VISCOELASTIC BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

Linear viscoelastic materials are described by the hered-itary constitutive relations (repeated suffix notation under-stood)

$$\sigma_{ij}(\underline{r},t) = 2\int_{-\infty}^{t} dt'\mu(t-t')\varepsilon_{ij}(\underline{r},t')$$

$$+ \int_{-\infty}^{t} dt'\lambda(t-t')\varepsilon_{kk}(\underline{r},t'), \qquad i,j = 1,2,3 \qquad (1.1)$$

in terms of the stress and strain tensors σ_{ij}, ε_{ij} at positio[n] $\underline{r} = (x,y,z) = (x_1,x_2,x_3)$ and time t, and the singular visco-elastic functions $\mu(t)$ and $\lambda(t)$, both zero for negative time in order to incorporate Causality. These are related to the relaxation functions for shear and volume deformations. In particular

$$\mu(t) = \frac{d}{dt}(G(t)H(t)) \qquad (1.2)$$

where H(t) is the Heaviside step function and G(t) is the she[ar] relaxation function, approximated perhaps by a constant plus exponential decay terms. If one exponential is sufficient, the material is referred to as a standard linear solid. A similar relation exists between $(2\mu(t) + 3\lambda(t))/3$ and the bul[k] relaxation function. This latter quantity may be taken to be a constant, or alternatively, proportional to G(t), for many materials. Equation (1.1) is combined with the dynamical equations

$$\sigma_{ij,j}(\underline{r},t) + \frac{\partial^2}{\partial t^2}u_i(\underline{r},t) = 0, \qquad i = 1,2,3 \qquad (1.3)$$

where $u_i(\underline{r},t)$ are the displacements. It is sometimes possibl[e] to neglect the acceleration term in (1.3). This non-inertial approximation will be adopted henceforth. In a body occupyin[g] volume V with boundary B, the boundary conditions may for example take the form

$$u_i(\underline{r},t) = d_i(\underline{r},t), \qquad \underline{r} \in B_u(t)$$

$$\sigma_{ij}(\underline{r},t)n_j(\underline{r}) = c_i(\underline{r},t), \qquad \underline{r} \in B_\sigma(t) \qquad (1.4)$$

$$B_u(t) \cup B_\sigma(t) = B$$

where $c_i(\underline{r},t)$ and $d_i(\underline{r},t)$ are specified functions. On taking the time Fourier transform of (1.1), the hereditary integrals become products and in fact both (1.1) and (1.3) reduce to the elastic form with Lamé's constants replaced by the so-called complex moduli. If the boundary regions B_u and B_σ are constant, this observation allows one to reduce any prob-lem to the corresponding one in Elasticity. This is the con-tent of the Classical Correspondence Principle.

Many interesting problems are however not in this cat-egory, for example those involving loads moving over a half-space; or the Normal Contact Problem where the load is stat-ionary but varying in magnitude; and also extending or closing crack problems. The basic complication is the following: if a displacement, for example, is known at time t on B_u, it does not follow that it is known at all previous times, so an hered-itary integral over this quantity is not necessarily known. An exception to this would be if B_u is non-increasing with time, since if a point is in B_u at time t, it was in it at all previous times. Elaborations on this observation allow certain extensions of the Classical Correspondence Principle (Graham [1] and references therein).

There are many problems however where the boundary regions vary in quite a complicated manner, for example exhibiting consecutive maxima and minima. A method for tackling such problems, involving a certain decomposition of hereditary integrals, was developed a long time ago by Graham [2] and Ting [3]. The main point of the present note is to give an alternative derivation of this decomposition, recently evolved and applied by Cecil Graham and myself, and to give a simple illustration of its use. This derivation leads to a form which appears to be easier to manipulate in certain contexts.

2. DECOMPOSITION OF HEREDITARY INTEGRALS

Let the two functions $u(t)$ and $v(t)$ be related by

$$v(t) = \int_{-\infty}^{t} dt'\ell(t-t')u(t')$$
$$u(t) = \int_{-\infty}^{t} dt'k(t-t')v(t') \qquad (2.1)$$

where $k(t)$ and $\ell(t)$, both zero for negative t, are inverses of each other in the sense that

$$\int_0^t dt'\ell(t-t')k(t') = \int_0^t dt'k(t-t')\ell(t') = \delta(t) \qquad (2.2)$$

in terms of the singular delta function $\delta(t)$. Let $\theta(t)$ be the set of the present and all past times $(-\infty,t]$, which we decompose into two sets $W_u(t)$, $W_v(t)$ where $u(t')$ is given for $t' \in W_u(t)$ and $v(t')$ is given for $t' \in W_v(t)$. In certain applications $v(t')$ may not be known on $W_v(t)$ but can be usefully represented. If we could decompose $v(t)$, for example, as follows

$$v(t) = \int_{W_u(t)} dt'\Pi_u(t,t')u(t') + \int_{W_v(t)} dt'\Pi_v(t,t')v(t') \qquad (2.3)$$

then everything on the right-hand side is known, provided that the sets $W_u(t)$ and $W_v(t)$ can be specified, so that $v(t)$ is given explicitly. A decomposition of this kind can be derived in the following manner. We first define the sets $W_u(t)$ and $W_v(t)$. Let $t_1,t_2,t_3,\ldots$ be the sequence of transition times earlier than t, marking when t' changes from $W_u(t)$ to $W_v(t)$ or vice-versa. We take it that $[t_1,t] \in W_u(t)$ since otherwise $t \in W_v(t)$ and $v(t)$ is known to begin with. Let us write $v(t)$ as

$$v(t) = \int_{t_1}^{t} dt'\ell(t'-t)u(t') + \int_{-\infty}^{t_1} dt'\ell(t-t')u(t')$$
$$= \int_{t_1}^{t} dt'\ell(t-t')u(t') + \int_{-\infty}^{t_1} dt'T_1(t,t')v(t') \qquad (2.4)$$

where

$$T_1(t,t') = \int_{t'}^{t} dt'\ell(t-t'')k(t''-t') \qquad (2.5)$$

from (2.1). The procedure can be repeated to obtain finally

$$\Pi_u(t,t') = T_0(t,t')R(t';t_1,t) + T_2(t,t')R(t';t_3,t_2) + T_4(t,t')R(t';t_5,t_4) + \ldots \qquad (2.6)$$
$$\Pi_v(t,t') = T_1(t,t')R(t';t_2,t_1) + T_3(t,t')R(t';t_4,t_3) + \ldots$$

where

$$R(t;t_2,t_1) = \begin{cases} 1, & t \in [t_2,t_1] \\ 0, & t \notin [t_2,t_1] \end{cases} \qquad (2.7)$$

for all t_2, t_1 and t. Also

$$T_0(t,t') = \ell(t-t') = \int_{t'}^{t_r} dt''T_{r-1}(t,t'')\ell(t''-t'), \qquad r \text{ even}$$
$$T_r(t,t') = \int_{t'}^{t_r} dt''T_{r-1}(t,t'')k(t''-t'), \qquad r \text{ odd} \qquad (2.8)$$

The number of terms in these series depends on the number of transition times t_r. If t_n is the final transition time, then where it is the upper bound in an integral, the lower bound is $-\infty$. In a similar manner, it can be shown that $u(t)$ can be decomposed in the form

$$u(t) = \int_{W_u(t)} dt'\Gamma_u(t,t')u(t') + \int_{W_v(t)} dt'\Gamma_v(t,t')v(t') \qquad (2.9)$$

where

$$\Gamma_v(t,t') = N_0(t,t')R(t';t_1,t) + N_2(t,t')R(t';t_3,t_2) + N_4(t,t')R(t';t_5,t_4) + \ldots \qquad (2.10)$$

$$\Gamma_u(t,t') = N_1(t,t')R(t';t_2,t_1) + N_3(t,t')R(t';t_4,t_3)$$
$$+ \ldots$$

where the quantities $N_r(t,t')$ are given by formulae akin to (2.8) but with $\ell(t)$ and $k(t)$ interchanged.

This apparently trite formalism is actually extremely powerful in the context of non-inertial boundary value problems. The decomposition was developed as mentioned to solve the Normal Contact Problem (2.3) and applied more recently to the steady state case [4]. A form of it is the fundamental ingredient required to write down an integral equation for moving load problems (a special case of which was derived some time ago [5,6]) and perhaps more general problems also. It arises also in the case of crack problems in a manner which we shall now discuss.

3. CLOSING CRACK PROBLEM

Consider a fixed length crack lying along the x-axis occupying the region $[-c,c]$, in an infinite viscoelastic medium. Let there be a constant pressure $p(t)$ acting on both faces while the crack is open, where $p(t)$ may change sign. In an elastic medium, such a sign change from positive to negative would lead to instant closure. This is not the case in a viscoelastic medium, which makes the problem non-trivial and leads to certain interesting effects. While the crack is open, this problem is one covered by the Classical Correspondence Principle and the solution may be immediately written down since the elastic solution is known [7]. In particular, we have that the gap is given by

$$g(x,t) = 2m(x)q(t) \tag{3.1}$$

where

$$m(x) = (c^2 - x^2)^{\frac{1}{2}}$$
$$q(t) = \int_{-\infty}^{t} dt'k(t-t')p(t')$$

The quantity $k(t)$, zero for negative t, is defined by the fact that its Fourier transform $\hat{k}(\omega)$ is given by

$$\hat{k}(\omega) = (1-\hat{\nu}(\omega))/\hat{\mu}(\omega) \tag{3.3}$$

where $\hat{\nu}(\omega)$ is a generalised Poisson's ratio of the material, expressible in terms of the complex moduli $\hat{\mu}(\omega)$ and $\hat{\lambda}(\omega)$ according to the standard formula. The stress intensity factor has the form

$$K_1 = c^{\frac{1}{2}}p(t) \tag{3.4}$$

Since the crack may be open while $p(t)$ is zero or negative, this quantity may also be zero or negative, in contrast to the elastic case. When the quantity $q(t)$ becomes zero, the crack closes and the pressure on $[-c,c]$ is no longer known. Thus, let $p(t)$ be denoted by $p_o(t)$, a known quantity, when the crack is open; and by $p_c(t)$ when the crack is closed. Note however that $q(t)$ is known when the crack is closed. It is in fact zero. But this is precisely the situation we were dealing with above, when deriving the decomposition. From (2.6) and (2.20) we can immediately write down explicit forms for $p_c(t)$ and $q(t)$:

$$p_c(t) = \sum_{r=1,3,5..} \int_{t_{r+1}}^{t_r} dt'T_r(t,t')p_o(t')$$
$$q(t) = \sum_{r=0,2,4..} \int_{t_{r+1}}^{t_r} dt'N_r(t,t')p_o(t') \tag{3.5}$$

the second equation referring to times when the crack is open. If the crack is closed, the condition for the next time of reopening is

$$p_c(t_0) = p_o(t_0) \tag{3.6}$$

while if the crack is open at time t, the condition for the time of next closing is

$$q(t_c) = 0 \tag{3.7}$$

These latter two equations may be used in conjunction with (3.5) to inductively determine the times of opening and clo The situation simplifies if steady state conditions under a periodic load are assumed. Thus (3.5) - (3.7) constitute complete solution to the problem, in principle, and they ar a simple application of the decomposition derived above. This problem has in fact been solved for the special case of standard linear solid under a sinusoidal load in [8,9] by means of a less general machinery, and detailed results were obtained for the case of a standard linear model. It may b shown, using the explicit forms for (2.6) and (2.10) given i [4], that the general formulae (3.5) - (3.7) reduce to the results obtained in that paper.

Applications of the study of viscoelastic boundary valu problems include the exploration of the phenomenon of hyster etic friction which can be modelled by considering loads mov over viscoelastic half-spaces. This effect may be signific in many contexts, notably that of a tyre skidding on a road surface. Normal contact problems are relevant to the study of impact phenomena, while crack problems contribute insights to fracture processes in real materials.

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