic are gradually becoming important in this area of algebra; a point to which we shall return later.

Throughout the paper all groups shall be additively written Abelian groups and rings shall be unital associative rings.

## A QUICK COURSE IN ABELIAN GROUP THEORY

The following concepts will be needed from Abelian group theory:

1. A group  $G$  is said to be *reduced* if  $G$  does not contain a subgroup isomorphic to the additive group of rationals,  $\mathbb{Q}$ , or the Prufer quasi-cyclic group  $\mathbb{Z}(p^\infty)$  for any prime  $p$  [This latter group is the additively written version of the (multiplicative) group of  $p^n$ th complex roots of unity, with  $n$  running over all integers  $\geq 0$ ].
2. An element  $g$  of a group  $G$  is said to be a *torsion* element if  $ng = 0$ , the identity of  $G$ , for some integer  $n \neq 0$ . If no such integer  $n$  exists then  $g$  is a *torsion-free* element. The torsion elements form a subgroup  $tG$  of  $G$  and the quotient  $G/tG$  is a torsion-free group.
3. A group  $F$  is said to be *free* if it has the form  $F = \bigoplus X_i$  where  $X_i \cong \mathbb{Z}$ , the additive group of integers, for all  $i \in I$ .  
(These are precisely the projectives in the category of

Abelian groups.)

4. The set of all endomorphisms of a group  $G$  (= homomorphisms from  $G$  to  $G$ ) form a ring  $E(G)$ , the *endomorphism ring* of  $G$ , if we set  $(\phi_1 + \phi_2)(g) = \phi_1(g) + \phi_2(g)$  and  $\phi_1\phi_2(g) = \phi_1(\phi_2(g))$  for endomorphisms  $\phi_1$  and  $\phi_2$ .
5. If  $G$  is a group then the subgroups  $nG$ , where  $n$  runs through all non-zero integers, form a base of neighbourhoods of 0 for a linear topology on  $G$ . This topology is called the  *$\mathbb{Z}$ -adic or natural topology*. If  $G$  is torsion-free then this topology is Hausdorff precisely if  $G$  is reduced. If subgroups of the form  $p^n G$  are chosen then the resulting topology is the  *$p$ -adic topology*.
6. The completion of  $\mathbb{Z}$  in its  $p$ -adic topology is the group  $J_p$  of  *$p$ -adic integers*. Elements of  $J_p$  can be regarded as formal infinite series  $s_0 + s_1 p + s_2 p^2 + \dots$  where  $s_i \in \{0, 1, \dots, p-1\}$ . We note that  $J_p$  can also carry a ring structure and that it has cardinality  $2^{\aleph_0}$ . (Further details may be found in, e.g. Fuchs [4]).
7. A group  $G$  is *indecomposable* if it cannot be written in the form  $G = A \oplus B$  for non-zero groups  $A$  and  $B$ . Note that if  $G = A \oplus B$  then  $E(G)$  has idempotents, viz. the projections.

## THE REALIZATION PROBLEM.

The basic realization problem can be stated as follows:

"Given a ring  $A$ , what conditions on  $A$  will ensure that there is an Abelian group  $G$  with  $E(G) \cong A$  as rings."

The basic problem can be modified (and made harder!) by insisting that the resulting group  $G$  should belong to some prescribed class of groups.

The fundamental result in this area is due to A.L.S. Corner [1] in 1963. (The original paper is a beautiful example of how mathematics should be written!)

Corner's Theorem. If  $A$  is a countable, reduced torsion-free ring then there exists a countable, reduced torsion-free group  $G$  with  $E(G) \approx A$ .

(We remark that properties such as reduced etc. attributed to a ring mean that the underlying group of the ring has the same properties.)

I shall not attempt to give any proof but merely indicate that all three conditions are necessary. Consider the following rings: (i)  $\mathbb{Q} \oplus \mathbb{Q}$ , (ii)  $\mathbb{Z}(p) \oplus \mathbb{Z}(p)$  ( $\mathbb{Z}(p)$  is the field of  $p$  elements), (iii)  $J_p \oplus J_p$ . Each of these rings satisfies two but not the third of the conditions on the ring  $A$  in the above theorem. However in case (i) if  $E(F) = \mathbb{Q} \oplus \mathbb{Q}$  then  $G$  would be a vector space over  $\mathbb{Q}$ . If  $\dim G = n$  then we would need  $n^2 = 2$  which is impossible. A similar vector space argument shows that (ii) is impossible. Finally if  $E(G) = J_p \oplus J_p$  then  $G$  is naturally a  $p$ -adic module and it will have finite rank. However it is known that a finite rank  $p$ -adic module which is reduced (in this case  $\mathbb{Q}$  must be replaced in the definition by the field of  $p$ -adic numbers) is free and this again leads after a little argument to solving  $n^2 = 2$ . So (iii) is also impossible.

A non-algebraist might reasonably ask why the above result is important. (I'm assuming the question comes in the context of pure mathematics and is not related to applications!) One answer is that the result can be used to construct some amazing examples of groups. These groups show that it is practically impossible to derive any analogue of Krull-Schmidt decomposition theory. We content ourselves with three examples which can be produced using Corner's result.

Example 1. There is an indecomposable group of infinite rank.

Take  $A = \mathbb{Z}[t]$ , the integral polynomials in the variable  $t$  and apply Corner's result. If  $E(G) = A$  then  $G$  is indecomposable since  $A$  has no idempotents.

Example 2. There is a superdecomposable group of countable rank

(i.e. a group which has no non-zero indecomposable direct summand.)

Let  $\Lambda = \{\lambda_r \mid r \in \mathbb{Q}, r \geq 0\}$  and define  $\lambda_r \lambda_s = \lambda_{\max(r,s)}$ . Let  $A = \mathbb{Z}\Lambda$ , the semigroup ring of  $\Lambda$  over  $\mathbb{Z}$ . It can be shown that the underlying group of  $A$  is freely generated by the  $\lambda_r$  and so  $A$  satisfies the conditions of Corner's Theorem. Hence we find a group  $G$  with  $E(G) = A$ . However a little calculation shows that if  $\epsilon$  is any non-zero idempotent in  $A$  then there is a non-zero idempotent  $\zeta$  such that  $\zeta = \epsilon\zeta = \zeta\epsilon \neq \epsilon$ . But now if  $G = B \oplus C$  then  $B = \epsilon(G)$  for some idempotent  $\epsilon$ . However  $D = \zeta(G)$  is then a summand of  $G$  and  $\zeta(G)$  is contained in  $B$  since  $\zeta(G) = \epsilon\zeta(G)$ . Thus  $B$  decomposes as  $B = D \oplus E$ , some  $E$ .

Example 3. There is a countable torsion-free group  $A$  such that  $A \approx A \oplus A \oplus A$  but  $A \not\approx A \oplus A$ .

Take  $\Lambda$  to be the semigroup with 1 generated by  $\rho_i, \sigma_i$  ( $i = 0, 1, 2$ ) subject to  $\rho_j \sigma_i = \delta_{ji}$ . Let  $R = \mathbb{Z}\Lambda$ , the integral semigroup ring of  $\Lambda$  (identifying the 0 of the semigroup with the 0 of  $A$ ). Again  $R$  is free as a group. If  $I$  denotes the principal ideal generated by  $\tau = 1 - \sigma_0 \rho_0 - \sigma_1 \rho_1 - \sigma_2 \rho_2$  then  $R/I$  is still freely generated as a group. Take  $A = R/I$  and use Corner's Theorem to exhibit  $G$  with  $E(G) = A$ . This group  $G$  will have the desired properties. (See Fuchs [4, Vol. 2, 91.6] for more details.)

Corner extended his theorem by using topological rings and he also produced a similar type of result for  $p$ -groups in 1969. I worked on the realization problem for  $p$ -adic torsion-free modules (1974) and produced a weak realization theorem there. Apart from some modifications and slight extensions of Corner's Theorem (arising mainly from Orsatti and his school in Padova) this was the state of the Realization Problem in 1974.

# ENTER SHELAH

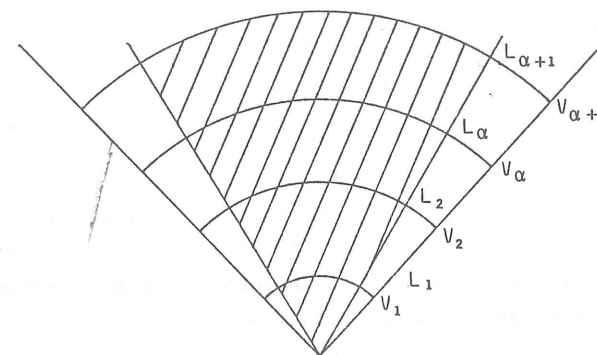
In late 1973 the situation changed dramatically. Sah Shelah applied techniques from model theory and logic to a number of problems in Abelian group theory [5] and produced some astonishing results. The most celebrated of these does not relate directly to the realization problem but is nonetheless important to our survey of this problem. I refer of course to his unexpected solution of the Whitehead Problem. This problem, which has its origins in topology, can be phrased as follows:

"If  $A$  is a torsion-free Abelian group with the property that every extension of  $\mathbb{Z}$  by  $A$  splits, must  $A$  be free

In other words if  $G$  is a group and  $G/\mathbb{Z} \cong A$  implies  $G \cong A \oplus \mathbb{Z}$ , must  $A$  be isomorphic to  $\bigoplus \mathbb{Z}$ ? Shelah's surprising answer is that the problem is undecidable in ordinary set theory!

To see what this means we must now make a small incursion into set theory. The most commonly used set theory consists of the axioms of Zermelo-Fraenkel. We do not need to consider these axioms individually; suffice it to say that they cover "naive set theory". If we include the Axiom of Choice then we have the basic everyday set theory which we denote by ZFC. To understand Shelah's answer to the Whitehead problem we need two additional axioms.

**Godel's Axiom.** If we let  $V_0 = \emptyset$ ,  $V_1 = \mathcal{P}(\emptyset)$ ,  $V_2 = \mathcal{P}(V_1)$  ... and in general  $V_{\alpha+1} = \mathcal{P}(V_\alpha)$  (with  $V_\sigma = \bigcup_{\alpha < \sigma} V_\alpha$  for a limit ordinal  $\sigma$ ) then we have the universe of sets  $V = \bigcup V_\alpha$  (where  $\alpha$  runs through all ordinals). Alternatively if  $X$  is a set, let  $\text{Def}X$  be the family of all subsets of  $X$  of the form  $\{a \in X \mid P(a)\}$  where  $P(a)$  is any property of sets expressed in the predicate calculus. Now let  $L_0 = \emptyset$ ,  $L_{\alpha+1} = \text{Def}(L_\alpha)$  (with  $L_\sigma = \bigcup_{\alpha < \sigma} L_\alpha$  for a limit ordinal  $\sigma$ ) and set  $L = \bigcup L_\alpha$  (where  $\alpha$  runs through all ordinals).  $L$  is the universe of constructible sets and in general is thought of as a "smaller" universe than  $V$ . (See Fig. 1).



$\phi = V_0 = L_0$

Godel's axiom is that  $V = L$ .

**Martin's Axiom.** This axiom arose originally in discussions of the Souslin problem. Before stating the axioms we need to recall some definitions relating to partially ordered sets.

## Definitions

- (i) If  $(P, \leq)$  is a partially ordered set then the elements  $p, q$  of  $P$  are compatible if there is  $r \in P$  with  $p \leq r$ ,  $q \leq r$ .
- (ii) A subset of  $P$  is compatible if every pair of elements is compatible.
- (iii) A subset  $D$  of  $P$  is dense in  $P$  if for all  $p \in P$ , there is a  $d \in D$  with  $p \leq d$ .
- (iv) A partially ordered set  $(P, \leq)$  satisfies the countable condition if every pairwise incompatible subset of  $P$  is countable.



We can now state Martin's axiom (MA):

Suppose  $(P, \leq)$  is a partially ordered set satisfying the countable chain condition. If  $\{D_i\}$  ( $i \in I$ ) is a family of dense subsets of  $P$  with  $|I| < 2^{\aleph_0}$ , then there is a compatible subset  $G$  such that  $G \cap D_i \neq \emptyset$  for all  $i \in I$ .

An observant naive set theorist will notice that MA follows from the continuum hypothesis (CH). However it has also been shown that (ZFC + MA + negation of CH) is consistent. (By consistent we mean that if ZFC is free from contradictions then so also is the above.) Indeed it is also known that ZFC + (V=L) is consistent.

Shelah's answer to the Whitehead problem was this: In (ZFC + MA + negation of CH) there is a group  $A$  (of cardinality  $\aleph_1$ ) which satisfies the conditions of Whitehead's problem but  $A$  is not free.

The outcome is, of course, that for naive set theorists the problem is undecidable! This of course was a considerable shock to most people working in Abelian groups. (See Eklof [3] for a very readable discussion of this area.)

#### RECENT DEVELOPMENTS

While the Whitehead Problem is of no direct importance for the Realization Problem, the techniques developed by Shelah in his 1974 paper (and subsequently extended by him) have become the major tool for tackling the problem. The following results indicate some of the many recent advances made:

1. (ZFC + (V=L)). Every cotorsion-free ring is an endomorphism ring. (Dugas and Gobel, 1981).

(A group is cotorsion-free if it is torsion-free, reduced and contains no copy of  $J_p$ , for any  $p$ .)

2. (ZFC). If  $A$  is any algebra over a complete discrete valuation ring  $R$  then there exists a  $R$ -module  $G$  having  $A$  as its "essential" endomorphism ring (Dugas, Gobel and Goldsmith, 1982).
3. (ZFC). Every cotorsion-free algebra is an endomorphism algebra (Dugas and Gobel, 1982).

The state of the art for the Realization Problem (in 1984) has been very elegantly presented in a unified approach by Corner and Gobel [2]. Their results are based on a combinatorial technique devised by Shelah. In very recent work, Dugas and Gobel and Gobel and Goldsmith have established (in  $V = L$ ) that most realizations can be obtained in classes of groups which are almost free (in the sense that all subgroups of cardinality less than the cardinal of the realizing group are free). Some of the results so obtained are undecidable in ZFC.

#### CONCLUDING REMARKS

One of the principal objectives in writing this paper is to convince non-logicians that set and model theory will have a role in our subjects once we deal with any uncountable structure. (Since  $\aleph_1$  is uncountable that takes in most of us!) This impact is perhaps most apparent in Abelian group theory but the reason for this is clear - finite Abelian groups are completely classified being direct sums of cyclic groups. However other areas of algebra, topology and analysis will slowly but surely become involved also.

#### REFERENCES

1. CORNER, A.L.S.  
"Every Countable Reduced Torsion-Free Ring is an Endomorphism Ring", *Proc. London Math. Soc.*, (3) 13 (1963) 867-710.

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