NUMERICAL METHODS FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

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A singular perturbation problem is a problem which depends on a parameter (or parameters) in such a way that solutions of the problem behave nonuniformly as the parameter tends toward some limiting value of interest. Such singular perturbation problems involving differential equations arise in many areas of interest, e.g. modelling of semi-conductor devices, aerodynamics, fluid mechanics, thin shells. We illustrate some of the nonuniformities that occur with some simple prototypes.

Example 1 This example deserves a prize as one of the most commonly used "first examples" in the literature.

$$\varepsilon y'' + y' = 0 \quad 0 < x < 1 \quad \varepsilon > 0$$

$$y(0) = 0 \quad y(1) = 1$$

Defining the limit function $\bar{y}(x) = \lim_{\varepsilon \to 0} y(x)$ we see that

$$\bar{y}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

which is discontinuous at $x = 0$. Thus

$$\lim_{x \to 0} \lim_{\varepsilon \to 0} y(x) \neq \lim_{\varepsilon \to 0} \lim_{x \to 0} y(x)$$

The solution of the reduced problem (obtained by setting $\varepsilon = 0$ and omitting the boundary condition at $x = 0$) is $y_0(x)=1$. For small ε, y is close (in the max norm) to y_0 except in a small neighbourhood of $x = 0$, called the "boundary layer" - because of a mathematical analogy with the boundary layers of fluid dynamics. The boundary layers in the flow of fluids

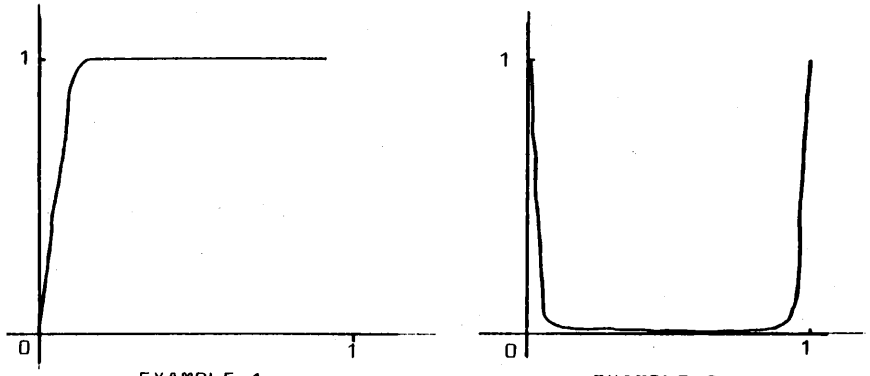
of low viscosity are narrow regions near certain parts of the boundary, where the flow velocity changes rapidly from zero at the boundary to values almost equal to those obtained for an inviscid flow.

Example 2 In a given singular perturbation problem more than one boundary layer can occur. This is illustrated by

$$\varepsilon y'' - y = 0 \quad 0 < x < 1 \quad \varepsilon < 0$$

$$y(0) = y(1) = 1$$

This problem has two boundary layers - one at each end of [0,1].



EXAMPLE 1 EXAMPLE 2

Example 3 The nonuniformity can also occur inside the domair

$$\varepsilon y'' + (y^2)' = 0 \quad -1 < x < 1 \quad \varepsilon > 0$$

$$y(-1) = -1 \quad y(1) = 1$$

The limit function $\bar{y}(x)$ is

$$\bar{y}(x) = \begin{cases} -1 & -1 < x < 0 \\ 0 & x = 0 \\ +1 & 0 < x < 1 \end{cases}$$

Thus $y(x)$ behaves nonuniformly at $x = 0$ and we say that there is a shock layer (or an interior transition layer) at the origin. The solution changes from concave to convex at this point - this change in convexity is one of the nasty features of shocks. In general, interior layers are a lot harder to wrestle with than boundary layers.

Example 4 The last example was a nonlinear problem. Shocks can also occur in linear problems.

$$\varepsilon y'' + 2xy' = 0 \qquad -1 < x < 1 \qquad \varepsilon > 0$$

$$y(-1) = -1 \quad y(1) = 1$$

The solution (as in Example 3) has a shock at $x = 0$. Linear problems where one of the coefficients has a zero inside the domain are called turning-point problems.

Example 5 The solution of a problem may behave uniformly throughout the domain, but its derivative can behave non-uniformly.

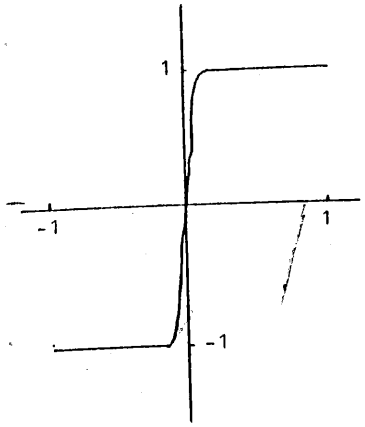
$$\varepsilon y'' + xy' - y = 0 \qquad -1 < x < 1 \qquad \varepsilon > 0$$

$$y(-1) = 1 \quad y(1) = 2$$

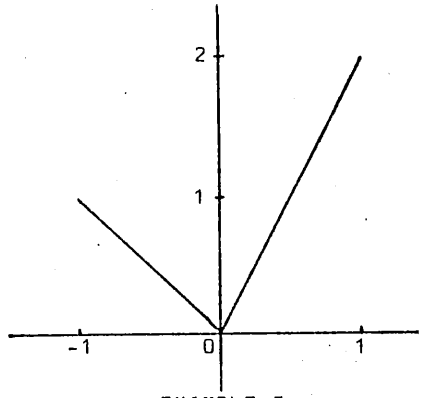
The limit function $\bar{y}(x)$ is

$$\bar{y}(x) = \begin{cases} -x & \text{for } x \leq 0 \\ 2x & \text{for } x > 0 \end{cases}$$

The transition between these two curves takes place in a corner layer near the origin.



EXAMPLE 3



EXAMPLE 5

Example 6 It is common practice to use the term 'singular perturbation problem' when referring to differential equations where the nonuniformity appears due to the loss of order in the reduced (i.e. $\varepsilon = 0$) differential equation. However, non-uniformities can occur in other ways.

$$(x + \varepsilon)^2 y' + \varepsilon = 0 \qquad x > 0 \qquad \varepsilon > 0$$

$$y(0) = 1$$

The exact solution is $y(x) = \varepsilon/(x + \varepsilon)$. Thus

$$\lim_{\varepsilon \to 0} y(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Usually, as $\varepsilon \to 0$, the reduced solution and the exact solution are close outside a small layer region. In this example the opposite occurs.

There are two main approaches to solving differential equations numerically:

(1) Finite Difference Methods

In one dimension, divide the interval [a,b] into N subintervals

$$a = x_0 < x_1 < \ldots\ldots\ldots < x_N = b$$

Replace y and its derivatives in the differential equation by suitable (difference) approximations

e.g. replace $y'(x_j)$ by $(u_{j+1} - u_j)/(x_{j+1} - x_j)$

and then replace the coefficients of the derivatives by an appropriate approximation.

e.g. on $[x_j, x_{j+1}]$ replace $a(x)$ by $a(x_j)$ or $a(x_{j+1})$

A system of algebraic equations is then solved to generate a set of points $\{u_j\}$ as an approximation to the set $\{y(x_j)\}$.

(2) Finite Element Methods

A function u(x) is generated by discretizing a weak form of the differential equation. This function approximates the solution y(x) globally.

In this note we will confine the discussion to finite difference methods.

Classical numerical methods perform badly (to say the least) when applied to singularly perturbed problems. In particular, their atrocious behaviour is most noticeable in non self-adjoint problems.

Consider Example 1 again. The solution of this is

$$y(x) = (1 - \exp(-x/\varepsilon))/(1 - \exp(-1/\varepsilon))$$

A classical finite difference scheme on a uniform mesh (i.e. $x_{j+1} - x_j = h$ for all j) for this problem would be

$$\frac{\varepsilon(u_{j+1} - 2u_j + u_{j-1})}{h^2} + \frac{u_{j+1} - u_{j-1}}{2h} = 0 \quad j=1,2,..,N-1$$

$$u_0 = 0 \qquad u_N = 1 \qquad h = 1/N$$

The solution of this difference scheme is

$$u_j = (1 - \lambda^j)/(1 - \lambda^N) \quad \text{where} \quad \lambda = (1 - h/2\varepsilon)/(1 + h/2\varepsilon).$$

It can be shown that:

(a) If N is odd (i.e. divide the interval into an odd number of subintervals) and

(i) if j is even then $u_j \to 0$ as $\varepsilon \to 0$;
(ii) if j is odd then $u_j \to 1$ as $\varepsilon \to 0$.

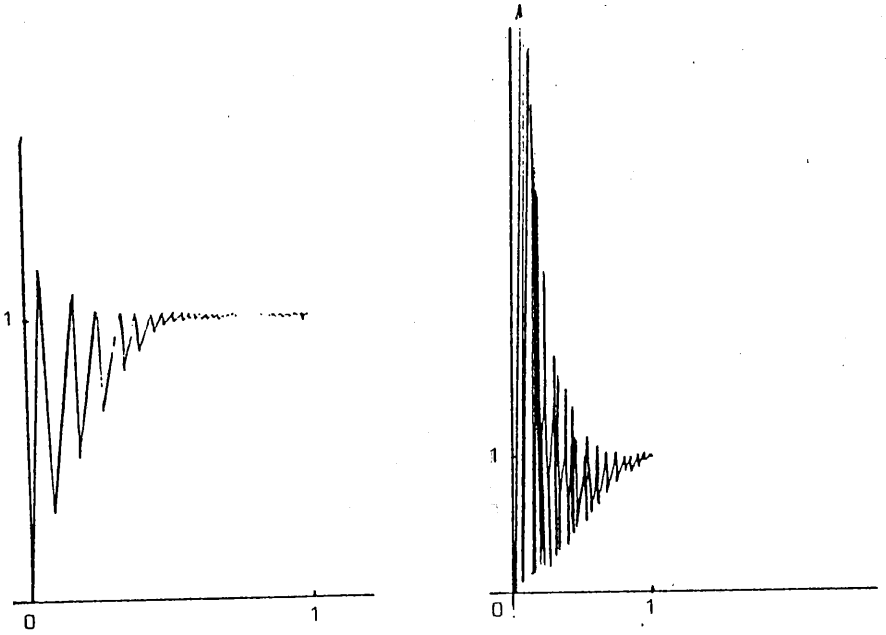
This results in a bounded oscillation between odd and even nodes.

(b) If N is even and

(i) if j is even then $u_j \to j/N$ as $\varepsilon \to 0$;
(ii) if j is odd then $u_j \to \infty$ as $\varepsilon \to 0$.

In this case, the odd/even separation is even more disastrous. The oscillations grow as $\varepsilon \to 0$.

These wild oscillations or "wiggles" (engineering jargon) also occur when classical finite element methods are used. Engineers working with "real-life problems" were the first to notice these wiggles. Their first concern was to somehow get rid of the oscillations. The problem is most noticeable when the ratio $h/2\varepsilon > 1$ (the above formula for u_j has problems when $h/2\varepsilon = 1$). Initially, they simply reduced h (i.e. increased the number of mesh points) in order to keep the ratio $h/2\varepsilon < 1$. However, for 'small' values of ε and in higher dimensions this restriction on the mesh size became too expensive. Their next idea was to use a nonuniform mesh - using a finer mesh in layer regions. This still placed restrictions on the mesh size.



CASE (a) CASE (b)

Then they hit upon 'upwinding'. This involves taking a suitable difference approximation to the first derivative:

if the layer is to the left, replace $y'(x_j)$ by $(u_{j+1}-u_j)/h$;

if the layer is to the right, replace $y'(x_j)$ by $(u_j-u_{j-1})/h$.

For Example 1, the upwind difference scheme is:

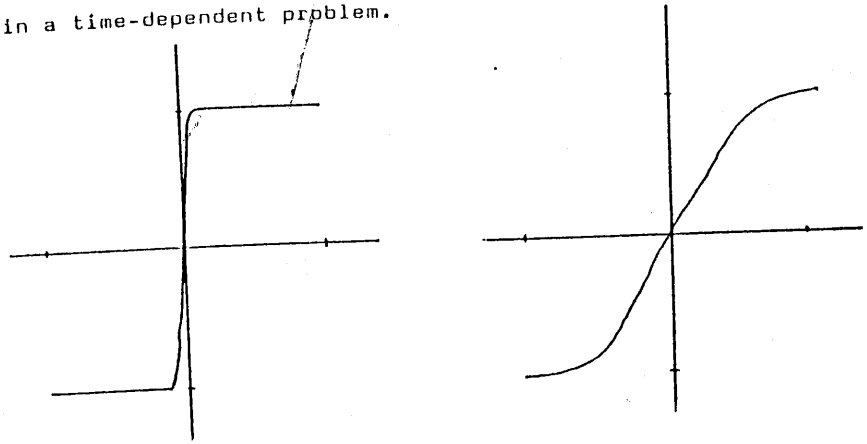
$$\frac{\varepsilon(u_{j+1} - 2u_j + u_{j-1})}{h^2} + \frac{u_{j+1} - u_j}{h} = 0 \quad j=1,2,\ldots,N-1$$

The solution of this difference scheme is

$$u_j = (1 - \tau^{-j})/(1 - \tau^{-N}) \text{ where } \tau = 1 + h/\varepsilon \quad j=0,1,2,..,N$$

As $\varepsilon \to 0$ $u_j \to 1$ for all $j \neq 0$, and $u_0 = 0$.

Upwinding does remove the wiggles and for small values of ε it appears to do the job, but for large ε it is not as accurate as classical difference schemes. Upwinding also tends to "smear" the abrupt change in a shock and makes it difficult to locate ("track") the position of a moving shock in a time-dependent problem.



TRUE UPWINDING

Upwinding is an improvement, but it still does not solve the problem.

We would like to find a difference scheme (on a uniform mesh) having the property that its solution $\{u_j\}$ is such that for all $j \geq 0$

$$|y(x_j) - u_j| \leq Ch^p$$

where $p > 0$ and C is a constant, both independent of j, h and ε. If we have such a difference scheme, then we say that its solution converges to the solution of the continuous problem uniformly in ε with order p. Upwinding does not converge uniformly.

In 1955, Allen and Southwell proposed a new method for a problem in fluid mechanics, based essentially on the form of the exact difference scheme for a constant coefficient

problem in one dimension. In 1969, the Russian A.M. Il'in examined the problem

$$\varepsilon y'' + a(x)y' = f(x) \qquad 0 < x < 1$$

$$y(0),\ y(1) \text{ given} \qquad a(x) \geq \alpha > 0$$

$$a(x),\ f(x) \text{ sufficiently smooth}$$

and showed that a difference scheme similar to Allen and Southwell's converged uniformly in ε with order one. In the 1970s, more uniformly accurate difference schemes were found. In recent times (1980-1985), three point difference schemes which are uniformly second accurate have been appearing. All these difference schemes, which are based essentially on being exact for constant coefficients, are called exponentially-fitted difference schemes (or "smart upwinding" by the engineers). Thus, apart from a few loose ends, good numerical methods for singularly perturbed differential equations in one dimension (which are linear and without turning-points) exist and the area seems to be all sown up. BUT what about two dimensions? Here there are no significant results whatsoever - either analytically or numerically. Since exponentially fitted difference schemes rely on being able to solve the constant coefficient analog of the problem, no extension to higher dimensions has yet been found. In two dimensions, there is a bottomless pit of viciously hard singularly perturbed problems. In one dimension, problems which exhibit shock behaviour have not been satisfactorily dealt with yet, and as for non-linear problems - well, it probably won't be till after the year 2001 that they'll be looked at seriously!!

REFERENCES

A fairly comprehensive analysis of the 'state of the art' up to 1980 can be found in [1]. The three conferences in [2] present some of the more recent results in the area. The papers [3] and [4] are considered to be the first major landmarks in this field. Asymptotic expansions, existence and uniqueness, bounds on the solution and its derivatives ... for the continuous problem are discussed in [5] - [11]. Numerical results for singularly perturbed problems may be found (for example) in the papers [12] - [14].

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ASPECTS OF HIPPARCOS

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1. INTRODUCTION

Some aspects of the HIPPARCOS space astrometry mission are presented in this article. The objectives of the mission and its broad principles of operation are described in Section 2. In Section 3, system level analyses of the mission, to which our company has contributed over the past few years, are outlined. Finally, in Section 4, as an illustration of the work, a specific, relatively simple, problem is discussed.

3. HIPPARCOS

HIPPARCOS is a space astrometry mission, sponsored by the European Space Agency (ESA), which is scheduled for launch in 1988. Its objective is to measure the astrometric parameters (positions, proper motions, trigonometric parallaxes) of about 100,000 pre-selected stars to a (very high) accuracy of 0.002 arcseconds.

The basic principle of measurement is to scan, continuously and systematically, the entire sky with a telescope capable of accurately measuring the angles between stars separated by a large angle. It is possible, by numerically combining several millions of such angular measurements, to derive the required astrometric parameters. The period of data collection is to be $2\frac{1}{2}$ years.

The telescope is equipped with two fields of view (FOVs) to enable measurement of the angles between widely separated stars. Each FOV is of dimension $0.9^\circ \times 0.9^\circ$. The angle between the FOVs, called the *basic angle*, is denoted by $\gamma = 58^\circ$