IRISH MATHEMATICAL SOCIETY

BULLETIN

THE IRISH MATHEMATICAL SOCIETY

President: *Prof. M. Newell* — Department of Mathematics, University College, Galway.

Vice-President: *Prof. S. Dineen* — Department of Mathematics, University College, Dublin.

Secretary: *Dr R. Timoney* — Department of Mathematics, Trinity College, Dublin.

Treasurer: *Dr G.M. Enright* — Department of Mathematics, Mary Immaculate College, Limerick.

Committee Members: *Drs N. Buttimore, P. Fitzpatrick, F.G. Gaines, B. Goldsmith, R. Critchley, P. McGill; Profs S. Tobin and T.T. West.*

CAL REPRESENTATIVES

Cork:	U.C.C.	*Dr G. Kelly (Statistics)*
	R.T.C.	*Mr D. Flannery*
Dublin:	Carysfort	*Dr J. Cosgrove*
	D.I.A.S.	*Prof. J. Lewis*
	Kevin St	*Dr B. Goldsmith*
	N.I.H.E.	*Dr M. Clancy*
	T.C.D.	*Dr R. Timoney*
	U.C.D.	*Dr P. Boland*
		Dr E. O'Riordan
Dundalk:	R.T.C.	*Dr J. Hannah*
Galway:	U.C.G.	*Dr G. Enright*
Limerick:	M.I.C.E.	*Dr R. Critchley*
	N.I.H.E.	*Mr J. Leahy*
	Thomond	*Prof. A. O'Farrell*
Maynooth:		*Mr T. Power*
Waterford:		

BULLETIN

EDITOR

Patrick Fitzpatrick

The aim of the *Bulletin* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

ASSOCIATE EDITOR

Martin Stynes

The *Bulletin* seeks articles of mathematical interest written in an expository manner. All areas of mathematics are welcome, pure and applied, old and new.

Detailed instructions relating to the preparation of manuscripts may be found on the inside back cover.

Correspondence relating to the *Bulletin* should be sent to:

Irish Mathematical Society Bulletin,
Department of Mathematics,
University College,
Cork.

CONTENTS

EDITORIAL

At the ordinary meeting of 20 December 1985 the Society decided to change the name of the Newsletter to the Irish Mathematical Society Bulletin. This move reflects a widespread perception that the former title did not do justice to the substance of the publication which has been developed over the past few years. It was also decided to continue the present numbering so that this issue maintains the sequence as number 16.

Perhaps a more significant change in circumstances arises as a result of a review by the NBST of its printing arrangements. In discussion with the Society it was agreed that the Board would print two issues in 1986, but they did not give any firm commitment to printing further issues. It should be observed that the NBST prints the Newsletter/Bulletin **free of charge** to the Society. The Society Membership Fee does not cover any of the running costs apart from typing. Therefore, any attempt to make the publication self-financing would involve the introduction of a subscription fee at a level determined by the cost of commercial printing.

We bring these points to the attention of our readers in order to stimulate a debate on the role of the Newsletter/Bulletin in the activities of the IMS, the question of financing and continuation, the frequency of publication and so on. Comments and contributions in the form of letters to the Editor would be welcome.

The deadline for copy for the September issue is 30 June 1986.

Pat Fitzpatrick

IRISH MATHEMATICAL SOCIETY

Committee Meeting, 12.15, 19th December, 1985, in 39 TCD

Present: M. Newell (President, in the Chair), N. Buttimore, P. Fitzpatrick, B. Goldsmith, R. Timoney (pp. A. O'Farrell).

Apologies: S. Dineen, G. Enright, F. Gaines, M. Stynes, S. Tobin, T. West.

1. The Treasurer's report (for the period 1st October, 1984 to 30th September, 1985) (audited by R. Critchley and J. Leahy) was approved.

2. The Vice-President, S. Dineen, negotiated with the NBST for continued printing of the Newsletter. As a result the NBST has agreed to print two further issues in 1986. It was agreed to investigate possible sources of support for printing the Newsletter such as an NBST cash grant, commercial sponsorship. Also the possibility of whether the Newsletter could be printed (for a fee) by the UCD printing facility or with the aid of the Printing School in the Dublin Institute of Technology, Bolton Street. P. Fitzpatrick explained that the job of printing the Newsletter was considered to be too small by most commercial printing houses but that the binding could be done commercially by a company in Dundalk.

3. The committee noted a report prepared by T. West on the joint meeting with the London Mathematical Society which is to take place in March, 1986. The speakers are W.B. Arveson (Berkeley), A. Connes (College de France), R.G. Douglas (Stonybrook) and E.C. Lance (Leeds). Total costs were estimated to amount to £2,430 of which the LMS has agreed to provide Stg £1,500 and the Royal Irish Academy Mathematical Symposium fund IR£250. The committee agreed that the balance of £380 would be provided by the Society, as requested.

4. In addition the committee agreed to the following expenditure on conferences:

Galway Groups (May, 1986) £100 + guarantee of £50;

Cork Analysis (May, 1986) £ 70 + guarantee of £30.

5. The committee noted a report from A. Seda on a meeting of the European Mathematical Council Data Base Committee. The committee welcomed the contribution being made by Dr Seda to the EMCDBC on behalf of the IMS and it was agreed to endorse strongly his application to the NBST for financial support.

6. In response to a request from M. Atiyah for the IMS to nominate a representative to attend an EMS meeting in Liblice, Czechoslovakia (in November, 1986) it was agreed to enquire whether the EMC could offer travel support to an IMS representative.

7. It was agreed, in response to a suggestion by the Treasurer, that lapsed members (as defined by the rules) should be charged a £10 re-instatement fee.

8. P. Fitzpatrick suggested that the overseas membership fee of £5.50 (or its equivalent) was too low.

9. It was agreed to write to the American Mathematical Society to enquire into the possibility of reciprocity of membership.

IRISH MATHEMATICAL SOCIETY

Ordinary Meeting, Friday December 20th, 1985

at

Dublin Institute for Advanced Studies

12.15 pm

The President, M. Newell took the chair and there were 14 members present.

1. The minutes of the previous ordinary meeting of April 14th, 1985, which had appeared in the October Newsletter, were agreed.

2. The Treasurer, G. Enright, presented his report which had been audited by R. Critchley and J. Leahy.

3. The Treasurer mentioned that 18 members of the Society had applied for reciprocal membership of the IMTA so far. The Treasurer raised the question of how much lapsed members should be required to pay if they wished to be re-admitted into the Society. There was considerable discussion on this point but no clear consensus was reached.

4. R. Timoney reported briefly on his activities on behalf of the Secretary, A.G. O'Farrell.

5. S. Dineen reported on discussions he had with the NBST concerning the printing of the Newsletter. These resulted in the NBST agreeing to print two issues of the Newsletter in 1986.

6. There was some discussion on whether it would be reasonable to publish the Newsletter only twice annually. The possibility of mailing the Newsletter to members individually was raised. It was agreed that a more

appropriate title for the Newsletter would be "Bulletin".

7. R. Timoney presented a report drawn up by T. West on the forthcoming joint meeting with the London Mathematical Society (on Friday afternoon/Saturday morning, March 21-22, 1986). The speakers are W. Arveson (Berkeley), A. Connes (College de France), R. Douglas (Stonybrook) and E. Lance (Leeds). The lectures will take place at Trinity College, Dublin. The Royal Irish Academy will host a Hamilton Exhibition and a reception on the Friday evening and that will be followed by a conference dinner in the Kildare Street and University Club. The total cost is estimated to be £2,430 of which the London Mathematical Society has agreed to contribute £1,500 sterling and the Royal Irish Academy Mathematical Symposium Fund £250. The balance of approximately £380 will be met by the Society.

8. The following were nominated, seconded and elected unopposed (for two-year terms):

Treasurer	-	G. Enright
Secretary	-	R. Timoney
Committee	-	R. Critchley
		P. Fitzpatrick
		P. McGill
		S. Tobin

9. Professor Maurice Kennedy was elected as the first honorary member of the Society.

IRISH MATHEMATICAL SOCIETY

Membership List Supplement 86-1

Compiled January 27th, 1986

Amendments;

85005 Fitzgerald, Mr G.C. (ex RTC, Cork)
Statistical Software Ltd, Farm Centre, Wilton, Cork.

85126 Rahman, Dr M.A. (NIHE, Limerick) Electronics Dept.

85175 Seifert, Dr B. (ex Paris)
Corpus Christi College, Oxford.

Additions:

86182	Harte,	Dr J.	RTC, Dundalk.
86183	O'Reilly,	Dr M.	RTC, Dundalk.
86184	Corbett,	Mr B.	RTC, Waterford.
86185	Higgs,	Dr R.J.	Mathematics Department, UCD.
86186	Burns,	Dr J.	St Patrick's, Maynooth.
86196	Brennan,	Mr M.	RTC, Waterford.

College of Technology, Kevin Street

86187	Gaffney,	Mr T.	86190	O'Shea,	Dr B.
86188	McCarthy,	Mr D.J.	86191	Tuite,	Dr M.J.
86189	O'Gallchobhair	Mr P.			

NIHE, Dublin

86192	Burzlaff,	Dr J.	86194	Lendach,	Dr B.
86193	Carroll,	Dr J.	86195	Murphy,	Dr A.

PERSONAL ITEMS

Dr Eugene O'Riordan of the Mathematics Department, Dundalk RTC, has been invited to be a keynote speaker at BAIL IV (4th International Conference on Boundary and Interior Layers) in Novosibirsk, USSR, 7-11 July 1986.

Dr Radoslav Dimitric (Belgrade) has recently taken up a Department of Education Postdoctoral Fellowship at the DIT, Kevin Street. Dr Dimitric was a student of Laszlo Fuchs in Tulane and works in the area of Abelian groups and module theory

****

OBITUARY

Professor James R. Timoney

It was with surprise and great sadness that we learned of the death of Professor J.R. Timoney on 11th August 1985. After his retirement from UCD in December 1979, he often visited the College and at the end of last June, he attended the customary lunch for the Extern Examiner in Mathematics.

Dick Timoney was born in Belleek, Co. Fermanagh in 1909 and educated at St Macartan's College, Monahan, Blackrock College, Co. Dublin and UCD. He entered UCD in 1927 and in his first year studied both Science and Engineering, obtaining scholarships in both at the end of the year. He then had to choose which one to continue in and he chose to study Mathematical Science. He graduated with a first class honours BA in 1930. Professor P.G. Gormley graduated the same year, also with first class honours. Dick obtained a scholarship to study for his MA, which he obtained in 1931 with first class honours. He obtained a prize in the Travelling Studentship examination which, together with a scholarship from UCD, enabled him to spend the session 1931-32 in Edinburgh, where he worked under the supervision of Sir Edmund Whittaker. On returning to Dublin in 1932, he was appointed Assistant in Mathematics in UCD. His duties involved teaching both Pure Mathematics and Mathematical Physics. He was appointed Statutory Lecturer in 1937 and Professor of Mathematical Analysis in 1966, which position he held until his retirement.

As a teacher, Dick Timoney was very highly regarded by his students. His lectures were presented with clarity and enthusiasm and he managed to convey to his audience the beauty and elegance of the subject. He developed a very good rapport with his classes. Despite his heavy teaching load and multitudinous administrative duties, Dick always found time to help students individually and also to give additional exercise

classes. For example, the Third Science class - a favourite of his which he taught for many years - used to have their examinations in Autumn and it was not unusual to see him giving a series of tutorials to these students in August. His good relations with students extended outside the circle of those studying Mathematics and he was sought after by student officers to be the obligatory academic representative on various student committees. He was a member of countless committees of this type. As a result he was one of the best known members of staff in College.

Dick Timoney became Head of the Department of Mathematics on the death of P.G. Gormley in 1973 and served in this capacity until 1976. However, for many years before that, much of the day-to-day administration of the Department was done by him. During his period as Head, the Department expanded and flourished. He started the holding of department meetings, thus making it possible for all members of staff to take part in decision-making. His approach inspired a sense of loyalty, commitment and co-operation from his staff and worked very successfully. His judgement was particularly sound, not only on matters affecting the Department but in matters affecting UCD as a whole. He was forward-looking in his views and was happy to adapt courses and methods to changing needs.

Dick involved himself very much in the development of Mathematics in the country as a whole. The worldwide movement to reform school Mathematics started in the 1960s. Dick was one of the lecturers at the first course on "Modern Mathematics for Teachers" in UCD - later published in book form. He was a former member of the Irish Mathematics Teachers Association and was Chairman of the Dublin Branch for many years. He also made important contributions to the development of its Newsletter. He was a judge at the Aer Lingus Young Scientists' Exhibition for many years. Also in Third Level, he contributed greatly to the development of Mathematics in Ireland. His relationship with the mathematicians in the other NUI colleges and also in TCD was excellent. In the early 1960s, he and Heini Halberstam worked closely together to improve the status of Mathematics in Dublin, and the good relations thus established between the Departments in UCD and TCD have continued to prosper. Dick also served on numerous appointment boards in the Civil Service and third level colleges, where his sound judgement has proved a great asset to the country.

In the general life of UCD, Dick Timoney was a major figure. He was a member of the Academic Council for thirty years, having been co-opted in 1949 while still a lecturer. He was a member of the Governing Body from 1964 to January 1979 and of its Buildings Committee from 1976-79. He served the Governing Body as a member of several subcommittees established to resolve specific problems. Some of these problems were of a difficult and sensitive nature and Dick's diplomatic talents and genial manner made him an obvious choice to carry out these tasks. He was Chairman of the Junior Academic Staff Association in 1947-48 and President of the Employers Association in 1953-58. He served for many years on the Senate of the NUI, having been elected by convocation in 1964, 1972 and again in 1977. He had an abiding love for UCD and on his many visits there since retirement, displayed great interest in what was happening there.

Dick always maintained his enthusiasm for Mathematics. He was especially fond of "hard" complex analysis - zeta function, Riemann hypothesis etc. He was always happy to show his colleagues the shortest or most elegant way to solve a particular problem. In his article in this Newsletter, Number 3, he quotes a motto of one of his teachers, Professor McWeeney, on the best way to approach a mathematical problem: "if you attack it judiciously, it will come out in a line". This characterised his own approach also, the "line" usually involved some particularly neat trick.

Dick is survived by his widow Nora and their family of four, two sons and two daughters. The eldest, Richard, carries on the tradition and is a lecturer in Mathematics at TCD Dick was always interested in engineering as a hobby - his expertise with cars is legendary - and the other son David carries on this interest, being a lecturer in Mechanical Engineering at UCD. The eldest daughter, Nicola, is an economist and the younger, Norma, is a dentist.

On behalf of the mathematical community we wish to express our deep sorrow at Dick's passing. For all of us, it is a great loss; for Nora and his family, it is a particularly sad one and we can but offer our condolences.

T.J. Laffey, Stephen O'Brien

ORDINARY MEMBERSHIP

The Ordinary Membership subscription for the session 1985/1986 is IR5.00 per person. Payment is now overdue and should be forwarded to the Treasurer without further notice.

NUMERICAL METHODS FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

E. O'Riordan

A singular perturbation problem is a problem which depends on a parameter (or parameters) in such a way that solutions of the problem behave nonuniformly as the parameter tends toward some limiting value of interest. Such singular perturbation problems involving differential equations arise in many areas of interest, e.g. modelling of semi-conductor devices, aero-dynamics, fluid mechanics, thin shells. We illustrate some of the nonuniformities that occur with some simple prototypes.

Example 1 This example deserves a prize as one of the most commonly used "first examples" in the literature.

$$\varepsilon y'' + y' = 0 \quad 0 < x < 1 \quad \varepsilon > 0$$

$$y(0) = 0 \quad y(1) = 1$$

Defining the limit function $\bar{y}(x) = \lim_{\varepsilon\to 0} y(x)$ we see that

$$\bar{y}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

which is discontinuous at $x = 0$. Thus

$$\lim_{x\to 0}\lim_{\varepsilon\to 0} y(x) \neq \lim_{\varepsilon\to 0}\lim_{x\to 0} y(x)$$

The solution of the reduced problem (obtained by setting $\varepsilon = 0$ and omitting the boundary condition at $x = 0$) is $y_0(x)=1$. For small ε, y is close (in the max norm) to y_0 except in a small neighbourhood of $x = 0$, called the "boundary layer" - because of a mathematical analogy with the boundary layers of fluid dynamics. The boundary layers in the flow of fluids

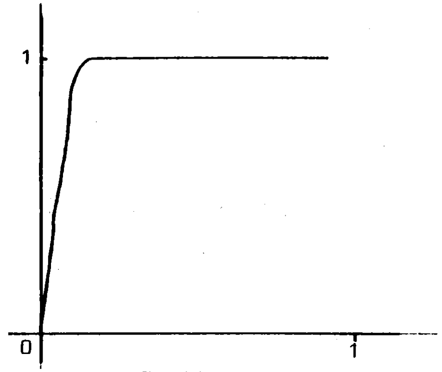
of low viscosity are narrow regions near certain parts of the boundary, where the flow velocity changes rapidly from zero at the boundary to values almost equal to those obtained for an inviscid flow.

Example 2 In a given singular perturbation problem more than one boundary layer can occur. This is illustrated by

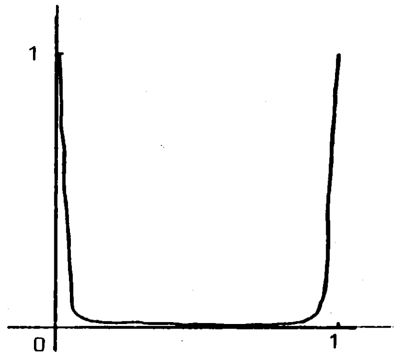
$$\varepsilon y'' - y = 0 \quad 0 < x < 1 \quad \varepsilon < 0$$

$$y(0) = y(1) = 1$$

This problem has two boundary layers - one at each end of [0,1].



EXAMPLE 1



EXAMPLE 2

Example 3 The nonuniformity can also occur inside the domair

$$\varepsilon y'' + (y^2)' = 0 \quad -1 < x < 1 \quad \varepsilon > 0$$

$$y(-1) = -1 \quad y(1) = 1$$

The limit function $\bar{y}(x)$ is

$$\bar{y}(x) = \begin{cases} -1 & -1 < x < 0 \\ 0 & x = 0 \\ +1 & 0 < x < 1 \end{cases}$$

Thus y(x) behaves nonuniformly at x = 0 and we say that there is a shock layer (or an interior transition layer) at the origin. The solution changes from concave to convex at this point - this change in convexity is one of the nasty features of shocks. In general, interior layers are a lot harder to wrestle with than boundary layers.

Example 4 The last example was a nonlinear problem. Shocks can also occur in linear problems.

$$\varepsilon y'' + 2xy' = 0 \qquad -1 < x < 1 \qquad \varepsilon > 0$$

$$y(-1) = -1 \quad y(1) = 1$$

The solution (as in Example 3) has a shock at x = 0. Linear problems where one of the coefficients has a zero inside the domain are called turning-point problems.

Example 5 The solution of a problem may behave uniformly throughout the domain, but its derivative can behave non-uniformly.

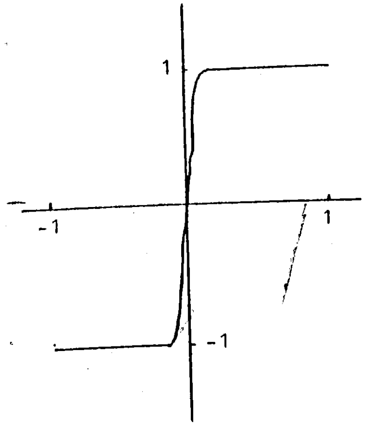
$$\varepsilon y'' + xy' - y = 0 \qquad -1 < x < 1 \qquad \varepsilon > 0$$

$$y(-1) = 1 \quad y(1) = 2$$

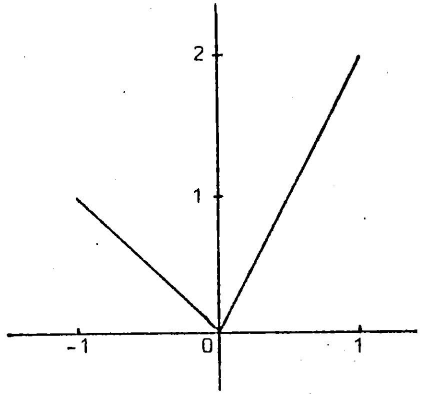
The limit function $\bar{y}(x)$ is

$$\bar{y}(x) = \begin{cases} -x & \text{for } x \le 0 \\ 2x & \text{for } x > 0 \end{cases}$$

The transition between these two curves takes place in a corner layer near the origin.



EXAMPLE 3



EXAMPLE 5

Example 6 It is common practice to use the term 'singular perturbation problem' when referring to differential equations where the nonuniformity appears due to the loss of order in the reduced (i.e. ε = 0) differential equation. However, non-uniformities can occur in other ways.

$$(x + \varepsilon)^2 y' + \varepsilon = 0 \qquad x > 0 \qquad \varepsilon > 0$$

$$y(0) = 1$$

The exact solution is $y(x) = \varepsilon/(x + \varepsilon)$. Thus

$$\lim_{\varepsilon \to 0} y(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Usually, as $\varepsilon \to 0$, the reduced solution and the exact solution are close outside a small layer region. In this example the opposite occurs.

There are two main approaches to solving differential equations numerically:

(1) Finite Difference Methods

In one dimension, divide the interval [a,b] into N sub-intervals

$$a = x_0 < x_1 < \dots\dots\dots < x_N = b$$

Replace y and its derivatives in the differential equation by suitable (difference) approximations

e.g. replace $y'(x_j)$ by $(u_{j+1} - u_j)/(x_{j+1} - x_j)$

and then replace the coefficients of the derivatives by an appropriate approximation.

e.g. on $[x_j, x_{j+1}]$ replace $a(x)$ by $a(x_j)$ or $a(x_{j+1})$

A system of algebraic equations is then solved to generate a set of points $\{u_j\}$ as an approximation to the set $\{y(x_j)\}$.

(2) Finite Element Methods

A function $u(x)$ is generated by discretizing a weak form of the differential equation. This function approximates the solution $y(x)$ globally.

In this note we will confine the discussion to finite difference methods.

Classical numerical methods perform badly (to say the least) when applied to singularly perturbed problems. In particular, their atrocious behaviour is most noticeable in non self-adjoint problems.

Consider Example 1 again. The solution of this is

$$y(x) = (1 - \exp(-x/\varepsilon))/(1 - \exp(-1/\varepsilon)$$

A classical finite difference scheme on a uniform mesh (i.e. $x_{j+1} - x_j = h$ for all j) for this problem would be

$$\frac{\varepsilon(u_{j+1} - 2u_j + u_{j-1})}{h^2} + \frac{u_{j+1} - u_{j-1}}{2h} = 0 \quad j=1,2,\dots,N-1$$

$$u \;\; = 0 \qquad u_N = 1 \qquad h = 1/N$$

The solution of this difference scheme is

$$u_j = (1 - \lambda^j)/(1 - \lambda^N) \quad \text{where} \quad \lambda = (1 - h/2\varepsilon)/(1 + h/2\varepsilon).$$

It can be shown that:

(a) If N is odd (i.e. divide the interval into an odd number of subintervals) and

(i) if j is even then $u_j \to 0$ as $\varepsilon \to 0$;
(ii) if j is odd then $u_j \to 1$ as $\varepsilon \to 0$.

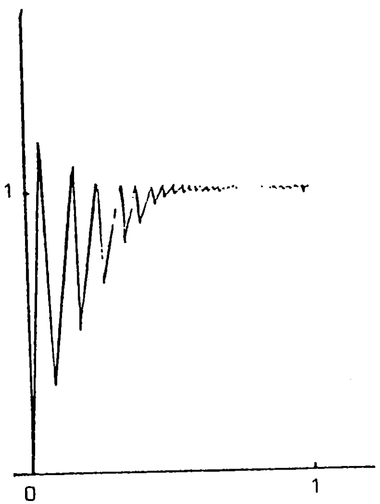
This results in a bounded oscillation between odd and even nodes.

(b) If N is even and

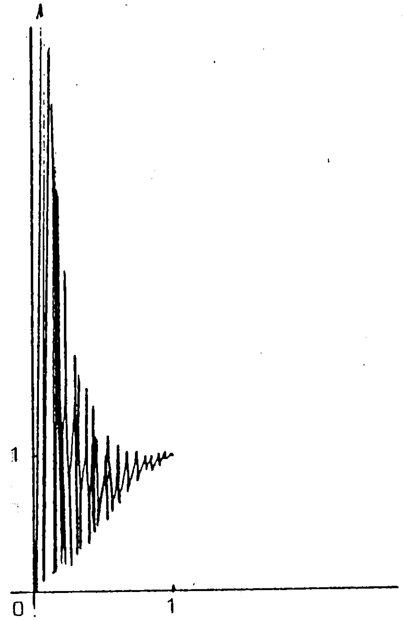
(i) if j is even then $u_j \to j/N$ as $\varepsilon \to 0$;
(ii) if j is odd then $u_j \to \infty$ as $\varepsilon \to 0$.

In this case, the odd/even separation is even more disastrous. The oscillations grow as $\varepsilon \to 0$.

These wild oscillations or "wiggles" (engineering jargon) also occur when classical finite element methods are used. Engineers working with "real-life problems" were the first to notice these wiggles. Their first concern was to somehow get rid of the oscillations. The problem is most noticeable when the ratio $h/2\varepsilon > 1$ (the above formula for u_j has problems when $h/2\varepsilon = 1$). Initially, they simply reduced h (i.e. increased the number of mesh points) in order to keep the ratio $h/2\varepsilon < 1$. However, for 'small' values of ε and in higher dimensions this restriction on the mesh size became too expensive. Their next idea was to use a nonuniform mesh - using a finer mesh in layer regions. This still placed restrictions on the mesh size.



CASE (a)



CASE (b)

Then they hit upon 'upwinding'. This involves taking a suitable difference approximation to the first derivative:

if the layer is to the left, replace $y'(x_j)$ by $(u_{j+1}-u_j)/h$;

if the layer is to the right, replace $y'(x_j)$ by $(u_j-u_{j-1})/h$.

For Example 1, the upwind difference scheme is:

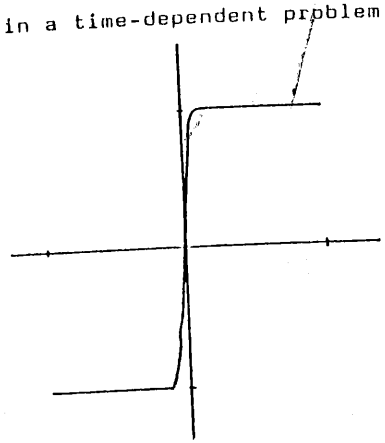
$$\frac{\varepsilon(u_{j+1} - 2u_j + u_{j-1})}{h^2} + \frac{u_{j+1} - u_j}{h} = 0 \qquad j=1,2,\ldots,N-1$$

The solution of this difference scheme is

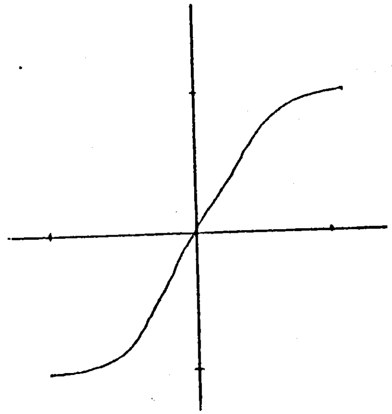
$$u_j = (1 - \tau^{-j})/(1 - \tau^{-N}) \text{ where } \tau = 1 + h/\varepsilon \quad j=0,1,2,\ldots,N$$

As $\varepsilon \to 0$ $u_j \to 1$ for all $j \neq 0$, and $u_0 = 0$.

Upwinding does remove the wiggles and for small values of ε it appears to do the job, but for large ε it is not as accurate as classical difference schemes. Upwinding also tends to "smear" the abrupt change in a shock and makes it difficult to locate ("track") the position of a moving shock in a time-dependent problem.



TRUE



UPWINDING

Upwinding is an improvement, but it still does not solve the problem.

We would like to find a difference scheme (on a uniform mesh) having the property that its solution $\{u_j\}$ is such that for all $j \geq 0$

$$|y(x_j) - u_j| \leq Ch^p$$

where $p > 0$ and C is a constant, both independent of j, h and ε. If we have such a difference scheme, then we say that its solution converges to the solution of the continuous problem uniformly in ε with order p. Upwinding does not converge uniformly.

In 1955, Allen and Southwell proposed a new method for a problem in fluid mechanics, based essentially on the form of the exact difference scheme for a constant coefficient problem in one dimension. In 1969, the Russian A.M. Il'in examined the problem

$$\varepsilon y'' + a(x)y' = f(x) \qquad 0 < x < 1$$

$$y(0),\ y(1) \text{ given} \qquad a(x) \geq \alpha > 0$$

$$a(x),\ f(x) \text{ sufficiently smooth}$$

and showed that a difference scheme similar to Allen and Southwell's converged uniformly in ε with order one. In the 1970s, more uniformly accurate difference schemes were found. In recent times (1980-1985), three point difference schemes which are uniformly second accurate have been appearing. All these difference schemes, which are based essentially on being exact for constant coefficients, are called exponentially-fitted difference schemes (or "smart upwinding" by the engineers). Thus, apart from a few loose ends, good numerical methods for singularly perturbed differential equations in one dimension (which are linear and without turning-points) exist and the area seems to be all sown up. BUT what about two dimensions? Here there are no significant results whatsoever - either analytically or numerically. Since exponentially fitted difference schemes rely on being able to solve the constant coefficient analog of the problem, no extension to higher dimensions has yet been found. In two dimensions, there is a bottomless pit of viciously hard singularly perturbed problems. In one dimension, problems which exhibit shock behaviour have not been satisfactorily dealt with yet, and as for non-linear problems - well, it probably won't be till after the year 2001 that they'll be looked at seriously!!

REFERENCES

A fairly comprehensive analysis of the 'state of the art' up to 1980 can be found in [1]. The three conferences in [2] present some of the more recent results in the area. The papers [3] and [4] are considered to be the first major landmarks in this field. Asymptotic expansions, existence and

uniqueness, bounds on the solution and its derivatives ... for the continuous problem are discussed in [5] - [11]. Numerical results for singularly perturbed problems may be found (for example) in the papers [12] - [14].

1. DOOLAN, E.P., MILLER, J.J.H. and SCHILDERS, W.H.A. *'Uniform Numerical Methods for Problems with Initial and Boundary Layers'*, Dublin, Boole Press (1980).

2. MILLER, J.J.H. (Ed.). *'Boundary and Interior Layers - Computational and asymptotic Methods'*, I, II, III, Boole Press, Dublin (1980, 1982, 1984).

3. de G. ALLEN, D.N. and SOUTHWELL, R.V. "Relaxation Methods Applied to Determine the Motion, in 2-D, of a Viscous Fluid Past a Fixed Cylinder", *Quart. J. Mech. Appl. Math.*, VIII (2) (1955) 129-145.

4. IL'IN, A.M. "Difference Scheme for a Differential Equation with a Small Parameter Affecting the Highest Derivative", *Math. Notes*, **Vol. 6**, No. 2 (1969) 596-602.

5. VISHIK, M.I. and LYUSTERNIK, L.A. "Regular Degeneration and Boundary Layer for Linear Differential Equations with a Small Parameter", *AMS Translations (2)*, 20 (1961) 239-264.

6. WASOW, W.R. *'Asymptotic Expansions for Ordinary Differential Equations'*, Interscience, New York (1965).

7. ECKHAUS, W. *'Matched Asymptotic Expansions and Singular Perturbations'*, North Holland, Amsterdam (1973).

8. O'MALLEY, Jr, R.E. *'Introduction to Singular Perturbations'*, Academic Press, New York (1974).

9. SMITH, D.R.
"The Multivariable Method in Singular Perturbation Analysis", *SIAM Rev.*, (1975) 221-273.

10. HOWES, F.A.
"Boundary-Interior Layer Interactions in Nonlinear Singular Perturbation Theory", *Memoirs AMS 203* (1978).

11. CHANG, K.W. and HOWES, F.A.
'Nonlinear Singular Perturbation Phenomena: Theory and Applications', Springer-Verlag, Berlin (1984).

12. KELLOGG, R.B. and TSAN, A.
"Analysis of Some Difference Approximations for a Singular Perturbation Problem Without Turning Points", *Math. Comp.*, 32 (1978) 1025-1039.

13. MILLER, J.J.H.
"Sufficient Conditions for the Convergence, Uniformly in ε, of a 3-point Difference Scheme for a Singular Perturbation Problem", *'Numerical Treatment of Differential Equations in Applications'*, Eds R. Ansorge and W. Tornig, Lecture Notes in Math., 679, Springer, Berlin (1978) 85-91.

14. NIIJIMA, K.
"A Uniformly Convergent Difference Scheme for a Semilinear Singular Perturbation Problem", *Numer. Math.*, 43 (1984) 175-198.

Department of Mathematics,
Regional Technical College,
Dundalk

ASPECTS OF HIPPARCOS

W.G. Tuohey

1. INTRODUCTION

Some aspects of the HIPPARCOS space astrometry mission are presented in this article. The objectives of the mission and its broad principles of operation are described in Section 2. In Section 3, system level analyses of the mission, to which our company has contributed over the past few years, are outlined. Finally, in Section 4, as an illustration of the work, a specific, relatively simple, problem is discussed.

3. HIPPARCOS

HIPPARCOS is a space astrometry mission, sponsored by the European Space Agency (ESA), which is scheduled for launch in 1988. Its objective is to measure the astrometric parameters (positions, proper motions, trigonometric parallaxes) of about 100,000 pre-selected stars to a (very high) accuracy of 0.002 arcseconds.

The basic principle of measurement is to scan, continuously and systematically, the entire sky with a telescope capable of accurately measuring the angles between stars separated by a large angle. It is possible, by numerically combining several millions of such angular measurements, to derive the required astrometric parameters. The period of data collection is to be 2½ years.

The telescope is equipped with two fields of view (FOVs) to enable measurement of the angles between widely separated stars. Each FOV is of dimension 0.9° x 0.9°. The angle between the FOVs, called the *basic angle*, is denoted by $\gamma = 58^{\circ}$

The FOVs scan the entire celestial sphere through a combination of two motions (see Fig. 1):

(a) A short period spin about the Z_G-axis (rate R = 11.25 rotations/day).

(b) A long period revolution (precession of the Z_G axis) which describes an axisymmetric cone about the line joining the satellite and sun. The half-cone angle is called the revolving scanning angle and is denoted by ζ (= 43°). The average precession rate is K (=6.4 revolutions/year).

There is a modulating grid at the focal plane of the telescope which, together with a photon counting detector, encodes the movement of a star as it crosses a FOV. This constitutes the primary instrument. In addition, there is another detector (called a 'star mapper') placed in the focal plane. Its purpose is to provide data for control of the satellite's attitude and to fulfil a supplementary scientific mission (named TYCHO). (By attitude is meant the pointing directions of the Z_G and X axes - see Fig. 1).

3. GENERAL SYSTEM ANALYSIS

Selection of Key Parameters

The values chosen for parameters ζ, K and R are limited by certain technical considerations. For example, the electrical power supply (solar panels) depends on ζ while a low value of K makes for ease of manoeuvrability; the choice of R is limited by data rates and on-board computer capability.

There are scientific constraints, also. These include a requirement for uniform sky coverage, optimisation of global accuracy and minimisation of interruptions (occultations) due to earth and moon. There are similar considerations for the

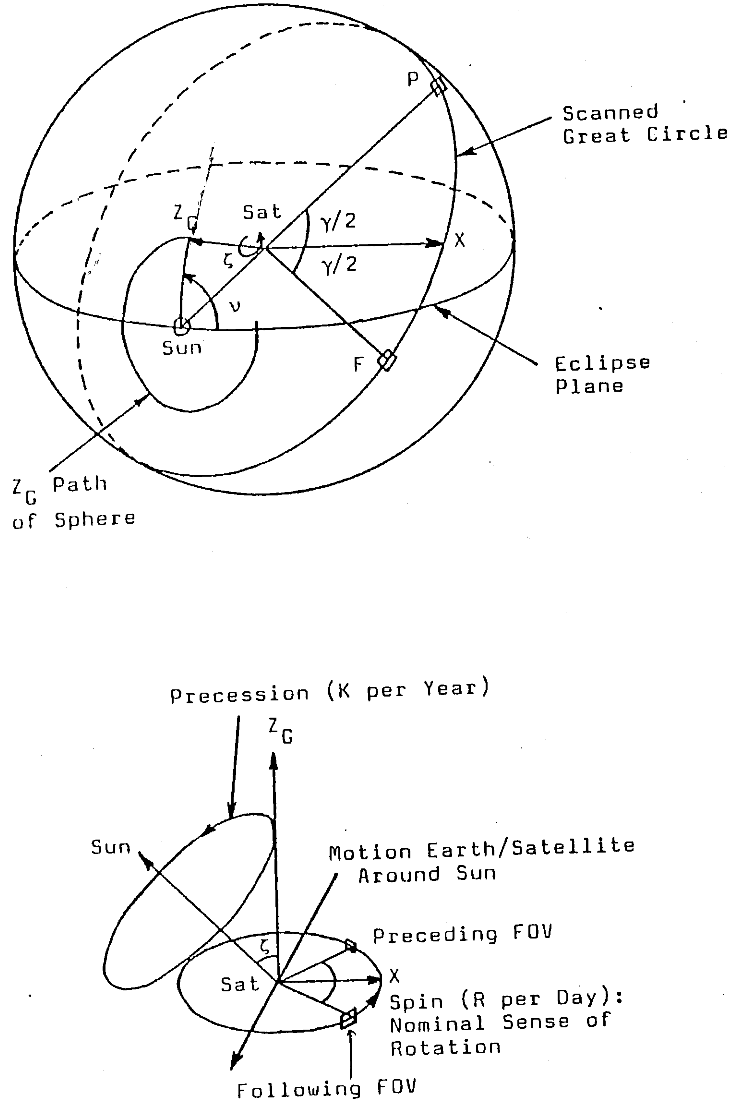


FIGURE 1: HIPPARCOS SCANNING MOTIONS

choice of γ; for example, integral divisors of π are found to exhibit undesirable behaviour in the data reduction process.

Accuracy Analysis

Accuracy analysis makes up a major part of systems activity. Its purpose is to assess the impact of different error sources on overall accuracy. Such assessments enable design trade-offs to be made, for example. Some of the major error sources considered are photon statistical noise, background noise, high frequency attitude jitter and irregularities of the grid.

Photometric Calibration

While we have contributed to the two foregoing topics, mainly at a computational level, our main activity has concerned *in-orbit calibration* of the satellite's payload. (Pre-launch calibrations are quite distinct.) There are two principal topics, photometric calibration and geometric calibration.

As an illustration of photometric calibration, consider

$$I_o = CI_{pf} \tag{1}$$

where I_o represents the photoelectron count rate observed by an instrument and I_{pf} the incident photon flux. The objective of the calibration is to estimate C, the instrument sensitivity.

In practice, C is not a simple constant. It may depend, for example, on position in the field of view (η, ξ), on star colour $(B - V)$, on time (t) and on count rate (non-linear effect). Thus a simple form for C might be

$$C = C_0 + C_1\eta + C_2\xi + C_3(B - V)$$
$$+ C_4\eta(B - V) + C_5\xi(B - V) + C_6t + C_7I_o \tag{2}$$

Therefore, the calibration task is to estimate parameters C_0 to C_7. A weighted least squares procedure is the method used for this estimation.

A major part of the work is to assess the performance of the calibration method. The main results of such an assessment are the accuracy achievable for a measurement period of given duration (i.e. for a given volume of data) and the appropriate form for function C. The assessment takes account of measurement error models, of predicted instrument response and of *a priori* errors on star magnitude and colour.

Geometric Calibration

Each star in a field of view is assigned a longitudinal (η) and transverse (ξ) coordinate; these define the *field* (sky) position of the star. This is illustrated in Fig. 2, which distinguishes preceding (p) and following (f) fields. For each star, there is a corresponding star image on the detector grid. This image is assigned *grid* coordinates (G,H).

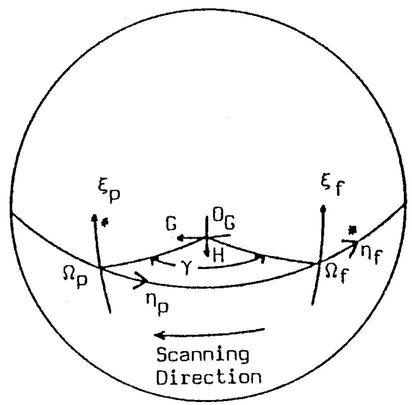


FIGURE 2: FIELD AND GRID COORDINATES

There is a mapping between field and grid which can be described in polynomial form. Thus, for the longitudinal grid coordinate one has

$$G = \sum_{n=0} \sum_{m=0} a_{mn}^{(\alpha)} \eta^m \xi^n \qquad (3)$$

where α = p or f according to field of view.

The various terms of (3) may be associated with such effects as grid defocusing and in-plane displacements, grid rotation and tilts and telescope mirror deformations. The basic angle (γ) can be included in (3), in the terms $a_{00}^{(p)}$, $a_{00}^{(f)}$.

The main part of the mapping, called the nominal field to grid transformation, is known pre-launch. However, it is an in-orbit calibration task to estimate the additional distortion induced post-launch.

The above polynomial form describes *large scale* distortions. In addition, it is necessary to determine *medium scale* distortions. The latter are described by a large matrix of components (≈ 150 x 150). However, the calibration method devised takes account of the good pre-launch knowledge of these components and, thereby, reduces the measurement time which would otherwise be necessary.

Among other geometric calibration tasks may be noted that of chromaticity calibration. In the present context, chromaticity refers to the displacement of a star image with respect to the image position of a star of average colour.

4. A SPECIFIC PROBLEM

Method

The main in-orbit calibration activities are carried out during the commissioning period. This commences some days

after launch and lasts about one month. However, prior to this, there is an initialisation phase during which attitude control of the satellite is acquired. As part of this process, a first calibration of the basic angle (γ) and of grid rotation (θ) is required; this topic is discussed in what follows.

The star mapper is the only detector operational during the initialisation phase. Therefore, its measurements must form the basis for the calibration method. Essentially, this detector measures the time at which a star crosses a particular reference line in the field of view (preceding or following, as appropriate). The distance between the reference lines is exactly γ (see Figs 2 and 3).

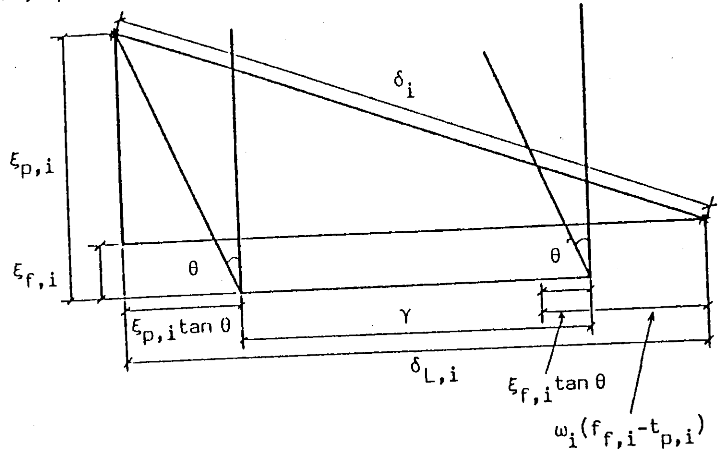


FIGURE 3: STAR SEPARATION

The method is based on measurements on a set of star pairs. Each pair is such that one member crosses the preceding reference line at approximately the same time as the other member crosses the following reference line (see Fig. 2). Hence, the separation between the pair is, approximately, γ.

Consider the i^{th} such pair, as in Fig. 3. Let the *a priori* value for their separation be δ_i. Thus, its longitudinal (along scan) component is

$$\delta_{L,i} = (\delta_i^2 - (\xi_{p,i} - \xi_{f,i})^2)^{\frac{1}{2}} \quad (4)$$

where $\xi_{p,i}$, $\xi_{f,i}$ are transverse coordinates.

Let $t_{p,i}$, $t_{f,i}$ be the transit times for the preceding and following stars, respectively. Then, if ω_i is the rotation rate, one has

$$\delta_{L,i} = \gamma + \omega_i(t_{f,i} - t_{p,i}) + \tan\theta\,(\xi_{p,i} - \xi_{f,i})$$

In fact, θ is small so that one may write

$$\gamma + \theta(\xi_{p,i} - \xi_{f,i}) = \delta_{L,i} - \omega_i(t_{f,i} - t_{p,i}) + \varepsilon_i \quad (5)$$

The added term, ε_i, represents error (from a number of sources).

The measurements from a number, N say, of such star pairs are collected, each giving rise to an equation of form (5). A weighted least squares method is applied to this set of equations to obtain estimates $\hat{\gamma}$ and $\hat{\theta}$.

Assessment

In order to assess the method, a number of simplifying, but not unrealistic, assumptions may be made. Thus, one may assume that observations are uncorrelated and that

$$\text{Var}\ (\varepsilon_i) = \sigma^2 \quad \forall\, i \quad (6)$$

Further, one may assume that $\xi_{p,i}$, $\xi_{f,i}$ are both random variables uniformly distributed in (-a,a) (a = 20 arcminutes for star mapper). It then follows, approximately, that

$$\hat{\gamma} = \frac{1}{N}\sum y_i \quad (7)$$

$$\hat{\theta} = \frac{1}{N}\left(\frac{3}{2a^2}\right)\sum y_i(\xi_{p,i} - \xi_{f,i}) \quad (8)$$

where

$$y_i = \delta_{L,i} - \omega_i(t_{f,i} - t_{p,i}) \quad (9)$$

Moreover,

$$\text{Var}\ (\hat{\gamma}) = \sigma^2/N \quad (10)$$

$$\text{Var}\ (\hat{\theta}) = \left(\frac{3}{2a^2}\right)\sigma^2/N \quad (11)$$

and

$$\text{Cov}\ (\hat{\gamma}, \hat{\theta}) = 0 \quad (12)$$

Thus, the achievable accuracy depends (unsurprisingly) on the ratio

$$\mu = \sigma^2/N \quad (13)$$

An approximate error analysis of the right-hand side of (5) yields

$$\sigma^2 = a + bT_s^2 \quad (14)$$

in which estimated values are available for a and b. T_s is the average interval between transits of a *suitable* star pair,

$$T_s = \overline{t_{f,i} - t_{p,i}} \quad (15)$$

Let the total number of candidate calibration stars, assumed uniformly distributed in the sky, be M. Then, the density per square degree is

$$\rho = M\pi/4(180^2) \quad (16)$$

Noting that the mean spin rate is 168.75° per hour and that the star mapper width is 40 arcminutes, it follows that an area

$$A = (168.75/3600)(40/60) = 0{,}03125 \quad (17)$$

square degrees is swept out in 1 second. Hence, the average rate of arrival of a candidate star in the star mapper is given by

$$\lambda = A\rho = 0.03125\rho \quad (18)$$

A minimum separation time of 10 seconds, between members of a star pair, is necessary to avoid ambiguity in identification. On imposing this constraint and on letting the maximum separation time be t_{max}, it can be shown that

$$T_p = \frac{1}{\lambda}[e^{-60\lambda} - e^{-2\lambda(t_{max}+20)}]^{-1} \quad (19)$$

where T_p is the average interval between *suitable* star pairs. Moreover, one may show that

$$T_s = \frac{1}{\lambda} + [10e^{-10\lambda} - t_{max}e^{-\lambda t_{max}}][e^{-10\lambda} - e^{-\lambda t_{max}}]^{-1} \quad (20)$$

The total number of *suitable* star pairs in a given period T_{tot} may then be calculated as

$$N = T_{tot}/(T_p + T_s) \quad (21)$$

It is clear from the foregoing that the two elements of μ (Equation (13)) depend on t_{max}. In order to optimise the method's performance one seeks to minimise μ. However, it is clear from the nature of the dependencies on T_s that there are conflicting objectives (of minimising σ^2 and maximising N). The value of t_{max} (and, hence, of T_s) which gives the best compromise between these objectives defines a *suitable* star pair.

ACKNOWLEDGEMENTS

I would like to acknowledge the contributions of my colleagues J. Campbell (CAPTEC) and E. O'Mongain (Experimental Physics Department, UCD). In addition, I wish to acknowledge the collaboration of E. Zeis and J.P. Gardelle (MATRA Toulouse) and of R.D. Wills (ESA). I wish, also, to thank the directors of CAPTEC for permission to publish this article.

REFERENCES

Much of the work on which this article is based is contained in contractors' reports and similar documents and is, therefore, somewhat inaccessible. However, the following may be of interest and are more readily available:

1. PERRYMAN, M.A.C.
"Ad Astra HIPPARCOS, The European Space Agency's Astrometry Mission", ESA BR-24, June 1985.

2. SCHUYER, M.
"The Hipparcos Satellite's Mission: The Objectives and Their Implementation", ESA Bulletin No. 42, May 1985.

CAPTEC,
Malahide,
Dublin

ERRATUM

In my paper "Capacities, Analytic and Other", *IMS Newsletter*, 13, pp. 48-56, the symbol $|| \, |\nabla\phi| \, ||_{L_1}$ appearing in line 6 of p. 54 should be replaced by $|| \, |\nabla\phi| \, ||_{H_1}$. Then in lines 10-13 of that page $W^{1,1}$ should be redefined as the space of L_1 functions with H_1 distributional derivatives.

A.G. O'Farrell

LOFTING THE VIADUCT WITH A MINIMUM OF EFFORT

George Kelly

The Chetwynd Viaduct lies astride the Cork-Bandon road, its gaunt, dilapidated structure dominating the adjacent countryside. Since its construction in 1849 it has presented a formidable challenge to bowl-players, namely, to loft a 28 oz. bowl over its forbidding height of 90 feet. It is claimed that a Mr Dan Hurley from Bandon accomplished this feat in 1900 and, likewise, a Mr Bill Bennet of Killeady, Ballinhassig, Co. Cork, in the 1930s. There are, however, no written records to support these claims.

The first official attempt was in 1955 when a crowd of over 6,000 spectators gathered to witness eleven competitors endeavouring to "loft the viaduct". Amongst them were the famous Barry brothers from Cork, Mick and Ned, the former being regarded as the greatest bowler of all time. Both brothers succeeded in hitting the upper part of the framework with a 28 oz. bowl, but failed to get it over.

In August 1985 interest was again renewed in the event when a well-known Cork brewery offered £5,000 for what had by now come to be regarded as a superhuman sporting feat - the lofting of the viaduct with a 28 oz. bowl. Shortly after this, in fact on September 8th, 1985, history was made when before a crowd of almost 10,000 spectators a 23-year-old German Hans Bohlken, succeeded in doing exactly that. Bohlken used a ramp, which apparently is standard practice in German bowling and made himself £5,000 richer in the process.

The question of lofting the viaduct with a minimum amount of effort gives rise to an interesting problem in mechanics. It is well known that the path of a projectile moving under gravity only is a parabola whose equation can take the form

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad (1)$$

Here x and y denote the horizontal and vertical directions respectively, and u at α with the horizontal is the initial velocity of projection. The two values of x for which the height is h are given by the quadratic

$$gx^2 - 2u^2\sin\alpha\cos\alpha + 2u^2h\cos^2\alpha = 0 \qquad (2)$$

and if 2d is the distance between these points, an easy calculation using Eqn (2) shows that

$$gd = u\cos\alpha(u^2\sin^2\alpha - 2gh)^{\frac{1}{2}} \qquad (3)$$

Eqn (3) defines u as a real function of α since the expression under the radical is always positive. The value of α which gives the least value of u may be obtained most easily by re-writing (3) in the form

$$u^4\cos^4\alpha - u^2(u^2 - 2gh)\cos^2\alpha + g^2d^2 = 0 \qquad (4)$$

and noting that Eqn (4) will have equal roots in $\cos^2\alpha$ if $u^2 = 2g(h\pm d)$. But this is precisely the condition that u be minimum and since $u^2 > 2gh$, this minimum value is given by

$$u^2 = 2g(h + d) \qquad (5)$$

The corresponding value of α is obtained from Eqn (4) in the form

$$\cos^2\alpha = \frac{d}{2(h + d)} \quad \text{or} \quad \sin^2\alpha = \frac{2h + d}{2(h + d)} \qquad (6)$$

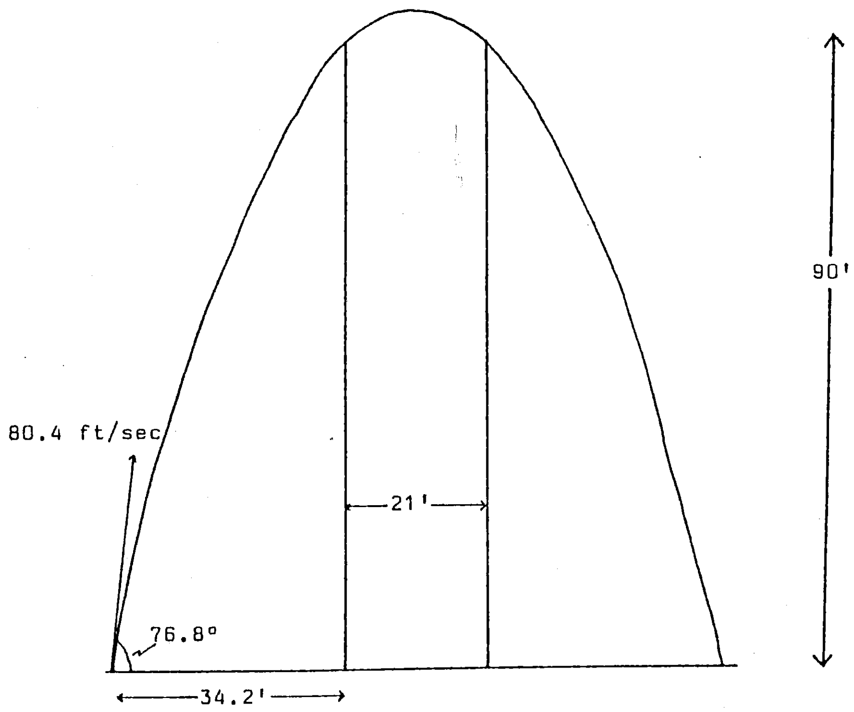
If x_0 and $x_0 + 2d$ are the two roots of Eqn (2), then using Eqn (6) and the familiar formula for the roots of a quadratic equation gives

$$x_0 + d = \frac{u^2}{g}\sin\alpha\cos\alpha = \sqrt{d(2h + d)} \qquad (7)$$

Equations (5), (6) and (7) are directly applicable to the viaduct problem upon taking $h = 90$ feet, $2d = 21$ feet and $g = 32.10$ ft/sec^2. The values obtained are

$$u = 80.4 \text{ ft/sec}, \quad \alpha = 76.79°, \quad x = 34.2 \text{ ft}, \qquad (8)$$

where x_0 is the distance from the viaduct at which the loft should be made (the value of 45 feet quoted in the *Cork Examiner*, 1977, is measured from the centre of the viaduct).



Regrettably, the use of a ramp by the German victor has given rise to some controversy. Since, however, the height of the ramp is small compared to the height of the viaduct, the overall results in (8) would be largely unchanged. A much more important factor is that running up the ramp generates vertical velocity which is automatically communicated to the bowl and makes it easier to attain the component $u \sin \alpha$ which is required by Eqn (8). On the other hand, there is no doubt that to loft from a ramp requires an extra degree of skill to use the ramp effectively. In fact, Bohlken has been described as having an "incredible technique".

It seems reasonably certain that further attempts will be made at lofting the viaduct.

Department of Mathematical Physics,
University College,
Cork.

WHAT IS A PROBABILISTIC PROOF?

Paul McGill

This note is aimed at those who ask naive, and sometimes not so naive, questions about 'probability'. I try to give the flavour of the approach. For that is what it is. A way of looking at problems 'probabilistically'.

Probabilistic arguments arise in all sorts of different situations. For example one comes across them in combinatorics, statistical physics, differential geometry, and especially in analysis. It is this last that I shall concentrate on, in an attempt to clarify the difference between a probabilistic and an analytic proof of the same result. One confusion is that an analytic proof for one person may be a probabilistic proof to another. My definition is the very purest of all. Namely that a probabilistic proof is one which is motivated in terms of the sample path (or individual trial).

I have found it helpful to think of 'probability' as a factorisation.

$$\textit{Problem} \mapsto \textit{Probabilistic Formulation} \overset{E}{\mapsto} \textit{Solution}$$

where E is of course the expectation operator. So, roughly speaking, one argues in terms of the sample path, then integrates to obtain the (analytic) answer. It is not claimed that this factoring is the easiest solution, but rather that it is sometimes more 'intuitive' (whatever that means) or maybe more 'natural'.

Example 1. Consider the probability density in t

$$q_x(t) = \frac{x}{\sqrt{2\pi t^3}}\exp\left(-\frac{x^2}{2t}\right) \qquad (t > 0)$$

where $x > 0$ is a parameter. It is not immediately obvious that

$$q_x * q_y = q_{x+y} \qquad (\dagger)$$

with $*$ denoting convolution. So we see that an **analytic** proof of $(\dagger)$ is the computation of the Laplace Transform of q_x, the result being

$$\int_0^\infty e^{-\lambda t} q_x(t)\,dt = e^{-\sqrt{2\lambda}x},$$

and NOW the answer is clear. But recall that if we add two independent random variables then the law of their sum is given by the convolution of the separate laws (we conveniently omit the proof!). Hence a **probabilistic** proof of $(\dagger)$ is possible if one can find two independent random variables H_x and H_y, such that $H_x + H_y = H_{x+y}$, where H_z has law q_x for all $z > 0$.

So there it is. All that needs to be done is to find the appropriate probabilistic setting in which the result will be obvious. To set up the answer we digress a little, and introduce the currently fashionable theory of martingales.

Example 2. Suppose that X_n is a sequence of i.i.d. (independent identically distributed) random variables such that X_1 has values ±1 with $P[X_1 = 1] = p$. We define the simple random walk as $s_n = \sum_{i=1}^n X_i$. One of the things to notice about this is the way the definition is sequential. Thus we define the sum s_n when we have observed the variables $X_1, X_2, \ldots X_n$ already. One thinks of this as tossing a (biased) coin successively, and the picture is one of dynamic probability (the universe unfolding, etc.). It is natural from this point of view to think not just of the process s_n itself, but of the pair consisting of the process s_n and the information which it has accumulated up to the time n, which we represent by the σ-algebra $S_n = \sigma(X_1, X_2, \ldots, X_n)$. Now THINK. Suppose we are betting on the value of s_n. Clearly it is more advantageous to know the value of s_{n-1} than it is to know that $s_0 \doteq 0$.

On the basis of 'latest is best' (intuition!) we agree that

$$E[s_n | S_{n-1}] = E[s_n | s_{n-1}],$$

and by definition

$$E[s_n | s_{n-1}] = s_{n-1} + E[X_n | s_{n-1}] = s_{n-1} + (2p - 1).$$

Iterating one obtains

$$E[s_n - (2p - 1)s_n | S_m] = s_m - (2p - 1)m. \qquad (m < n)$$

We have written it in this way to emphasise that the process (new word!) $s_n - (2p - 1)n$ is stable under the operation of taking the conditional expectation. Before leaving this example we introduce the notion of a random time. Consider the first time τ_1 that the random walk goes strictly positive (sometimes called the hitting time of 1). Then we might be interested in computing the distribution of τ_1. The question is how.

Definitions (1) An increasing family $\mathcal{F}_n$ of σ-algebras of events in a probability space is called a filtration.

(2) A sequence of random variables X_n is said to be adapted to $\mathcal{F}_n$ if each X_n is measurable w.r.t. $\mathcal{F}_n$.

Thus 'adapted' has connotations of being observable in the filtration at the appropriate time. The filtration S_n defined in Example 2 above is called the natural filtration of the random walk s_n.

A martingale M_n in the filtration $\mathcal{F}_n$ is a process which is adapted and stable under the conditional expectation operation, i.e.

$$E[M_n | \mathcal{F}_m] = M_m. \qquad (m < n)$$

Recall how this means that $E[M_n 1_A] = E[M_m 1_A]$ for every $A \in \mathcal{F}_m$.

In words: 'averaging M_n over members of $\mathcal{F}_m$ yields M_m'. The only way to understand this definition is to work with it. We do this later on. But for the moment be content with a few examples.

Examples 3

(a) Suppose that the probability space is $([0,1],\mathcal{B},m)$ where $\mathcal{B}$ is the Borel σ-algebra and m is Lebesgue measure. Consider the sequence of Rademacher functions $f_n = \operatorname{sgn}(\sin 2^n \pi x)$, each of which has mean zero. We define g_n almost everywhere by

$$g_0 = 0$$

$$g_1 = 1_{[0,\frac{1}{2}]} - 1_{[\frac{1}{2},1]} = f_1$$

$$g_2 = \frac{3}{2}1_{[0,\frac{1}{4}]} + \frac{1}{2}1_{[\frac{1}{4},\frac{1}{2}]} - \frac{1}{2}1_{[\frac{1}{2},\frac{3}{4}]} - \frac{3}{2}1_{[\frac{3}{4},1]} = g_1 + 2^{-1}f_2$$

$$\vdots$$

The general formula being $g_n = g_{n-1} + 2^{-(n-1)}f_n$.
Let $\mathcal{F}_n = \sigma(f_i : 1 \le i \le n)$. Up to null sets $\mathcal{F}$ is just unions of the dyadic intervals $\{[k2^{-n}, (k+1)2^{-n}] : 0 \le k \le 2^n - 1\}$. Then

$$E[g_n | \mathcal{F}_{n-1}] = g_{n-1} + 2^{-(n-1)}E[f_n | \mathcal{F}_{n-1}] = g_{n-1}$$

so by induction g_n is an $\mathcal{F}_n$ bounded martingale.

(b) If X is an integrable random variable (so that one can define conditional expectations) then the sequence

$$X_n = E[X | \mathcal{F}_n]$$

is a martingale in the filtration $\mathcal{F}_n$. This is an example of a closed martingale.

(c) Let u_n be a random walk whose i.i.d. steps are now normal $N(0,1)$. By using the formula

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\{-\frac{x^2}{2t} + \sqrt{2\lambda}x\}dx = e^{\lambda t}$$

one sees that $\exp\{\sqrt{2\lambda}u_n - \lambda n\}$ is a martingale for the natural filtration of u_n.

Definition An integer-valued random variable $T \geq 0$ is said to be a stopping time for the filtration $\mathcal{F}_n$ if $\{T = n\} \in \mathcal{F}_n$ for all $n \geq 0$.

Example 4 The first passage time τ_1 defined in Example 2 is a stopping time for the filtration S_n. To see why note that

$$\{\tau_1 = n\} = \{s_1 < 1, s_2 < 1, \ldots, s_{n-1} < 1, s_n = 1\} \in S_n$$

Thus a stopping time is one which can be observed 'as soon as it happens'. Notice that the last zero before time τ_1 cannot. Nor can the minimum before τ_1. Both of these facts are all too familiar to gamblers.

The most important thing about martingales is not so much the celebrated martingale convergence theorem, but rather the fact that the definition can be made to work for stopping times also. Notice that, by definition of the conditional expectation, if M_n is a martingale then $E[M_n] = E[M_0]$.

Doob's Optional Stopping Theorem If M_n is an $\mathcal{F}_n$ martingale which is uniformly bounded up to the $\mathcal{F}_n$ stopping time T then

$$E[M_T] = E[M_0].$$

Let us now construct a martingale which gives us the law of τ_1. We will look for a function f such that $M_n = e^{-\lambda n}f(s_n)$ is a martingale (for the natural filtration S_n of s_n). Let

us suppose that $f(x) = e^{\mu x}$. Then computing the conditional expectation we have

$$E[f(s_n)e^{-\lambda n}|S_{n-1}] = e^{-\lambda(n-1)}e^{\mu s_{n-1}}[pe^{\mu} + e^{-\mu}(1 - p)]e^{-\lambda}.$$

From which the condition for a martingale is that

$$e^{\lambda} = pe^{\mu} + (1 - p)e^{-\mu}.$$

This is a quadratic equation in e^{μ}, with two solutions. We want our martingale to be bounded up to the time τ_1 so choose (for $\mu > 0$) the positive square root

$$\mu = \mu(\lambda) = \log\left(\frac{e^{\lambda} + e^{2\lambda} - 4p(1 - p)}{2p}\right).$$

With this choice of μ we can apply the Doob Theorem at the stopping time τ_1 and get

$$E[e^{-\lambda\tau_1 + \mu s_{\tau_1}}] = 1$$

Notice how we ignore the set $\{\tau_1 = +\infty\}$ since the martingale is zero there. But if $\{\tau_1 < +\infty\}$ then $s_{\tau_1} = 1$ and so $E[e^{-\lambda\tau_1}] = e^{-\mu(\lambda)}$. From this information we make various computations. Note that

$$\mu(0) = \log \frac{1 + |2p - 1|}{2p}$$

which gives $P[\tau_1 < +\infty] = e^{-\mu(0)} = (p/(p-1)) \wedge 1$. If we look at the (interesting) balanced case $p = \frac{1}{2}$ then we compute that

$$E[e^{-\lambda\tau_1}] = \frac{1}{e^{\lambda} + \sqrt{e^{2\lambda} - 1}}$$

so by differentiation, putting $\lambda = 0$, we get $E[\tau_1] = +\infty$. Thus the expected waiting time for first positive passage is infinite, although the time itself is finite.

This is an example of the probabilistic method. It is clearly formulated in terms of the sample path, and in the end the answer comes by taking an expectation.

Important Remark The boundedness condition in Doob's Theorem is essential. Consider the example of the simple random walk when $p = 1/2$. Then τ_1 is a stopping time but we have

$$1 = E[s_{\tau_1}] \neq E[s_0] = 0.$$

We are now ready to finish off this circle of ideas. We begin with the martingale of Example 3(c) above $M_n = \exp(\sqrt{2\lambda}u_n - \lambda n)$. There is a continuous time analogue of this martingale $M_t = \exp(\sqrt{2\lambda}B_t - \lambda t)$, where B_t is called Brownian motion (and we take $B_0 = 0$ here). There are two structural facts that we need, both of them difficult to prove.

(1) The process B_t varies continuously with time. This result is due to Wiener.

To state the second one we define the random time

$$T_x = \inf(t > 0 : B_t = x).$$

(2) $(B_{T_x+t} - x : t \geq 0)$ is a process with the same law as B_t which is independent of the process $B_{T_x \wedge t}$. This is a particular case of the strong Markov property.

It is a FACT that we can apply the Doob theorem at time T_x to the martingale M_t. Which gives us

$$E[M_{T_x}] = E[M_0] = 1 = E[\exp(\sqrt{2\lambda}B_{T_x} - \lambda T_x)]$$

But using (1) $B_{T_x} = x$ (at least when T_x is finite) so we get

$$E[e^{-\lambda T_x}] = e^{-\sqrt{2\lambda}x}.$$

Going back to Example 1 we find that T_x has law q_x. But now (2) shows that $\tilde{T}_y = \inf(t > 0 : B_{T_x+t} - x > y)$ has the same law as T_y, while at the same time being independent of T_x. Since we have the sample path identity

$$T_{x+y} = T_x + \tilde{T}_y$$

the conclusion $q_x * q_y = q_{x+y}$ is immediate.

As we have written it here the probabilistic proof seems to depend on the analytic proof. However one can see that T_x has law q_x directly, by using (2) and the reflection argument of Désiré André. This reasoning is too subtle for the casual reader.

In conclusion we point out how this typifies the ingredients of a probabilistic proof. It is certainly harder than the original, but has an undeniable charm and utility since we have a diagram for 'why' the result holds.

Department of Mathematics,
Maynooth College,
Co. Kildare

CONVEXITY AND SUBHARMONIC FUNCTIONS

Stephen Gardiner

This article gives a simple account of some of the ways in which notions of convexity are related to the study of subharmonic functions. Several recent results in this area are included in the discussion. The article is based on a lecture given at the December 1985 meeting of the DIAS Mathematical Symposium.

1. Notation

We shall be concerned with Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, points of which are denoted by $X = (x_1, \ldots, x_n)$. We write $|X| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$, and denote the open ball of radius r centred at X by $B(X,r)$. The closure and boundary of a subset E of $\mathbb{R}^n$ will be denoted respectively by $\overline{E}$ and ∂E.

2. Harmonic Functions

A function u on an open subset ω of $\mathbb{R}^n$ is called harmonic if it is twice continuously differentiable and satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \equiv 0.$$

(Harmonic functions arise naturally in gravitation, electrostatics, hydrodynamics and the theory of analytic functions). Alternatively, letting $M(u,X,r)$ denote the mean value of u over the sphere $\partial B(X,r)$ whenever $\overline{B}(X,r) \subset \omega$, a function u is harmonic in ω if and only if:

(i) $-\infty < u < +\infty$ in ω;

(ii) u is continuous in ω;

(iii) $\overline{B}(X,r) \subset \omega \Rightarrow u(X) = M(u,X,r)$.

3. Subharmonic Functions

By subdividing (i) - (iii) above we arrive at the dual notions of sub- and superharmonicity.

SUBHARMONIC FUNCTION

(ia) $-\infty \leq u < +\infty$ in ω $[u \not\equiv -\infty$ on any component of $\omega]$;

(iia) u is upper semicontinuous (u.s.c.), i.e. $\{X \in \omega : u(X) < a\}$ is open $\forall a \in \mathbb{R}$;

(iiia) $\overline{B}(X,r) \subset \omega \Rightarrow u(X) \leq M(u,X,r)$.

SUPERHARMONIC FUNCTION

(ib) $-\infty < u \leq +\infty$ in ω $[u \not\equiv +\infty$ on any component of $\omega]$;

(iib) u is lower semicontinuous, i.e. $\{X \in \omega : u(X) > a\}$ is open $\forall a \in \mathbb{R}$;

(iiib) $\overline{B}(X,r) \subset \omega \Rightarrow u(X) \geq M(u,X,r)$

Such functions arise naturally in many situations. For example, if f is analytic in $\mathbb{C}$, then $\log|f|$ is subharmonic. Again, the gravitational potential energy due to a mass distribution is superharmonic. We can immediately make the following observations:

(I) u is subharmonic if and only if $-u$ is superharmonic;

(II) u is harmonic if and only if both u and $-u$ are subharmonic;

(III) if u, v, are subharmonic and $a, b > 0$, then $au + bv$ is subharmonic.

An equivalent formulation of the definition of a subharmonic function is obtained if we replace (iiia) above by:

(iiia') for any open set W with compact closure in ω, and for any continuous function h on $\overline{W}$ which is harmonic in W and satisfies $h \geq u$ on ∂W, we have $h \geq u$ in W.

It is this condition which accounts for the name subharmonic.

4. One-Dimensional Potential Theory

Laplace's equation for the real line is simply $d^2u/dx^2 \equiv 0$, so that harmonic functions are just linear functions of the form $ax + b$ $(a,b \in \mathbb{R})$. In view of (iiia') above, the concept of a subharmonic function on a subset of $\mathbb{R}$ is equivalent to the idea of a convex function. Thus subharmonic functions are a generalization to higher dimensions of convex functions. This explains why notions of convexity recur so frequently in the study of subharmonic functions.

5. Spherical Means

Spherical means of functions have played a fundamental role in potential theory since the pioneering work of F. Riesz [6] in 1926. It is natural to consider how $M(u,X,r)$ behaves as a function of r. Riesz showed that, if $n = 2$, then $M(u,X,r)$ is convex as a function of $\log r$ and, if $n \geq 3$, then $M(u,X,r)$ is convex as a function of r^{2-n}. The functions $\log|X|$ $(n = 2)$ and $|X|^{2-n}$ $(n \geq 3)$ arise as solutions of Laplace's equation in $\mathbb{R}^n \setminus \{0\}$.

Thus, when we modify a subharmonic function (by taking its mean over a sphere of radius r and fixed centre) so that it depends only on one variable (r), convex functions reappear. It is worth pointing out that the same convexity properties hold for

$$\sup\{u(Y) : |Y - X| = r\}, \quad \log M(e^u,X,r), \text{ and}$$

$$\{M(u^p,X,r)\}^{1/p} \text{ for } u \geq 0 \text{ and } p > 1.$$

6. Composition Properties

If we begin with functions of one real variable, we can make the following simple observations of functions:

$$[\text{Convex}] \circ [\text{Linear}] = [\text{Convex}]$$

$$[\text{Increasing Convex}] \circ [\text{Convex}] = [\text{Convex}].$$

('Increasing' is to be interpreted in the wide sense, i.e. non-decreasing). It is well known that these properties carry across to higher dimensions as follows:

$$[\text{Convex}] \circ [\text{Harmonic}] = [\text{Subharmonic}] \qquad (1)$$

$$[\text{Increasing}] \circ [\text{Subharmonic}] = [\text{Subharmonic}]. \qquad (2)$$

However, it has only recently (see [3], [5]) been noticed that this is a special case of the more general, but equally elementary, result stated below (for applications, see [3]).

THEOREM 1. The function $v\phi(u/v)$ is subharmonic in each of the following cases:

(i) u is harmonic, v is positive and harmonic, ϕ is convex;

(ii) u is subharmonic, v is positive and harmonic, ϕ is convex and increasing;

(iii) u is subharmonic, v is positive and superharmonic, ϕ is convex, increasing, and $\phi(x) = 0$ for $x \leq 0$.

By taking $v = 1$, it is clear that (i) and (ii) include (1) and (2) above. The proof is quite short and we outline it below.

LEMMA 1. If $\{u_\alpha : \alpha \in I\}$ is a family of subharmonic functions on ω and $\sup_\alpha u_\alpha$ is u.s.c. and less than $+\infty$, then $\sup_\alpha u_\alpha$ is subharmonic in ω.

Proof of Lemma: $\bar{B}(X,r) \subset \omega \Rightarrow u_\beta \leq M(u_\beta, X, r) \leq M(\sup_\alpha u_\alpha, X, r)$

$$\Rightarrow \sup_\alpha u_\alpha \leq M(\sup_\alpha u_\alpha, X, r),$$

so $\sup_\alpha u_\alpha$ satisfies conditions (ia) - (iiia) of Section 3.

Sketch Proof of Theorem: Corresponding to each part of the theorem, ϕ can be written as:

(i) $\phi(x) = \sup\{ax + b : a,b \in \mathbb{R} \text{ s.t. } at + b \leq \phi(t) \quad \forall t \in \mathbb{R}\}$;

(ii) $\phi(x) = \sup\{ax + b : a \geq 0,\ b \in \mathbb{R} \text{ s.t. } at + b \leq \phi(t)$ $\forall t \in \mathbb{R}\}$;

(iii) $\phi(x) = \sup\{ax + b : a \geq 0,\ b \leq 0 \text{ s.t. } at + b \leq \phi(t)$ $\forall t \in \mathbb{R}\}$.

Thus $v\phi(u/v)$ can be written as

$$\sup_{a,b} v[a(u/v) + b] = \sup_{a,b} [au + bv]$$

and $au + bv$ is subharmonic for the appropriate values a,b in each of the three cases. It is quite easy to check that $v\phi(u/v)$ is u.s.c., and clearly $v\phi(u/v) < +\infty$, so the result now follows from Lemma 1.

Remark: Theorem 1 and its proof transfer easily to the axiomatic setting of harmonic spaces, and so can be applied to sub-solutions of a wide class of elliptic and parabolic p.d.e.'s. This is particularly interesting because (1) and (2) do not hold for harmonic spaces, the reason being that the constant function 1 is not necessarily harmonic in the general setting.

7. Convex Domains

Let $\Omega \neq \mathbb{R}^n$ be a domain (connected, non-empty open set) in $\mathbb{R}^n$, and let u be the signed distance function given by

$$u(X) = \begin{cases} -\text{dist}(X, \Omega) & \text{if } X \in \bar{\Omega} \\ \text{dist}(X, \Omega) & \text{if } X \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

The following recent result is due to Armitage and Kuran [11].

THEOREM 2. The function u is subharmonic in $\mathbb{R}^n$ if and only if the domain Ω is convex.

The "if" part of the result is straightforward and was already known, at least implicitly. For example, when $n = 2$, let L denote an arbitrary straight line $a_L x_1 + b_L x_2 = c_L$ in $\mathbb{R}^2 \setminus \Omega$, $(a_L^2 + b_L^2 = 1)$, and let u_L be the signed distance function from L given by $u_L = \pm(a_L x_1 + b_L x_2 - c_L)$, the sign being chosen so that $u_L < 0$ in Ω. Since each u_L is harmonic, $u = \sup_L u_L$ and u is real-valued and continuous, it follows from Lemma 1 that u is subharmonic in $\mathbb{R}^n$.

The "only if" part requires a longer argument and is genuinely new. A surprising fact about this result is that more can be said when $n = 2$:

THEOREM 3. The function u is subharmonic in $\Omega \subset \mathbb{R}^2$ if and only if Ω is convex.

Armitage and Kuran give a counterexample to show that Theorem 3 fails in higher dimensions. For example, when $n = 3$, let Ω be the torus obtained by rotating the disc $D = \{(0,x_2,x_3) : (x_2 - 2)^2 + x_3^2 < 1\}$ about the x_3-axis. Then it can be shown that u is subharmonic in Ω yet Ω is clearly not convex.

8. Generalized Means

Convexity properties of spherical means of subharmonic functions (Section 5) have analogues for "weighted means" of such functions over other surfaces. To take a simple example, if u is subharmonic in the upper half-plane $\{(x_1,x_2) : x_2 > 0\}$ and $u \leq 0$ on the x_1-axis, then (using polar coordinates)

$$r^{-1}\int_0^\pi \sin\theta\ u(r,\theta)d\theta$$

is convex as a function of r^{-2}.

More generally, various authors over the past 15 years have shown convexity properties for weighted means over the boundaries of half-balls and truncated cones (of varying radii) and bounded cylinders (of varying height or varying radii). In fact, these separate studies have recently ([4]) been unified into a general convexity theorem. The general mean is defined in terms of harmonic measure, and the surface over which it is defined is obtained as the level surface of the quotient of two harmonic functions. For example, in the above case of the half-plane, the appropriate harmonic functions are x_2 and $x_2 r^{-2}$, so the semi-circular means arise as integrals over level surfaces of r^{-2} and convexity is in terms of r^{-2}.

Finally, we remark that convexity properties are not confined to integrals of subharmonic functions over bounded surfaces, for (see [2], for example) if $u \geq 0$ is subharmonic on $\mathbb{R}^{n-1} \times (a,b)$ and does not grow "too quickly" as $|X|$ becomes large, then

$$x_n \mapsto \int_{\mathbb{R}^{n-1}} u(X)\,dx_1 dx_2 \ldots dx_{n-1} \qquad (a < x_n < b)$$

is a convex function provided it is finite on a dense subset of (a,b).

REFERENCES

1. ARMITAGE, D.H. and KURAN, Ü.
"The Convexity of a Domain and the Superharmonicity of the Signed Distance Function", *Proc. Amer. Math. Soc.*, 93 (1985) 598-600.

2. BRAWN, F.T.
"Hyperplane Mean Values of Subharmonic Functions in $\mathbb{R}^n \times]0,1[$ ", *Bull. London Math. Soc.*, 3 (1971) 37-41.

3. GARDINER, S.J.
"A Maximum Principle and Results on Potentials", *J. Math. Anal. Appl.*, 109 (1985) 507-514.

4. GARDINER, S.J.
"Generalized Means of Subharmonic Functions", *Ann. Acad. Sci. Fenn. Ser. A1* (to appear in 1986).

5. GARDINER, S.J. and KLIMEK, M.
"Convexity and Subsolutions of Partial Differential Equations", *London Math. Soc.* (to appear in 1986).

6. RIESZ, F.
"Sur les Fonctions Subharmoniques et leur Rapport à la Théorie du Potentiel", *Acta Math.*, 48 (1926) 329-343.

Department of Mathematics,
University College,
Dublin

CAYLEY: GROUP THEORY BY COMPUTER

Patrick Fitzpatrick

INTRODUCTION

CAYLEY is a sophisticated programming language for working with algebraic structures. Its principal domain is in the area of group theory but it may also be applied to rings, fields, modules and vector spaces. Not only does it contain a large compendium of preprogrammed group theoretical algorithms but also it provides the user with the facility to write and develop new programs. Cayley has taken up residence on the VAX 11/785 at UCG and is therefore accessible to group theorists throughout the country via the Higher Education Authority's network HEANET.

The purpose of this note is to give the reader a brief introduction to the main elements of Cayley and to provide a few examples to illustrate the power of the language. We avoid detailed technical discussion of syntax and format and make no claim to be comprehensive; rather it is our aim to whet the reader's appetite for "hands on" experience. Complete information on the current version of Cayley may be found in [1] and its updates (see also [2]).

Definition and Manipulation of Groups

There are various ways to define a group in Cayley: using generators and relations, as a permutation group, and as a matrix group.

Example (a) The program segment

```
dih4 : free (a,b);
dih4.relations : a↑2 = b↑2 = (a*b)↑4 = 1;
```

Financial assistance from the Faculty of Arts, UCC, is gratefully acknowledged.

defines the dihedral group of order 8 giving it the name *dih4*.

(b) The program

```
g : permutation group(8);
g.generators : (1,2,3), (4,5,6), (1,3,8);
```

defines g as the subgroup of Sym(8) generated by the given three elements.

(c) To define the group of 2x2 matrices of determinant 1 over GF(3) and call it *sl23* we need:

```
k : field(3);
v : vector space (2,k);
sl23 : matrix group (v);
sl23.generators : x = (1,1 : 0,1), y = (1,0 : 1,1);
```

The generating matrices are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Observe that each Cayley statement ends with a semicolon and that no distinction is made between upper and lower case letters (although they may of course be used for clarity in programming). Also note that the symbol ↑ may be ^ on some terminals.

Elements are defined and manipulated in the obvious ways. Thus if *u* and *v* are previously defined elements of some group then

```
u↑2 * v↑-1, (u*v↑3)↑-1   and   u↑v
```

represent the elements u^2v^{-1}, $(uv^3)^{-1}$ and u^v $(=v^{-1}uv)$ respectively. Furthermore the instruction

$$h = <u,v>;$$

defines h as the subgroup generated by (u,v). A lot of information about a group g and its subgroups may be obtained using the *PRINT* command:

PRINT order(g), exponent(g), classes(g);

for example.

STRUCTURED PROGRAMMING

Cayley is a high-level language which allows structured programming in a group theoretic context. Thus we can implement the following constructions:

(i) *IF expression*
THEN statements
[ELSE statements]
END;

(ii) *FOR i = 1 to 100 DO*
statements
END;

(iii) *FOR EACH x IN s DO*
statements
END;

(iv) *WHILE expression DO*
statements
END;

(v) *REPEAT*
statements
WHILE expression;
END;

Here the term *expression* denotes a variable which has the logical (Boolean) value *true* or *false* and in each of (iii) - (v) the program continues as long as *expression* takes the value *true*. The constructions are all subject to the usual constraints regarding their use in combinations such as nested sequences. Other commands *GOTO*, *LOOP* and *BREAK* may be used to transfer control from one part of a program to another. The *GOTO* command has its usual meaning while *LOOP* forces a continuation of the next cycle of a loop without carrying out the remainder of the statements associated with it and *BREAK* transfers control to the next statement after the *END* of the loop. Use of these constructions is particularly important in minimising the execution time of a given program.

STANDARD FUNCTIONS

One of the aspects which makes Cayley such an attractive working environment for the group theorist is the large library of standard functions available. Many of the group theoretic concepts that appear in everyday use are included: normalizer, centralizer, normal closure, core, conjugacy classes, generators, Sylow p-subgroup, upper and lower central series, derived and Frattini series, orbit, block, stabilizer for example. An expression of the form

normalizer(G,H)

represents the group $N_G(H)$ and can be used as it stands in a program. Some of the standard functions are restricted to groups defined in certain ways, but even these limitations can often be overcome by judicious programming.

In addition to the standard functions represented by keywords there are also built into Cayley several standard group theoretical algorithms for working with finitely presented groups: for instance, the Todd-Coxeter, Nilpotent Quotient and Reidemeister-Schreier algorithms are included. A recent addition to the supply of programs available constructs the finite simple groups of order $<10^6$ both as permutation groups and by generators and relations.

LIBRARY PROGRAMS

As soon as one begins to develop programs in Cayley it beccomes imperative to store and edit programs and subroutines (or procedures as Cayley calls them). This is achieved through the use of a library file which has the form

LIBRARY name;
statements
FINISH;

Before entering the Cayley environment (for which the command is CAY) the programmer creates a library (or recalls one created at a previous working session). For example

PLIB groups/c

sets up such a library and calls it *groups*. Creating a library file called *permgp* and inserting it in *groups* requires

ADD permgp

The writer then edits the file in the usual way using the editor on the host machine. On exit from the editor Cayley conveniently prompts

Add problem to library (Y/N)?

(It calls library files problem files.) Modification can be achieved by

MOD permgp

which extracts the file from the library and allows it to be edited. In order to run *permgp* the user enters Cayley and then

LIBRARY permgp;

will execute the statements between the initial and final lines of the file. To change the file again it is necessary to exit from Cayley (using *QUIT;*) and then *MOD* again.

AN EXAMPLE

Given a finite group G and a subgroup H determine whether or not H is subnormal in G and if it is find its defect. We count the sequence of subgroups

$$G_0 = G,\ G_1 = H^{G_0},\ G_2 = H^{G_1},\ \ldots$$

and determine whether this series terminates in H or stabilizes at some larger subgroup. We avoid using the keyword *normal closure* in order to illustrate better the technique of nesting. The expression *invariant (U,V)* takes the Boolean value *true* if *U* is a normal subgroup of *V* and *false* otherwise. The complete program is placed in a library file called *defect*. Text entered within double quotation marks is regarded as comments and ignored by Cayley. A detailed listing of the program is given in the appendix.

HEANET

Finally we look briefly at the network aspect of using Cayley at UCG. At the time of writing HEANET connects UCG, UCC, UCD, TCD, NIHE (Dublin) and NIHE (Limerick). Local advice and permission is obviously required in order to avail of the network and in addition the prospective user will require permission and assistance from the Computer Centre at UCG (in particular to obtain a special command file for running Cayley). In practice the user calls UCG via his own computer terminal, logs on there as if his terminal were connected directly, carries out his working session and then reverses the procedure to break the connection. In the interests of economy it is important to minimise the time spent working with the network connected. This can be achieved by writing all library files as text files in the local machine and using the *TRANSFER* option on the network to send them fully written to UCG. Illustrating with the example of the previous section the user would then have a file *defect.txt* in his directory at UCG. The command

LIBRARY/TEXT/LOG GROUPS defect.txt

(which is incidentaLLY part of VAX/VMS not Cayley) will then insert the file as a library file in the library *groups*.

CONCLUDING REMARKS

There is no doubt that this brief summary does not provide sufficient information for the reader to exploit fully the power of Cayley. However I hope that it will serve as an introduction to the rudiments. I will be happy to correspond in more detail with anyone who is interested and Ted Hurley at UCG has assured me that he is willing to help out with enquiries there.

REFERENCES

1. CANNON, John J.
   CAYLEY: A Language for Group Theory (Preprint 1982),
   University of Sydney.

2. CANNON, John J.
   "An Introduction to the Group Theory Language CAYLEY", in
   'Computational Group Theory', (Durham 1982), 145-183,
   Academic Press, L ndon - New York, 1984.

Mathematics Department,
University College,
Cork

APPENDIX

```
library defect;
"determine whether a subgroup H is subnormal in a group G and if it is
 find its defect"
print 'the group should be called G and the subgroup H';
print G,H;
U=G;
defect=0;
notdone = true;
"notdone is true until the seqeunce
      G0 = G, G1 = H^G, G2 = H^G1, ...
stabilises"
while notdone do
      V=H;
      notnorm = true;
"notnorm is true until the subgroup V is normal in U"
      while notnorm do
            for each x in generators(V) do
              for each y in generators(U) do
                 V=<V,x^y>;
              end;
              if invariant(U,V) then
                 notnorm = false;
              end;
            end;
      end;
 "if V=U then sequence has stopped at a subgroup containing H strictly,
  if V=H then sequence has reached H,
  otherwise continue"
      if V eq U then
            break;
      else
            if V eq H then
                  notdone = false;
            end;
      end;
      defect = defect+1;
      U=V;
end;
"if notdone=true then the loop was broken, otherwise H is (sub)normal"
if notdone then
      print 'the subgroup is not subnormal';
else
      if defect eq 1 then
            print 'the subgroup is normal';
      else
            print 'the subgroup is subnormal with defect',
            defect;
      end;
end;
finish;
```

MATHEMATICS EDUCATION

ON TEACHING MATRIX ALGEBRA BY COMPUTER

Bob Critchley and Gordon S. Lessells

Since 1983, an experiment in computer-assisted learning (CAL) has been running at NIHE, Limerick, whereby students on various courses have had the opportunity of learning the basics of Matrix Algebra at a computer terminal.

Origins

In 1983, a project funded by Shannon Free Airport Development Company (SFADCO) was set up, based at NIHE, to create a CAL package called "Costing for Small Business". The aims of the project were (1) to investigate the potential for CAL in management and third-level education and (2) to decide on the feasibility of setting up a company to produce CAL software. A NIHE initiated project, viz. to create a CAL package in Matrix Algebra, was chosen to run in conjunction with "Costing for Small Business".

System

Control Data Corporation (CDC) have been heavily involved with computer-based education using the PLATO system in the United States. The CDC-110 stand-alone microcomputer-based system was chosen for the production of both our courses. In 1983, the CDC-110 system allowed the creation of lessons according to three different models:

(1) Tutorial Learning Model (TLM)
(2) Drill and Practice Model (DPM)
(3) Situation Simulation Model (SSM)

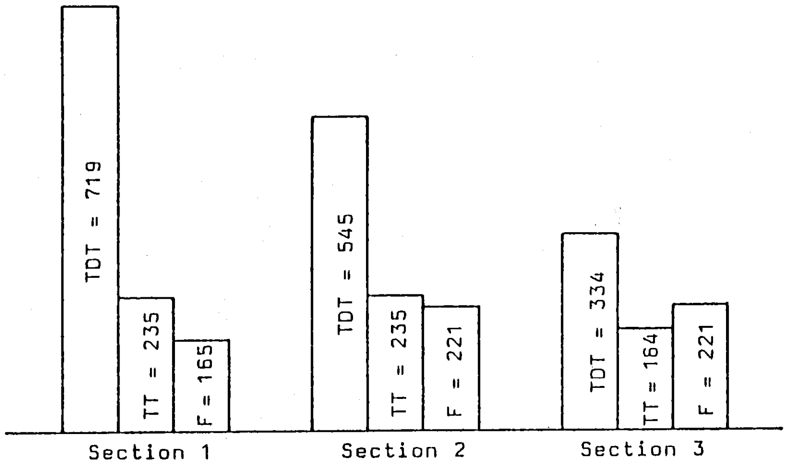
The Tutorial Learning Model was closest to the lecture/tutorial

style of teaching and was chosen for our course. By means of a menu-driven system a lesson could be created as a sequence of frames without using the underlying micro-TUTOR language. However, acquiring a mastery of the command language was almost equivalent to learning a new programming language.

Creation

Two Applied Mathematics students, Donal Crosse and David Nash, on their Co-operative Education placement, were employed by the project for a period of six months to assist us in creating the CAL package. The textual material was written by ourselves, with the design and layout of the frames carried out by Donal and Dave. The material was written in an interactive style in order to retain the attention of students. To this end we were able to take advantage of the touch-sensitive screen. In order not to confront the student with a large quantity of material at once, frames were built up gradually, requiring a key press from the student to proceed. Questions could also be followed by different material according to the response given. By suitable highlighting and animation it was possible to pinpoint the student's attention at the desired section of the screen thus giving the textual material more life than, say, a programmed learning text.

The finished product was a lesson of about six hours duration comprising three sections (1) Matrix Basics, (2) Linear Combinations of Matrices, (3) Multiplication of Matrices. A student is routed through the course having to pass a criterion test at the end of each subsection to allow him/her to proceed. Failure of a criterion test gives the student the option of redoing the previous subsection or reviewing back-up material before taking a new criterion test. Records of time spent and performance in the tests are kept by the computer. In all there were about 600 frames, although each student would only see a fraction of the total. A breakdown of the development time with evidence of the "learning curve" is given below.



TDT - Total Development Time (hours)

TT - Terminal Time (hours)

F - Number of Frames

Reactions to Course

As a first experiment in CAL, matrix algebra was found to be a very suitable topic for a number of reasons:

(1) The topic is widely taught throughout the institute and can be taught independent of applications. The same course can, therefore, be used by both engineering and business students.

(2) The prerequisites were minimal, being a knowledge of Leaving Certificate pass level algebra.

(3) The material lent itself to visual presentation. The use of blackboard and chalk is awkward when large matrices are involved.

(4) The material involved understanding concepts as well as

acquiring calculating skills.

Reaction of students to the course was gauged by means of interviews and questionnaires. Some of the observed benefits were:

(1) Students could go at their own pace.

(2) 12-hour access to the computers allowed students to choose their own study times.

(3) Some students also felt that the interactive nature of the course allowed them to learn better than by traditional methods.

On the debit side, some students were more concerned about the absence of a tutor whereas other students were more concerned about repetition of the same material because of failure in a criterion test. Various tests and subsequent exam results have indicated that the course provided a good foundation for further work on Matrix Algebra. We do, however, believe that the CAL approach is only suitable for a limited number of areas of mathematics, vectors, more matrices, probability, graph theory being four suggestions for possible investigation.

CDC - DEC

The original CDC system had a number of disadvantages:

(1) The lesson models were too restrictive.

(2) The use of floppy disks was unsuitable for use by large numbers.

(3) The system was slow and noisy.

(4) Customer service was poor.

(5) The system was incompatible with other micros.

For these and other reasons, the institute has adopted the Digital Equipment Corporation's DEC PRODUCER system for its

CAL laboratory. The original course has been reprogrammed and is now available in colour to students on DEC PRO-350 microcomputers or DEC VT 220 terminals linked to a Microvax 11. The increased computing power has reduced the time taken by students and the greater availability has meant that large classes can now be accommodated. Anyone interested in seeing (or purchasing!) the course should contact the authors.

Finally, we wish to acknowledge the assistance of Dr Joe Smyth, Dr Mark Burke, Eamonn Murphy, Mary Davern, Anna Kinsella and Brenda Sugrue who have all made valuable contributions along the way.

Applied Mathematics Department,
National Institute for Higher Education,
Limerick.

UNDERGRADUATE PROJECTS IN GROUP THEORY: AUTOMORPHISM GROUPS

T. Porter

In my earlier note [3], I described a project undertaken by a 3rd year student involving commutativity ratios. The basic tools needed were an intuitive idea of a presentation of a group, and some modular arithmetic. That project was the equivalent of a half paper in the final exams. The following year the system was modified and projects were enlarged so as to be equivalent to a full paper in the final exams. Here I will describe briefly a project involving calculation of automorphism groups from a presentation. The groups studied were the dihedral groups, which have a fairly easy presentation readily available:

$$D_n = \langle x,y : x^n = y^2 = (xy)^2 = e\rangle \text{ for } n \geq 3.$$

The idea of the project was as follows:

If $\alpha: D_n \to D_n$ is an automorphism then

$$\alpha(x) = x^i y^j$$
$$\alpha(y) = x^k y^\ell$$

for some $0 \leq i,j \leq n-1$ and $0 \leq j,\ell \leq 1$ since any element of D_n has a representation in the form $x^a y^b$, with $0 \leq a < n$, $0 \leq b < 2$. If one wants to build automorphisms, therefore, one may attempt to do so by picking "suitable" i,j,k,ℓ. Of course saying that α is a homomorphism and specifying $\alpha(x)$ and $\alpha(y)$ will say where each $x^a y^b$ is to go provided that the relations are compatible with the choice of i,j,k,ℓ. By this we mean that

$$x^n = e \Rightarrow (\alpha(x))^n = e \text{ that is } (x^i y^j)^n = e$$

$$y^2 = e \Rightarrow (\alpha(y))^2 = e \text{ that is } (x^k y^\ell)^2 = e$$

and the last relation implies $(x^i y^j x^k y^\ell)^2 = e$.

As one knows that

$$yx = x^{-1}y$$

(coming simply from $(xy)^2 = e$), one can simplify these expressions to get relations between i, j, k and ℓ.

Thus just using modular arithmetic, one can identify which i,j,k,ℓ correspond to an endomorphism, α, defined on generators as above. (The basic detailed abstract theory behind this depends on von Dyck's theorem, see Johnson [1] and was summarised by the student in her dissertation.) Now one has merely to observe that an endomorphism $\alpha: D_n \to D_n$ is an automorphism if and only if Ker α is trivial to have a method of extracting which (i,j,k,ℓ) correspond to automorphisms.

The problems in the modular arithmetic led the student to reduce the problem to the closely related one of calculating $Aut(C_n)$ for any n. As the aim was to give a presentation of $Aut(D_n)$, it was clearly insufficient to note merely that $Aut(C_n) \cong U_n$ the Abelian group of units in the ring $\mathbb{Z}/n\mathbb{Z}$, one needed a presentation of U_n. A search through many group theory books produced no easily readable discussion of this, however from various sources, the student pieced together a reasonable account. In her write-up of this, she included tables illustrating some of the principal differences between the various cases: for example n a power of 2, n a product of odd prime powers, etc. These tables listed explicit generators for $Aut(C_n)$ and presentations for all n up to 32.

The student then returned to studying $Aut(D_n)$. With some help on using *Mathematical Reviews*, the student had found a reference to a paper by G.A. Miller [2] dealing precisely with automorphisms of D_n. The solution of the problem for $Aut(C_n)$ cleared the way for finding a presentation for $Aut(D_n)$. That done, she found which of the automorphisms were inner. At this point a discrepancy was noticeable between her calculations and Miller's description. She found that D_3, D_4 and D_6 are the only dihedral groups, which are their own groups of automorphisms. Miller only mentions D_3 and D_4 in this context. Again she found that there are only three dihedral groups (D_4, D_5 and D_6) with outer automorphism group isomorphic to C_2; Miller seems only to mention D_4 and a metacyclic group of order 20. (For non-group theorists $Out(G) = Aut(G)/Inn(G)$ is the outer automorphism group of G.)

She continued with the study, giving explicit generators, and presentations for $Out(D_n)$ for all n, again tabulating the results for $n \leq 32$.

At various points in the project, it became useful to explore semi-direct products, parts of the theory of free groups (e.g. Johnson [1], Ch. 1), inner and outer automorphisms of groups in general, holomorphs, and some not so elementary modular arithmetic. Some of this material is usually considered too abstract by group theorists to be introduced in undergraduate courses, but here one was faced with a need for certain terminology, notation and ideas to simplify the description of $Aut(D_n)$. Perhaps this material is only viewed as being too abstract because it is often presented without it being necessary for the development or simplification of a solution to a problem. When one considered that it is this sort of situation that leads to new ideas and new concepts in mathematics, it is worth wondering if a small change in emphasis might not allow students some insight into the reason for the "menu" rather than being shown only the "finished meal" in mathematics courses.

Perhaps I should mention a slight disadvantage about projects of the form I have described in these two notes. After a student has been exposed to this sort of mathematics, where concepts are seldom studied, or introduced, unless necessary for further development, synthesis, simplification etc., it can happen that the usual style of lecture course seems to them hopelessly unmotivated, irrelevant and needlessly abstract. Even though one might like all pure mathematics courses to be presented in a better way, realistically one has to be cynical and warn a student, who knows how to do mathematics, but not necessarily how to remember unmotivated chunks of theory, that not all lecture courses in group theory are presented in this way. (I admit to exaggerating here to make a point. I should also mention that group theory is probably not the worst offender in this way.)

Finally I would mention that another student, this year, is attempting a similar analysis of automorphism groups of dicyclic groups. Also a glance through some of the older (pre 1930) group theory books provides some idea of the wealth of material in this general area which may be useful when planning out projects in group theory. (I suspect the same is true for other areas as well but my personal experience of projects has been more or less solely in this area.)

REFERENCES

1. JOHNSON, D.L.
"Topics in the Theory of Group Presentations", L.M.S. Lecture Notes No. 42, Cambridge University Press.

2. MILLER, G.A.
"Automorphisms of the Dihedral Groups", *Proc. Nat. Acad. Sci. U.S.A.*, 28 (1942) 368-371.

3. PORTER, T.
"Undergraduate Projects in Group Theory: Commutativity Ratios", *I.M.S. Newsletter*, No. 14 (1985) 44-49.

Department of Pure Mathematics,
University College of North Wales,
Bangor.

BOOKS RECEIVED

"INTRODUCTION TO DIFFERENTIAL GAMES AND CONTROL THEORY"

By *V.N. Lagunov*

Published by *Heldermann Verlag*, Berlin, 1985, vii + 285 pp.
DM 88. ISBN 3-88538-401-9

The main aim of the present book is to give a game-theoretic introduction to zero-sum two-person differential games. It is elementary and concise, not demanding from the reader any preliminary game-theoretic preparation and not requiring mathematical knowledge exceeding the modern technical-college course of higher mathematics.

To make it easier for the beginner to understand such a complex mathematical subject as a differential game the material is initially divided into two parallel streams: the elements of the general theory of games and the elements of the mathematical theory of optimal control. In the subsequent treatment both streams merge into a single channel: differential games.

"SECOND-ORDER SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE"

By *L.K. Hua, W. Liu And C.-Q Wu*

Published by *Pitman Publishing*, London, 1985, 291 pp.
Stg £16.50. ISBN 0-273-08645-6

This research note presents new results in the theory of pairs of second-order partial differential equations in the plane, with applications. Second-order systems of PDEs are reduced to their canonical form, from which the systems can be easily classified as elliptic, hyperbolic, parabolic or composite. Boundary value problems, initial value problems and also the more complicated mixed problems are investigated.

Attention is paid both to bi-analytical function theory governed by elliptic systems and to applications in elasticity. The discrete phenomena of the uniqueness of the characteristic problems for hyperbolic systems are discussed; also, the spline finite strip method and some numerical analyses for functional equations are provided.

Readership: Researchers and graduate students working in PDEs, generalized hyperanalytic function theory and functional equations. Also engineers who use the method of PDEs to solve engineering problems, particularly in elasticity and electrostatics.

"MULTIGRID METHODS FOR INTEGRAL AND DIFFERENTIAL EQUATIONS"

By *D.J. Paddon and H. Holstein*

Published by *Clarendon Press*, Oxford, 1985, xii + 323 pp.
Stg £30. ISBN 0-19-853606-2

Many problems in numerical analysis are reducible to the numerical solution of a system of algebraic equations. The multigrid method is a promising new technique for such problems which has been developed since the late 1970s. This volume contains the proceedings of a Summer School/Workshop on Multigrid Methods held at the University of Bristol in September 1983 and attended by many leading researchers in the field (most of the papers were revised later to include the authors' views and research up to July 1984).

BOOK REVIEWS

"THE BOOLE-DE MORGAN CORRESPONDENCE 1842-1864"

By *G.C. Smith*

Oxford Logic Guides, Published by *Oxford University Press*,
1982, Stg £19.00. ISBN 019-853183-4.

G.C. Smith of Monash University, Australia, has done mathematics and the history of mathematics a great service by editing the 90 or so letters between George Boole and Augustus De Morgan during the period 1842-1864. Smith has wisely divided the letters into the following categories:

1. Getting acquainted 1842-1845;
2. Mathematical logic and Ireland 1847-1850;
3. Probability and eccentricity 1851;
4. The laws of thought and marriage 1852-1856;
5. Books old and new; and homeopathy 1859-1861;
6. The controversy with Hamilton's successors; and the Jews 1861-1862;
7. From differential equations to spiritualism 1863-1864.

The book also includes short biographies of Boole and De Morgan, extensive commentary on the letters, almost complete bibliographies of both men, an appendix on Boole's theorem on definite integration and a historical epilogue concerning De Morgan's efforts to secure a pension for Boole's widow. All in all, Smith has crammed an incredible amount of information into 156 pages and the volume is handsomely produced by Oxford University Press.

The book contains the text of all the letters available to the author, though not De Morgan's reference for Boole's application for the professorship in Cork (see [1]). Smith has done a very fine job in painstakingly deciphering the handwriting of both Boole and De Morgan, a difficult job at the best of times. He comments with great depth and perception on both the mathematical and personal content of each letter and in particular he examines very closely the trains of thought of both men in the crucial period 1847-1850 while symbolic logic was taking shape in their minds, albeit in different forms. To my mind, Smith has done a fine job and his book is indispensible to those interested in either Boole or De Morgan or indeed the history of mathematics in general. I can recommend the book very strongly and it should find a place in every University Library so that students can see the actual evolution of mathematical concepts.

Much as I would like to give a book such as this unqualified praise, I must draw attention to the number of misprints and elementary errors it contains. These are all the more surprising when one realises that the book has emanated from Oxford University Press, but thankfully there is nothing that a careful proof-reading of a second edition could not remedy. The following is a list of potential corrections:

1. Page 2 Boole was married in 1855 not 1856.
2. Page 3 "obituary" is mispelled.
3. Page 33 Archbishop MacHale's first name was John not William.
4. Page 38 "be" should be "by".
5. Page 40 The title of Boole's major work was "An Investigation of the Laws of Thought", not "An Investigation into the Laws of Thought".
6. Page 142 "Edition" is mispelled.
7. Page 148 "Cambridge" is mispelled.

The bibliography of Boole's printed works contains over twenty errors and slips, all of them minor, which I have attempted to correct in [1].

The misprints and errors to which I have referred detract only slightly from the book which I regard as a fine piece of scholarship and welcome warmly. Rumour has it that the author is at present working on a companion volume on the correspondence of Boole and William Thomson, Lord Kelvin. I look forward eagerly to its publication.

REFERENCE

1. MacHALE, Desmond, *'George Boole - His Life and Work'*, Boole Press, Dun Laoghaire, 1985. ISBN 0-906783-05-4.

Des MacHale,
University College,
Cork

"THE ONE-DIMENSIONAL HEAT EQUATION"

(Encyclopaedia of Mathematics and its Applications - Vol. 23, Section : Analysis)

By *John Rozier Cannon*

Published by *Addison-Wesley Publishing Co.*, 1984, xxiii + 483 pp., Stg £61.20. ISBN 0-201-13522-1

A few weeks ago, a colleague presented the following problem to a number of Applied Mathematicians, including myself:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f ; \qquad 0 < x < 1, \qquad t > 0$$

$$u(x,0) = 0, \qquad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0$$

Are there any simple conditions on f that ensure the existence of a solution for which u, $\partial u/\partial x$, $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ are continuous in $[0,1] \times [0,t]$?

This is a question of regularity and such problems are usually tricky. However, it relates to the one-dimensional heat equation and what could be simpler? Many helpful suggestions were offered but nobody knew the answer. I set about my task of reviewing Cannon's book with added interest. I was not disappointed. The solution to the above and many more complex problems may be found there. The main thrust of the book is towards problems of existence, uniqueness, stability and other properties of solutions. It consists of a blend of the research of Cannon and others with the classical material and is written in the form of a monograph.

An undergraduate background in real and complex analysis, Lebesgue integration, Fourier series and transforms, familiarity with the concept of a Banach space and an elementary course in partial differential equations would be more than adequate to enable a comfortable reading of the book. This is noteworthy since most modern monographs which address the type of material in Cannon's book require considerably more mathematical maturity on the part of the reader.

Cannon starts by collecting together a number of basic inequalities and results from real and complex analysis and Lebesgue integration for reference purposes in the preliminary Chapter 0. He then develops in Chapter 1, the Weak Maximum, Comparison and Uniqueness theorems (for solutions continuous in the closure of the space-time domain in which the differential equation holds - the *parabolic domain*) and states without proof the Extended Comparison and Uniqueness theorems for solutions which admit a finite number of discontinuities on the boundary of the parabolic domain (*the parabolic boundary*). The latter theorems are fundamental to the development of the subject matter (mainly for uniqueness proofs).

The pure initial-value problem and initial-boundary value problems are analyzed in Chapters 3 through 6. He starts with the fundamental solution and develops boundary-integral representations over the parabolic boundary of the solutions which provide the basis for existence/uniqueness proofs for quite general data. In Chapters 6 and 7 he reduces a wide range of initial-boundary value problems to the solution of systems of boundary-integral equations and in Chapter 8 makes use of this formulation to deduce existence/uniqueness, continuous dependence on the data and *a priori* bounds. Chapter 9 covers pure boundary-value problems and periodic solutions. These chapters provide an excellent introduction to the subject. The analysis is always very clear, rigorous and well-organized.

The author takes the reader on a different route via Chapter 2 and 10 through 12. The Cauchy problem, which is ill-posed, is followed by analyticity properties of solutions which are used to study continuous dependence on the data for some ill-posed problems. Chapter 12 contains results of numerical experiments for some of the problems. These chapters are novel. Discussion of such matters is usually brief.

Chapter 13 deals with the inverse problem of measuring time-dependent diffusivity. The measurement problems are formulated by over-specifying the data.

The reader is introduced to moving-boundary problems in Chapters 14, 17 and 18. An elegant analysis of existence/uniqueness and various properties of the free-boundary is given for one-phase Stefan problems.

Finally, Cannon carries out an analysis of various initial-boundary-value problems for the inhomogeneous equation in Chapter 19 (wherein lay the solution of my colleague's problem) using techniques of integral representation and extends it to some quasilinear equations in Chapter 20.

The list of references to the literature covering the period 1800-1982 is truly encyclopaedic (taking up over 120 pages) and well-subclassified.

The author has certainly made all the above topics accessible to the first year graduate student. The mathematical background needed does not extend far beyond the list of results in Chapter 0. A few results on the convergence properties of Foruier series (assumed in Chapter 13) and in real and complex analysis (assumed in Chapter 10) could, profitably, have been included in the list. However, his achievement in leading the reader with such a modest background to such impressive results is remarkable. Considerable care and patience must have been exercised in the preparation of the book.

The book contains a wealth of problems. Some of them are set at the end of a chapter but most appear as proofs left to the reader. They vary in difficulty from straightforward extensions of the text to the fairly challenging. I have only one negative note to sound in this regard - one of the most frequently utilized results in the book (the Extended Comparison Theorem) was relegated to an exercise at the end of Chapter 15. Such an important result should have been part of the text. The last monograph on this topic of which I am aware is that of Widder in 1975 [7]. The scope of that book was a good deal narrower than that of Cannon's and contained no exercises. Cannon's problems provide a welcome pedagogic addition to the literature.

The book is of interest for a variety of reasons. It is classified under the section heading: Analysis. As a textbook on applied analysis it is excellent. Standard theorems are really put to work. The Arzela-Ascoli Tehorem, for example, was used several times in the establishment of an existence proof. There is considerable interest at the present time in the development and application of Boundary-Integral Equation (B.I.E.) methods to dynamic problems in Heat

Transfer [3,4] and related problems. The abstract method which has been analysed in Cannon's book is known in the B.I.E. community as the 'indirect B.I.E. method' and is less popular than the so-called 'direct method' [2]. The material in the book could provide a basis for the numerical analysis of indirect methods and could possibly be extended to cover direct methods also. I am unaware of any work along these lines.

The approach to parabolic equations which has been demonstrated by the author is not, of course, the only one. A different domain of ideas (which, for example, finds application in the Finite Element solution of parabolic equations) is based on a distributional approach (involving concepts of Sobolev spaces and Semigroups). The mathematical apparatus needed to deal with such ideas is a good deal deeper than that required by Cannon [1,5,6]. The quality of production (layout, print, diagrams) is excellent. I counted about thirty errors of various kinds (trivial misprints, references to incorrect or non-existent equations/theorems, a few incorrect statements of a minor nature).

In summary, I would consider this to be among the best books I have read on partial differential equations. Every university library should have a copy.

REFERENCES

1. AUBIN, J.P.
   'Applied Functional Analysis', Wiley Interscience (1979).

2. BANERJEE, P.K. and BUTTERFIELD, R.
   'Boundary Element Methods in Engineering Science', McGraw-Hill (1981).

3. BANERJEE, P.K. and BUTTERFIELD, R.
   'Developments in Boundary Element Methods', Applied Science Publishers (1979).

4. BREBBIA, C.A. and MAIER, G.
'Boundary Elements VII, Volumes I and II', Springer-Verlag (1985).

5. LIONS, J.L. and MAGENES, E.
'Non-Homogeneous Boundary Value Problems and Applications, Volumes I, II and III', Springer-Verlag (1972).

6. SHOWALTER, R.E.
'Hilbert-Space Methods for Partial Differential Equations', Pitman (1979).

7. WIDDER, D.V.
'The Heat Equation', Academic Press (1975).

James J. Grannell,
Department of Mathematical Physics,
University College, Cork

"ADVANCED ENGINEERING MATHEMATICS"

By *Ladis D. Kovach*

Published by *Addison-Wesley Publishing Co., Inc.*, 1982, xi + 706 pp. Stg £14.50. ISBN 0-906783-05-4

What is the engineer's role in society? How does mathematics assist the engineer? What mathematical skills does an engineer need? Who is best equipped to teach him these skills? Should he receive a shallow treatment of a great variety of different mathematical topics or a thorough treatment of a few?

These are but a few of the many questions that must constantly occupy the minds of faculty members in any institutions that train future engineers; and anybody intending to write a

mathematics textbook for students of modern engineering science must address himself to them. The resulting book will, inevitably, reflect the author's perceptions of what constitutes a suitable mathematical training for the engineer who will tackle tomorrow's problems.

In the preface of the book under review, the author declares "that *design* is the primary function of an engineer"; and that "a prerequisite for design is *analysis*". He goes on to announce his purpose in writing the book: "our objective is to demonstrate in a number of ways how an engineer might strip a problem of worldly features that are unimportant complexities, approximate the problem by means of a mathematical representation, and analyze *this*." In this respect, the author's intentions are no different to those of writers of similar books in which mathematical modelling is used to come to grips with physical problems.

A number of mathematical techniques that are used in engineering analysis are discussed in the text. The chapter headings may convey some idea of the material covered in the book:

1. First-Order Ordinary Differential Equations;
2. Higher-Order Differential Equations;
3. The Laplace Transformation;
4. Linear Algebra;
5. Vector Calculus;
6. Partial Differential Equations;
7. Fourier Series and Fourier Integrals;
8. Boundary-Value Problems in Rectangular Coordinates;
9. Boundary-Value Problems in Other Coordinate Systems;
10. Complex Variables.

The material is arranged so that a topic is not introduced until it is needed. Thus, applications of the Laplace transformation to the solution of linear systems of differential equations motivate an examination of systems of algebraic equations; hence the reason why Chapter 4 follows from Chapter 3. Again, conformal mapping is treated in the last chapter because a need for it was anticipated in the earlier chapters. In this way, the author carries out his plan to write the text so that the topics flow from one to the next.

Over 2000 exercises are given. Some of these are meant to elucidate points in the text, others are designed to provide drill for the student, while a third group is meant to challenge the student's understanding of the techniques used. Answers and hints to selected exercises are presented. Several of the exercises are incorrectly stated.

The book is well-written, very readable and has been very carefully proofread. I detected only five typographical errors - on p. 443, ℓ.11; p. 597, ℓ.13; p. 621, ℓ.17; p. 622, ℓ.1 and p. 635, ℓ.1 - and these are obvious and not likely to trouble the reader.

A feature of the book is the number of very brief biographical sketches that the author gives, either in the body of the text or in footnotes. Indeed, I know of very few books where one is likely to learn the names and origins of so many mathematicians whose work has made an impact in engineering. However, I could not help noticing that the author could not decide on Sir William Rowan Hamilton's nationality: he is referred to as an English mathematician on p. 227 and as an Irish mathematician on p. 275. On the other hand, George G. Stokes is referred to as an Anglo-Irish mathematical physicist. Still, it is nice to see homage paid to our predecessors'.

A few "howlers" have crept into the book. For instance, it is mentioned on p. 597 that " ... some sets cannot be classified as open or closed. The set (of complex numbers z satisfying) $1 \le \mathrm{Re}\, z \le 2$ is such a set (Exercise 1)." The answer to Exercise 1 reveals why: "The set is not closed since it has no boundary in the y-direction"! Again, on p. 621 it is asserted that Cauchy's integral formula (giving the value of an analytic function at a point interior to a simple closed curve in terms of its values on the curve) " ... is called a *formula* because it shows that the value of an analytic function at an isolated singularity can be calculated by means of a contour integral!

Considering the importance of Fourier analysis in applications, it is a little surprising to find on p. 422 the statement "that the convergence problem for a Fourier series is still unsolved." One would have thought that, by now, Lennart Carleson's 1967 result about the almost everywhere convergence of the Fourier series of a square-integrable function would have filtered through to most mathematicians who teach engineering students.

How does the book compare with others on the market? I tested it against two books with exactly the same title, one by Erwin Kreyszig, published by John Wiley & Sons Inc., 1979, 4th ed., and the other by C. Ray Wylie and Louis C. Barrett, published by McGraw-Hill International Book Company, 1982, 5th ed. The first editions of both of these appeared in 1962, and are very highly regarded. In my opinion, the book under review is unlikely to challenge either of them in the market place.

Finbarr Holland,
Mathematics Department,
University College,
Cork

PROBLEM PAGE

First of all, here's a simple arithmetical problem.

1. The recurring decimal $0.\dot{0}0\dot{1}$ represents a rational number. How long is the recurring block of digits in the square of this number?

Next, a problem sent in by Des MacHale.

2. Prove that at least one of the numbers $\pi + e$, πe, is transcendental.

Finally, a quickie on infinite series.

3. Suppose that $a_n \geq 0$, for $n = 1,2, \ldots$. How large can

$$\sum_{n=1}^{\infty} \frac{a_n}{e^{a_1+a_2+\ldots+a_n}}$$

be?

Now here are the solutions to some earlier problems.

1. If $1 < p \leq 2$ and $\alpha = \frac{\pi}{2p}$ show that

$$\left(\frac{\cos\theta}{\cos\alpha}\right)^p \geq 1 + \tan\alpha\cos p\theta, \qquad 0 \leq \theta \leq \alpha .$$

Since there is equality when $\theta = \alpha$, it is enough to prove that

$$-\sin\theta(\cos\theta)^{p-1} \leq -\sin\alpha(\cos\alpha)^{p-1}\sin p\theta, \qquad 0 \leq \theta \leq \alpha$$

which follows immediately if the function

$$\frac{\sin\theta\,(\cos\theta)^{p-1}}{\sin p\theta}$$

is decreasing for $0 < \theta \leq \alpha$. On differentiation this reduces

to proving that, for $0 < \theta \leq \alpha$,

$$1 \leq (p-1)\tan^2\theta + p\frac{\tan\theta}{\tan p\theta}$$

$$\Leftrightarrow \sec^2\theta \leq p\left[\tan^2\theta + \frac{\tan\theta}{\tan p\theta}\right]$$

$$\Leftrightarrow \sin p\theta \leq \frac{p}{2}\left[(1 - \cos 2\theta)\sin p\theta + \sin 2\theta\cos p\theta\right]$$

$$\Leftrightarrow (2-p)\sin p\theta \leq p\sin(2-p)\theta .$$

Since $0 \leq 2-p < p$ and $p\theta \leq \pi/2$, this final inequality holds because $\sin t/t$ is decreasing for $0 < t \leq \pi/2$. The proof is complete.

Remarks

1. The special case $\theta = 0$ can be written as:

$$1 \geq (\cos\alpha)^p + (\cos\alpha)^{p-1}\sin\alpha, \qquad 1 < p \leq 2,$$

where $\alpha = \pi/2p$. Is there a simpler proof of this?

2. The inequality appears in a paper by Matts Essen ("A Superharmonic Proof of the M. Riesz Conjugate Function Theorem, *Ark. for Mat.*, 22 (1984) 241-249). It is needed to show that a certain function is superharmonic and thus to prove the M. Riesz theorem (with the best possible constant). The 'elementary' proof of the inequality, given above, was first found by Wolfgang Fuchs.

2. Suppose that $0 = \phi_1 \leq \ldots \leq \phi_n \leq \pi$, that $A = [\sin(|\phi_i - \phi_j|)]$ and that $||A|| = \max\{||Ax|| : ||x|| = 1\}$. Show that

$$||A|| \leq \cot\left(\frac{\pi}{2n}\right)$$

and characterize the case of equality.

This problem was kindly sent in by Bob Grone of Auburn University. He also supplied the following solution, which depends on two results which are not particularly well known. I'll state these results first and discuss them in more detail after showing how they solve the problem.

Result 1 An nxn matrix A with non-negative entries has an eigenvalue λ with the following properties:

(i) $\lambda \geq |\mu|$, for all eigenvalues μ of A, and

(ii) λ has an associated eigenvector $x = [x_1, \dots x_n]^T$ of unit norm for which

$$x_i \geq 0, \qquad i = 1, \dots, n.$$

Result 2 If $a_1, \dots, a_n$ are the lengths of the sides of a plane closed polygon and θ_{ij} is the angle between the positive directions of the sides a_i and a_j, then the area of the polygon is

$$\tfrac{1}{2} \sum_{i \neq j} a_i a_j \sin\theta_{ij}.$$

To solve the problem, note that $A = [\sin|\phi_i - \phi_j|]$ is symmetric and so one can form an orthogonal basis for $\mathbb{R}^n$ of eigenvectors of A. Using this one easily sees that

$$||A|| = \lambda = x^T A x = 2\sum_{i>j} x_i x_j \sin(\phi_i - \phi_j),$$

where λ is the eigenvalue and x the eigenvector described in Result 1.

According to Result 2, the sum on the right is exactly twice the area of the centro-symmetric 2n-gon whose first n consecutive edges have lengths $x_1, \dots, x_n$ and are inclined at angles $\phi_1, \dots, \phi_n$ to the x-axis. By the isoperimetric inequality, this is at most twice the area of the regular 2n-gon with perimeter $2(x_1 + \dots + x_n)$. Hence

$$||A|| \leq \frac{(x_1 + \dots + x_n)^2}{n}\cot(\tfrac{\pi}{2n}).$$

Among all positive unit vectors x, the maximum value of $(x_1 + \dots + x_n)^2$ equals n, and this is obtained uniquely at $(1/\sqrt{n}, \dots, 1/\sqrt{n})$. Thus

$$||A|| \leq \cot(\tfrac{\pi}{2n}),$$

with equality if and only if $\phi_{i+1} = i\frac{\pi}{n}$, $i = 1, \dots, n$.

Result 1 belongs to the Perron-Frobenius theory of non-negative matrices (see, for example, E. Seneta, "Non-Negative Matrices and Markov Chains, Springer). To prove it one puts

$$K = \{x \in \mathbb{R}^n : ||x|| = 1, x_i \geq 0, i = 1, \dots, n\}$$

and

$$r(x) = \min_i \frac{1}{x_i} \sum_j a_{ij} x_j \qquad x \in K.$$

One then shows that r is bounded above on K and that

$$\lambda = \sup\{r(x) : x \in K\}$$

is attained at an eigenvector x with associated eigenvalue λ.

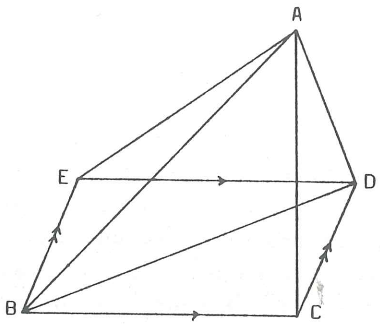
To complete the proof note if $y = [y_1, \dots, y_n]^T$ is any eigenvector of unit norm with associated eigenvalue μ then

$$|\mu y_i| \leq \sum_j a_{ij}|y_j|, \qquad i = 1, \dots, n.$$

and so

$$|\mu| \leq \min_i \frac{1}{|y_i|} \sum_j a_{ij}|y_j| \leq \lambda.$$

Result 2 is easily proved by induction using the following fact about areas:



Area ΔABC = Area ΔBCD + Area ΔAED.

Jim Clunie points out that Result 2 appears in Hobson's 'Plane Trigonometry' and also remarks that 'They don't write books like that anymore!"

Phil Rippon,
Open University,
Milton Keynes.

THE IRISH MATHEMATICAL SOCIETY

Instructions to Authors

The Irish Mathematical Society seeks articles of mathematical interest for inclusion in the *Bulletin*. All areas of mathematics are welcome, pure and applied, old and new.

In order to facilitate the editorial staff in the compilation of the *Bulletin*, authors are requested to comply with the following instructions when preparing their manuscripts

1. Manuscripts must be typed on A4-size paper and double-spaced.
2. Pages of the manuscripts should be numbered.
3. Commencement of paragraphs should be clearly indicated, preferably by indenting the first line.
4. Facilities are available for italics and bold-face type, apart from the usual mathematical symbols and Greek letters.

   Words or phrases to be printed in italics should be singly underlined in the manuscript; those to be printed in bold-face type should be indicated by a wavy underline.
5. Diagrams should be prepared on separate sheets of paper (A4-size) in black ink. Two copies of all diagrams should be submitted: the original without lettering, and a copy with lettering attached.
6. Authors should send two copies of their manuscript and keep a third copy as protection against possible loss.

If the above instructions are not adhered to, correct reproduction of a manuscript cannot be guaranteed.

Correspondence relating to the *Bulletin* should be addressed to:

Irish Mathematical Society Bulletin,
Department of Mathematics,
University College,
Cork.