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## NEWSLETTER

EDITOR

*Patrick Fitzpatrick*

The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

ASSOCIATE EDITOR

*Martin Stynes*

The *Newsletter* also seeks articles of mathematical interest written in an expository manner. All areas of mathematics are welcome, pure and applied, old and new.

Detailed instructions relating to the preparation of manuscripts may be found on the inside back cover.

Correspondence relating to the *Newsletter* should be sent to:


Irish Mathematical Society Newsletter,
Department of Mathematics,
University College,
Cork.

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## *Editorial*

*I am very happy to take over as editor of the Newsletter, and it is my first and most pleasant duty to thank Donal Hurley for the unstinted dedication, energy and professionalism that he brought to the editorial task during his period of tenure. He has achieved a remarkable transformation of the Newsletter: from A4-format, stencilled and duplicated broadsheet, to the present high-quality publication. The measure of his success is the great enjoyment with which the Newsletter is received by the mathematical community in Ireland. We wish him well as he moves on to other activities.*

*It is a pleasure also to welcome Martin Stynes as associate editor. We intend to continue the editorial policy already established, encouraging contributors to write in an expository style as far as possible. The current issue shows that, to a large extent, our aims in this direction are being met by authors.*

*It is interesting to notice, as well, that this issue contains a large proportion of articles and papers <u>not</u> written at the invitation of the editors. We would like to encourage this trend - perhaps it is a sign of increasing acceptance of the Newsletter as a repository for interesting mathematics, with a bias towards exposition aimed at a wide readership, rather than research. So keep those articles coming!*

*As always thanks are due to our typist Leslie Brookes, whose knowledge of mathematics ensures that proofreading becomes a very minor task, and to the NBST for printing the Newsletter on behalf of the IMS.*

*Pat Fitzpatrick*

# IRISH MATHEMATICAL SOCIETY

## I.M.S. COMMITTEE, 17th DECEMBER 1984

Based on Minute Notes by N. Buttimore
Modified Following Comments on Draft of 19/4/'85

Attendance: N. Buttimore, P. Boland, G. Enright, A. O'Farrell.

Remark : Quorum = 5.

Apologies : P. Fitzpatrick, D. Hurley, M. Stynes, F. Holland, S. Tobin.

A.G. O'Farrell (President and Acting Secretary) in the Chair.

1. The joint meeting with the I.M.S. was discussed. The following committee was set up to superintend it: T.T. West, D. O'Donovan, A.G. O'Farrell.

2. From now on, headed notepaper is to be produced at Maynooth, using TEX. It is to be Xeroxed from masters. Any officer needing paper, contact me. Note that UCD can copy at 2p/page.

3. Correspondence. The Acting Secretary had replied to some routine letters in routine fashion. This was approved.

4. Expenses of delegate. Our representative on the R.I.A. National Committee should claim expenses from the R.I.A., rather than the Society.

5. Treasurer's Business

5.1 G.M. Enright, the Treasurer, explained the proposed changes to the Constitution and Rules of the Society. These changes were tabled at the ordinary meeting in April 1984, and are to be voted upon at the forthcoming ordinary meeting on 21st December 1984.

5.2 There are 163 members. Student nominees on institutional members will appear in the March supplement to the Membership List.



5.3 The following proposed changes in subscriptions (to take effect for the session 1985-'86) were agreed. They have to be submitted to the Society.

Ordinary and Overseas Members: 5 Irish Pounds;
Institutional Members: 35 Irish Pounds;
Library Subscriptions: 20.00 US Dollars
or 20 Irish Pounds.

5.4 The Treasurer gave a preview of the accounts for 1983/'84 which were prepared for the forthcoming ordinary meeting, and which showed a balance at 30/9/'84 of £655.26.

6. P. Fitzpatrick was appointed Editor of the *Newsletter*, and M. Stynes as Assistant Editor.

7. It was decided to arrange the proposal of M. Newell as President and A.G. O'Farrell as Secretary at the ordinary meeting, subject to the agreement of M. Newell in the meantime.

*A.G. O'Farrell*

Draft of 30/5/'85

## IRISH MATHEMATICAL SOCIETY

Ordinary Meeting, Tuesday, 4/4/'85, at DIAS

The meeting commenced at 12.15 p.m. There were 11 members present. The President, Professor M. Newell, sent his apologies, and the Vice-President, Professor S. Dineen, took the Chair.

1. Minutes. The minutes of the meeting of 21/12/'84 were taken as read.

2. IMTA Reciprocity. The meeting approved the agreement, which had been negotiated by M. Clancy and recommended by the Committee of the Society. Subject to ratification by the IMTA delegate conference, the agreement provides that a member of one society will become an associate member of the other upon payment of £1.50 to the latter (this amount being subject to change from time to time). Such an associate member of either society shall be entitled to one copy of that society's journal, but is not entitled to voting rights in that society.

   It was agreed to ask the Treasurer to coordinate the payment of these fees to the IMTA, by means of an appropriate modification to the procedure for collecting annual dues.

3. IMS-LMS Joint Meeting. The meeting was informed of the state of plans for this first joint meeting, devoted to C*-algebras and Operator Theory, which is to take place in Dublin on Friday and Saturday, the 21st and 22nd of March, 1986. The meeting is planned to involve four invited speakers, and the final details should be available by the end of the year. It was noted with satisfaction that the response of the LMS to the suggestion of a joint meeting had been enthusiastic and generous. The Chairman noted the Society's debt to R. Timoney, who was responsible for the initial work on this project.



Professor Quinlan suggested that the DIAS School of Theoretical Physics might wish to sponsor the event.

4. Orlov-Shcharansky Campaign. Following the Society's support for this campaign, the Secretary is ensuring the circulation within Ireland of the Newsletter of the campaign. He asked that he be informed by members if their institution is not receiving this Newsletter.

5. Other Business. The Chairman noted that there were still a number of prominent institutions which had not become institutional members of the Society. He asked members to ensure that their institutions joined, and he also suggested that other schools might adopt the practice of his department, which is to make the gift of an introductory year's membership to all visitors.

The meeting ended at 12.35 p.m.

*Anthony G. O'Farrell*, Secretary

## ERRATA

The Constitution and Rules of the I.M.S. published in *Newsletter* 13 (March 1985) unfortunately contained two errors which are corrected as follows:

1. Constitution, Paragraph 5: substitute "terms" for "years".
2. Rules, Paragraph 4: insert a second sentence: "The President and Vice-President may not continue in office for more than two consecutive terms."

# NEWS AND ANNOUNCEMENTS

## SUMMARY OF RESULTS OF 1985 IRISH NATIONAL MATHEMATICS CONTEST

The Seventh Irish National Mathematics Contest was held on Tuesday, February 26, 1985. A total of 1,630 pupils from 86 schools entered for the contest. Last year, 1,634 pupils from 84 schools competed.

Returns were received from 62 schools on behalf of 1,231 pupils who participated this year. A preliminary analysis of the results shows that the overall average mark was about 48. (It is our intention to publish the results of a more elaborate analysis of this year's contest and of the previous six contests at a later date.) Thirty-five participants scored 80 marks or better, a considerable improvement on last year's results when only 21 reached the same level. The names of the top eleven contestants are shown on the accompanying Roll of Honour.

This year's INMC winner is:

Stephen O'Brien,
Gonzaga College,
Ranelagh,
Dublin 6.

Stephen scored 108 out of a possible 150, a highly commendable achievement.

Gonzaga College - which, incidentally, produced last year's winner as well - also returned the highest team score (the sum of the highest three scores by individual contestants), viz., 267. The winning team was composed of Stephen O'Brien, Gavin O'Sullivan and Malachy McAllister. Second place was taken by Belvedere College, Dublin 1, with a score of 267, and third by O'Connell School, Dublin 1, who scored 261.



A prize-giving ceremony will be arranged early in December to honour the top scorers. On that occasion Stephen O'Brien will be presented with an Award Pin on behalf of the Mathematical Association of America, whose examination materials for the American High School Mathematics Examination, have been used in all previous years' contests.

### ROLL OF HONOUR

| Candidate | School | Score |
|---|---|---|
| Stephen O'Brien | Gonzaga College, Ranelagh, Dublin 6 | 108 |
| Garrett Brennan | Oatlands College, Mount Merrion, Blackrock, Co. Dublin | 96 |
| Liam C. O'Suilleabhain | Belvedere College, Gt Denmark Street, Dublin 1 | 95 |
| James Farrelly | Franciscan College, Gormanstown, Co. Meath | 92 |
| Jeremy W. Bolton* | Academical Institution, Coleraine, Co. Londonderry | 91 |
| Hugh T. McManus | Belvedere College, Gt Denmark Street, Dublin 1 | 91 |
| Paul O'Farrell | St Benildus College, Upper Kilmacud Road, Dundrum, Dublin 14 | 91 |
| Alex T. Bradley | Clongowes Wood College, Naas, Co. Kildare | 90 |
| Colum Lawlor | C.B.S., The Green, Tralee, Co. Kerry | 90 |
| Yip W. Lee* | Coleraine Academical Institution, Coleraine, Co. Londonderry | 90 |
| Brian Salmon | O'Connell School, North Richmond Street, Dublin 1 | 90 |

* Did not participate in the IIMC

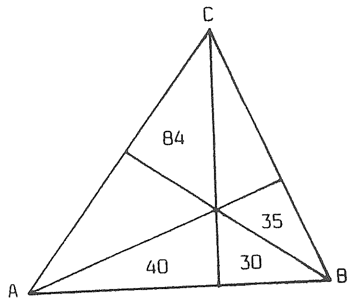
## SUMMARY OF RESULTS OF 1985 IRISH INVITATIONAL MATHEMATICS CONTEST

The Third Irish Invitational Mathematics Contest was held on Tuesday, March 19, 1985. Those who scored 80 or more in the INMC were invited to take part in the IIMC; returns were received on behalf of 29. Contestants had three hours to answer fifteen questions with integer solutions; partial credit was not given. Colm Morgan from Abbey Grammar, Newry, Co. Down, was our top scorer; Colm got eight questions correct, a very fine performance on what was a difficult test.

Here is a selection of the questions:

(3) Find c if a, b and c are positive integers which satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$.

(6) As shown in the figure on the right, $\Delta ABC$ is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of $\Delta ABC$.




(7) Assume a, b, c and d are positive integers such that $a^5 = b^4$, $c^3 = d^2$ and $c - a = 19$. Determine $d - b$.

(8) An ellipse has foci at (9,20) and (49,55) in the xy-plane and is tangent to the x-axis. What is the length of its major axis?

(This would have been beyond the scope of most Irish students. It is, however, a very nice question. Tom Laffey thinks that the principles behind the solution make a theorem that was missed by our predecessors. I myself incorporated the result into a course on Conics, which I gave to B.A. students this year.)



(12) Let A, B, C and D be vertices of a regular tetrahedron, each of whose edges measures 1 metre. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = n/729$ be the probability that the bug is at vertex A when it has crawled 7 metres. Find the value of n.

*Finbarr Holland*

## PERSONAL ITEMS

*Dr Ray Ryan* of the Mathematics Department, UCG, is visiting the Department of Mathematics at Kent State University for the academic year 1985-'86.

*Professor Robin Harte* of the Mathematics Department, UCC, will be at the University of Iowa on sabbatical leave during the academic year 1985-'86.

*Dr Gabrielle Kelly* of the Statistics Department, UCC, will be at Columbia University on sabbatical leave during the academic year 1985-'86.

*Dr Niall Ó Murchadha* of the Department of Experimental Physics, UCC, will spend the period September 1985-March 1986 on leave at the University of British Columbia.

*Mr Micheál Ó Searcóid* has been appointed to a temporary lectureship at the Mathematics Department, UCC, for the academic year 1985-'86. His research interests are in Operator Theory

*Dr Alastair Wood* has been appointed to the Westinghouse Chair of Applied Mathematical Sciences at NIHE, Dublin.



## A MATRIX JOKE

*Robin Harte*

1. If $x = (x_{ij}) \in A^{n,n}$ is an nxn matrix with entries $x_{ij}$ in a ring A with identity 1, under what conditions does it have a two-sided inverse $x^{-1} \in A^{n,n}$? If the ring A is commutative, then the answer is very nearly the same as for the real or the complex numbers:

$$x \text{ invertible in } A^{n,n} \iff |x| \text{ invertible in } A, \qquad (1.1)$$

where $|x|$ denotes the *determinant* of x, defined [5, Chapter 5] in any one of the usual ways. If the ring A is not commutative then the formulae for the determinant become ambiguous, unless we restrict to matrices $x = (x_{ij})$ which are *commutative*, in the sense that their entries form a commutative set $\{x_{ij}\}$. With this restriction implication (1.1) was demonstrated for 2x2 matrices of Hilbert space operators by Halmos [1, Problem 55], extended to nxn matrices of Banach algebra elements using the spectral mapping theorem [3, Example 2.4], and is now given in full generality by Halmos again [2, Problem 70]. In this note we will demonstrate that (1.1) holds separately for left and right inverses, at least for 2x2 matrices: the argument seems to depend on a joke.

2. Suppose that $x = (x_{ij})$ is a commutative nxn matrix over the ring A, with determinant $|x| \in A$, and *cofactor* $x^{\sim} \in A^{n,n}$, in the sense of the usual 'adjugate' or 'classical adjoint' matrix of x: then we recall *Cramer's rule*,

$$x^{\sim}x = xx^{\sim} = |x|\underline{1}, \qquad (2.1)$$

and

$$\underline{1}^{\sim} = \underline{1},$$

where $\underline{1} = (\delta_{ij})$ is the identity matrix. If also $y = (y_{ij})$ is another commutative matrix, and if in addition *the entries of*

$y$ *commute with the entries of* $x$, then we have the product formula

$$(xy)^\sim = y^\sim x^\sim, \qquad (2.3)$$

and hence also

$$|xy| = |x||y| = |y||x|. \qquad (2.4)$$

Backward implication in (1.1) is clear from (2.1); conversely if a commutative matrix $x$ has a two-sided inverse $x^{-1}$ in $A^{n,n}$, and if $x^{-1} = y$ is commutative and has its entries commuting with those of $x$, then (2.4) will guarantee that $|x|$ is invertible in $A$. The second Halmos argument [2, Problem 70] demonstrates this by noting that if $z \in A^{n,n}$ and $t \in A$ are arbitrary then there is implication

$$xz = zx \Rightarrow zx^{-1} = x^{-1}z \qquad (2.5)$$

and

$$x(t\underline{1}) = (t\underline{1}) \Longleftrightarrow \mathrm{AND}_{ij}(x_{ij}t = tx_{ij}). \qquad (2.6)$$

3. The analogue of (1.1) holds separately for left and right inverses: if $x \in A^{n,n}$ is commutative then

$$x \textit{ left invertible in } A^{n,n} \Longleftrightarrow |x| \textit{ left invertible in } A \qquad (3.1)$$

and

$$x \textit{ right invertible in } A^{n,n} \Longleftrightarrow |x| \textit{ right invertible in } A. \qquad (3.2)$$

We shall confine ourselves to the proof of (3.1) when $n = 2$:

THEOREM *If* $a, b, c, d$ *are mutually commuting elements of* $A$ *then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \textit{ left invertible in } A^{2,2} \Longleftrightarrow ad-bc \textit{ left invertible in } A. \qquad (3.3)$$



Proof. From (2.1) we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & \\ 0 & \end{pmatrix}\begin{pmatrix} 0 \\ ad-bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad (3.4)$$

which gives backward implication in (3.1). Conversely if

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (3.5)$$

with no commutativity assumptions on $a'$, $b'$, $c'$, $d'$ in $A$, then (3.4) gives

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} ad-bc & \\ 0 & \end{pmatrix}\begin{pmatrix} 0 \\ ad-bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \qquad (3.6)$$

We now come to what we think is the joke: if you take apart (3.6) and then reassemble its four constituent equations, you get

$$\begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}\begin{pmatrix} ad-bc & \\ 0 & \end{pmatrix}\begin{pmatrix} 0 \\ ad-bc \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (3.7)$$

The joke is now over: another application of (3.5) gives two (possibly equal) left inverses for $ad-bc$ in $A$:

$$(a'd'-b'c')(ad-bc) = (d'a'-c'b')(ad-bc) = 1 \qquad (3.8)$$

The analogue of (3.3) for right inverses, or indeed for left and for right zero-divisors, may be left to the reader. It is also possible to extend the argument of (3.3) to 3x3 matrices, although the joke is not nearly so funny. We shall give elsewhere [4] an inductive proof of (3.1) and (3.2) based on a proof of (1.1) due to Tom Laffey.

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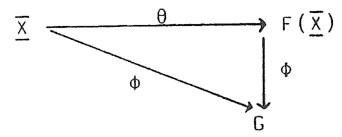
# FREE TOPOLOGICAL GROUPS

*Bernard R. Gelbaum*

The purpose of this paper is to provide a brief expository sketch of [1].

If $\underline{X}$ is any set, the free group $F(\underline{X})$ is defined abstractly as follows: $F(\underline{X})$ is a group such that if $G$ is any group and if $\phi : \underline{X} \mapsto G$ is any map of $\underline{X}$ into $G$ then there is a homomorphism $\Phi : F(\underline{X}) \mapsto G$ so that the diagram below commutes:




(*)

The embedding $\theta$ is fixed and is independent of $\phi$ and of $G$.

The existence of $F(\underline{X})$ is assured by the construction described next.

A *word* is a finite sequence $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ in which $x_i$ is an element of $\underline{X}$ and each $\varepsilon_i = \pm 1$. The product of two words $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ and $y_1^{\delta_1} \ldots y_m^{\delta_m}$ is the word $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} y_1^{\delta_1} \ldots y_m^{\delta_m}$. The collection $W$ of all words is thus an associative semigroup. The subsemigroup $S$ generated by all words of the form $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ in which $x_1 = x_2 = \ldots = x_n$ and

$$\sum_{i=1}^{n} \varepsilon_i = 0$$

leads to the quotient structure $W/S$, a group $F(\underline{X})$ for which $x_n^{-\varepsilon_n} \ldots x_1^{-\varepsilon_1}$ is a representative of the inverse of the element represented by $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$.

If $\underline{X}$ is a topological space, the natural object corresponding to $F(\underline{X})$ is a topological group for which the same diagram (*) obtains and where $\theta$ is a fixed topological embedding, $G$ is a topological group and $\phi, \Phi$ are continuous. Since a topological group $\Gamma$ is completely regular (if $x$ is a point and $U$ is an open set containing $x$ there is a continuous map $f : \Gamma \to [0,1]$ such that $f(x) = 1$, $f(\Gamma \setminus U) = 0$) $\underline{X}$ must be completely regular. The questions of existence and uniqueness are answered by the following theorem which is proved independently in [3] and [4] and can be proved in still another way [1] as outlined below.

Theorem. *If $\underline{X}$ is completely regular space there is a topological group $F(\underline{X})$ a topological embedding $\theta : \underline{X} \to F(\underline{X})$ such that for $\phi$ a continuous map into the topological group $G$ the diagram* (*) *commutes and $\Phi$ is a continuous homomorphism.*

The proof consists of a number of simple steps:

1. Let $\mathbb{H}_1$ be the group of all quaternions of norm 1.

2. Let $C(\underline{X}, \mathbb{H}_1)$ be the set of all continuous maps $f : \underline{X} \to \mathbb{H}_1$.

3. For each $f$ in $C(\underline{X}, \mathbb{H}_1)$ let $\mathbb{H}_{1f}$ be $\mathbb{H}_1$ and let $\mathcal{H}$ be the Cartesian product $\prod_f \mathbb{H}_{1f}$.

Then $\mathcal{H}$ is a compact topological group and the map $\theta : \underline{X} \to \{f(x)\}$(the *index* is $f$) is a topological embedding of $\underline{X}$ into $\mathcal{H}$ (here the complete regularity enters for the first time)

4. The group $\mathbb{H}_1$ and the group $\mathcal{R}$ of rotations of $\mathbb{R}^3$ are isomorphic (according to the correspondence: for $R$ in $\mathcal{R}$ and $(x,y,z)$ in $\mathbb{R}^3$ let $xi + yj + zk$ be the corresponding quaternion; then the quaternion corresponding to $R(x,y,z)$ is $g(xi + yj + zk)g'$ for a unique $g$ in $\mathbb{H}_1$ ($g'$ = conjugate of $g$).

5. The group $\mathcal{R}$ and hence $\mathbb{H}_1$ contains an infinite set $\{g_1, g_2, \ldots\}$ that is *free* [2]. That is, if $g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n} = 1$ then the word $g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n}$ is in the subsemigroup $S$ of the semigroup $W$ of words generated by the set $\{g_1, g_2, \ldots\}$.



6. If $\{p_k\}_{k=1}^{n}$ is a set of $n$ different points in $\underline{X}$ there is in $C(\underline{X}, \mathbb{H}_1)$ an $f$ such that $f(p_k) = g_k$. (Here the complete regularity of $\underline{X}$ enters again.)

Thus the group $F_0(\underline{X})$ generated in $\mathcal{H}$ by $\theta(\underline{X})$ is a free and topological group $(F_0(\underline{X})$ (for the *set* $\underline{X}$) in the topology inherited from $\mathcal{H}$). Thus $\mathcal{T}$, the set of all *group topologies* on $F(\underline{X})$, is not empty. The topology $\sup(\mathcal{T})$ makes $F(\underline{X})$ a topological group described in the conclusion of the theorem stated above.

Remarks. 1. The nub of the proof is found in 5 and 6 above. The existence of *some* group topology on $F(\underline{X})$ permits the conclusion (*via* $\sup(\mathcal{T})$) that $F(\underline{X})$ may be topologized to conform to the diagram (*).

2. If $\underline{X}$ is set, $F(\underline{X})$ is unique and if $\underline{X}$ is a completely regular space $F(\underline{X})$ is unique.

3. A topological group $G$ is *free* by definition if whenever $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} = 1$ then the word $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is in the subsemigroup $S$ in the semigroup $W$ of words constructed from the *set* $G$. Thus $F_0(\underline{X})$ is a topological group and is free but does not, *a priori*, conform to the requirements of (*) if $\underline{X}$ is a completely regular space. Indeed, if $G = F(\underline{X})$, *the* free topological group, then for (*) to be valid $F(\underline{X})$ must indeed be endowed with the topology $\sup(\mathcal{T})$.

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# HYPERBOLIC BEHAVIOUR OF GEODESIC FLOWS*

*Donal Hurley*

## INTRODUCTION

Geodesic flows, particularly those on manifolds of negative curvature, have been a rich source in the determination and display of possible types of macroscopic behaviour of motions in dynamical systems. Their study goes back to Hadamard and Poincare who considered the existence of periodic geodesics on some classes of surfaces. Later, in the 1930s Hedlund, Hopf and Morse studied the topological and ergodic properties of the flows on compact surfaces of negative curvature [H]. Already, they recognised the special role of the local instability of trajectories and proved that this was closely linked with the statistical (ergodic) behaviour of the flows.

One of the ways of expressing this local instability is the hyperbolic behaviour of the derivative of the flow. The central idea is that close to any fixed trajectory, the behaviour of neighbouring trajectories resembles the behaviour of trajectories in the neighbourhood of a saddle point singularity. Anosov [A] was the first to give an explicit formulation of hyperbolicity. He then used this condition as a basic assumption to study a class of dynamical systems which are now referred to as *Anosov systems*. The geodesic flow on compact manifolds of negative curvature is a very important example of these flows.

The conditions formulated by Anosov in 1967 are the strongest type of hyperbolic conditions. In 1977, Pesin [P] formulated a weaker set of hyperbolic conditions and studied the dynamical systems satisfying these conditions. Again, the geodesic flows on a class of manifolds without focal points

* This is the text of a talk given by the author to a Mathematical Symposium held at the Dublin Institute for Advanced Studies, on April 18th, 1984.

provide an important example of a system satisfying Pesin's conditions.

In this article I wish to outline the study of the hyperbolic behaviour of geodesic flows.

## Preliminaries

### 1.1 Notation

Let $M$ be a smooth compact Riemannian manifold of dimension $n \geq 2$. The tangent and unit tangent bundles of $M$ will be denoted by $TM$ and $SM$, respectively, with corresponding fibers $T_mM$ and $S_mM$ at $m \in M$. The projection map from these bundles to $M$ will be denoted by $\pi$. Finally $\langle\ ,\ \rangle$ and $\rho(\ ,\ )$ will denote the Riemannian metric and the corresponding distance function.

### 1.2 Geodesics

A *geodesic* is a curve $c(t)$ on $M$ whose tangent vectors are parallel. This is expressed in terms of the Riemannian connection of $TM$ by the equation

$$\nabla_{\dot{c}}\dot{c} = 0$$

or in local coordinates by

$$\frac{d^2c^i}{dt^2} + \Gamma^i_{jk}\frac{dc^j}{dt}\frac{dc^k}{dt} = 0$$

where $\Gamma^i_{jk}$ are the Christoffel symbols.

If $c(t)$ is a geodesic, then $\langle\dot{c}(t),\dot{c}(t)\rangle$ is constant and we assume it has the value 1, that is, the geodesics are parameterized by arc length. Since $M$ is compact, the geodesics are infinitely extendable in both directions so that $c(t)$ is a curve from $\mathbb{R}$ to $M$. For any pair $x$ and $y$ of distinct points of $M$ there exists a geodesic joining $x$ to $y$ (generally speaking, not unique). Among such there is always one whose length is equal to $\rho(x,y)$. If $v \in S_mM$ for some $m \in M$, there is a unique geodesic $c(t)$ satisfying the initial conditions $c(0) = m$ and $\dot{c}(0) = v$. We will denote this geodesic by $c_v(t)$.



### 1.3 Geodesic Flow

The geodesic flow is defined on the unit tangent bundle $SM$ as follows. The flow map $\Phi: \mathbb{R} \times SM \to SM$ is given by

$$\Phi(t,v) = \dot{c}_v(t).$$

Geometrically the flow map takes the tangent vector to a geodesic, and moves it a distance $t$ along that geodesic. We will assume that the metric on $M$ is smooth and thus the map $\Phi$ is smooth. The vector field of the geodesic flow is called the *geodesic spray* and denoted by $S$.

To facilitate studying the hyperbolic properties of the geodesic flow, it is necessary to consider the derivative

$$T\Phi_t : T(SM) \to T(SM)$$

where $\Phi_t : SM \to SM$ is the map $\Phi_t(v) = \Phi(t,v) = \dot{c}_v(t)$. A convenient formulation of the map $T\Phi_t$ is got by considering the geometry of $T(SM)$ and Jacobi vector fields along geodesics of $M$.

### 1.4 Geometry of T(SM)

If $v \in SM$, then the tangent space $T_v(SM)$ is decomposed into two complementary subspaces as follows. The first is the vertical subspace which is the $(n-1)$-dimensional subspace given by the kernel of the map $T\pi|_{T_v(SM)} : T_v(SM) \to T_{\pi(v)}(M)$ while the second is the horizontal subspace which is the $n$-dimensional subspace given by the kernel of the connection map $K|_{T_v(SM)} : T_v(SM) \to T_{\pi(v)}M$ [Eb]. (The connection map $K : T_v(TM) \to T_{\pi(v)}M$ is defined as follows: let $\xi \in T_v(TM)$ and let $X : (-\varepsilon,\varepsilon) \to TM$ be a curve with initial velocity $\xi$, then

$K\xi = \nabla_{\dot\sigma(0)}X(0)$ where $\sigma = \pi \circ X : (-\varepsilon,\varepsilon) \to M$ is the footpoint curve).

If $v \in S_mM$ and $v^\perp$ is the orthogonal complement in $T_mM$, then $K : T_v(SM) \to v^\perp$ and the map $i_v : T_v(SM) \to T_mM \oplus v^\perp$ given by

$$i_v\xi = (T\pi\xi, K\xi)$$

is a linear isomorphism. The *Sasaki metric* on SM is defined by $\langle\langle \xi,\eta \rangle\rangle = \langle T\pi\xi, T\pi\eta\rangle + \langle K\xi, K\eta\rangle$ for $\xi, \eta \in T(SM)$. Then $i_v$ is an isometry. The Riemannian volume $\mu$ on SM defined by the Sasaki metric is called the *Louiville measure* and it is invariant under the geodesic flow [A+S].

## 1.5 Jacobi Fields

Let $c(t)$ be a fixed geodesic on M. A vector field $Y(t)$ on $c(t)$ is a *Jacobi field* if

$$\nabla^2 Y + R(\dot c, Y)\dot c = 0$$

where $\nabla$ is covariant differentiation along c and R is the Riemannian curvature tensor on M. Jacobi fields are the variational vector fields of variations of c by geodesics.

If $\xi \in T_v(SM)$ then $\xi$ determines the unique Jacobi field $Y_\xi(t)$ along the geodesic $c_v(t)$ with initial conditions $Y_\xi(0) = T\pi\xi$ and $\nabla Y_\xi(0) = K\xi$.

If $\xi(t) = (T\Phi_t)\xi$, it can be shown [Eb] that

$$T\pi\xi(t) = T\pi \circ T\Phi_t\xi = Y_\xi(t)$$

and

$$K\xi(t) = K \circ T\Phi_t\xi = \nabla Y_\xi(t)$$

This gives a bijection between $T_v(SM)$ and the Jacobi fields on $c_v$. Further, if $Z(v)$ is the subspace of $T_v(SM)$ spanned by the geodesic spray vector field $S(v)$ we have

$$\xi \in Z(v) \iff Y_\xi = a\dot c_v(t) \text{ for some } a \in \mathbb{R}.$$



If $T_v^\perp SM$ is the orthogonal complement of $Z(v)$ in $T_v(SM)$ with respect to the Sasaki metric, we have

$$\xi \in T_v^\perp(SM) \iff Y_\xi \text{ is a perpendicular Jacobi field on } c_v \text{ [Eb].}$$

Thus the two subbundles Z and $T^\perp SM$ are $T\Phi_t$-invariant.

## 1.6 Stable and Unstable Jacobi Fields

We now restrict M to be a manifold without conjugate points. Thus if $Y(t)$ is a Jacobi vector field along a geodesic $c(t)$ which is not identically zero, then $Y(t) = 0$ at no more than one point along $c(t)$. This class includes manifolds of non-positive curvature and manifolds without focal points.

Let $v \in SM$, let $w \in v^\perp$, and let $Y_{w,s}(t)$ be the unique Jacobi field on $c_v(t)$ such that

$$Y_{w,s}(0) = w \text{ and } Y_{w,s}(s) = 0.$$

Then the limit $Y_w^-(t) = \lim_{s\to\infty} Y_{w,s}(t)$ exists and is a Jacobi vector field on $c_v(t)$ [Eb]. Clearly $Y_w^-(0) = w$ and $Y_w^-(t) \neq 0$ for $t > 0$. We call $Y_w^-$ a *stable Jacobi field*.

The *unstable Jacobi fields* $Y_w^+(t)$ along $c_v(t)$ are got by considering the limits as $s \to -\infty$,

$$Y_w^+(t) = \lim_{s\to-\infty} Y_{w,s}(t)$$

For each $w \in v^\perp$, there is a unique $\xi^-(w) \in T_v^\perp(SM)$ for which $Y_{\xi^-(w)}(t) = Y_w^-(t)$ and a unique $\xi^+(w)$ such that $Y_{\xi^+(w)}(t) = Y_w^+(t)$ [Eb].

Using these limiting Jacobi fields we now get the *stable* and *unstable subspaces* of $T_v(SM)$ which are defined as follows:

$$X_s(v) = \{\xi \in T_v^\perp(SM) : Y_\xi(t) \text{ is stable, i.e. } Y_\xi(t) = Y_w^-(t)$$

where $w = T\pi\xi\}$.

$X_u(v) = \{\xi \in T_v^{\perp}(SM) : Y_\xi(t)$ is unstable, i.e. $Y_\xi(t) = Y_w^+(t)$

where $w = T\pi\xi\}$.

The subspaces $X_s(v)$ and $X_u(v)$ are $(n-1)$-dimensional subspaces of $T_v(SM)$ which are invariant under the geodesic flow. The two subspaces coincide and consist of the space of perpendicular parallel vector fields on $c_v(t)$ in the case of M having sectional curvature $K \equiv 0$. If M has no focal points, then a Jacobi field $Y(t)$ is stable (unstable) if and only if $||Y(t)||$ is bounded for $t \geq 0$ $(t \leq 0)$ [Eb], [Es]. (If M has no focal points, then for any Jacobi field $Y(t)$ along a geodesic $c(t)$ such that $Y(t_o) = 0$, we have $||Y(t)||$ strictly increasing as $t \to \infty$.) We will show later that in the case of manifolds with strictly negative curvature, $X_s(v) \cap X_u(v) = \{0\}$.

## 2.1 Anosov Flows

The strongest type of hyperbolic condition is the following which was first formulated by Anosov [A].

Let N be a smooth manifold and let $\phi : \mathbb{R}\times N \to N$ be a complete flow which is smooth. Then it is an *Anosov flow* if the following holds: there are two continuous nontrivial distributions $E^-$ and $E^+$ of TN such that

(i) $T_nN = E^-(n) \oplus E^+(n) \oplus Z(n)$, where $Z(n)$ is the subspace of $T_nN$ generated by the flow vector field.

(ii) $T\phi_t(E^+(n)) = E^+(\phi_t(n))$ and $T\phi_t(E^-(n)) = E^-(\phi_t(n))$ for any $n \in N$, $t \in \mathbb{R}$.

(iii) there exist constants $a \geq 1$, $b > o$ such that for $n \in N$

$$||T\phi_t(v)|| \leq a||v||e^{-bt} \quad \text{if } v \in E^-(n)$$

$$||T\phi_t(v)|| \geq a^{-1}||v||e^{bt} \quad \text{if } v \in E^+(n)$$



The subspaces $E^-(n)$ and $E^+(n)$ are called the *stable* and *unstable subspaces*.

These conditions mean that at each point $n \in N$ the tangent space $T_nN$ can be decomposed in an invariant way into three subspaces $E^-(n)$, $E^+(n)$ and $Z(n)$ such that $T\phi t|_{E^-(n)}$ is a contraction, $T\phi t|_{E^+(n)}$ is an expansion and $Z(n)$ is the subspace generated by the flow vector field. Furthermore the coefficients of contraction or expansion are uniform on N. Near any fixed trajectory $\{\phi_t(n)\}$ the behaviour of neighbouring trajectories resembles the behaviour of trajectories in the neighbourhood of a saddle point.

## 2.2 Geodesic Flows of Anosov Type

Returning to the geodesic flow, we see that the subspaces $X_s(v)$ and $X_u(v)$ of $T_v^{\perp}(SM)$ are candidates for the subspaces $E^-(v)$ and $E^+(v)$ required by the Anosov conditions. If M has negative sectional curvature they do satisfy the condition.

Theorem [A]. Let M be a compact manifold with negative sectional curvature. Then the geodesic flow satisfies the Anosov conditions.

Proof. Since M is compact there are constants $r_1$ and $r_2$ such that

$$-r_1^2 \leq K_m(P) \leq -r_2^2$$

for all sectional curvatures $K_m(P)$. Then for any $v \in SM$, $w \perp v$, the stable Jacobi field $Y_w^-(t)$ along the geodesic $c_v(t)$ satisfies the inequalities

$$||w||e^{-r_1 t} \leq ||Y_w^-(t)|| \leq ||w||e^{-r_2 t} \quad \ldots (1)$$

[H + H].

We also have the following bound for the covariant derivative of $Y_w^-(t)$ [Eb]:

$$\|\nabla_w Y^-(t)\| \leq r_1 \|Y_w^-(t)\| \qquad \ldots (2)$$

Now let $\xi \in X_s(v)$ and $w = T\pi\xi$. Then

$$\|T\Phi_t\xi\|^2 = \|(Y_w^-(t), \nabla Y_w^-(t))\|^2$$

$$= \|Y_w^-(t)\|^2 + \|\nabla Y_w^-(t)\|^2$$

which, by (1) and (2) $\leq \|w\|^2 e^{-2r_2 t} + (r_1)^2 \|w\|^2 e^{-2r_2 t}$

$$\leq \|\xi\|^2 e^{-2r_2 t}(1 + r_1^2).$$

Thus $\|T\Phi_t\xi\| \leq \sqrt{1+r_1^2}\, e^{-r_2 t}\|\xi\|$ and so we have the required contraction for $X_s(v)$. The required inequality for $X_u(v)$ is got by using the fact that $X_u(v)$ may be identified with $X_s(-v)$ [Eb].

Finally if $\xi \in X_s(v) \cap X_s(u)$, then $Y_\xi$ is a parallel Jacobi field [Esch], i.e. $\nabla Y_\xi = 0$. Then

$$\|T\Phi_t\xi\| = \|\xi\| \quad \text{for } t \in \mathbb{R}$$

and so

$$X_s(v) \cap X_u(v) = \{0\}.$$

Thus the geodesic flow is an Anosov Flow.

While the above theorem shows that strict negative curvature is sufficient to ensure that the geodesic flow is Anosov, it is not a necessary condition. Eberlein [Eb] gave an example of a manifold, with non-positive curvature containing open subsets where the sectional curvature is zero on all tangent planes, and yet the geodesic flow is Anosov. Klingenberg [K] proved that if the geodesic flow is Anosov, then M has no conjugate points, and Eberlein then gave the following necessary and sufficient conditions.

Theorem [Eb]. Let M be a compact manifold without conjugate points. Then the following are equivalent.

(a) The geodesic flow is Anosov.

(b) $X_s(v) \cap X_u(v) = \{0\}$ for all $v \in SM$.

(c) There exists no nonzero perpendicular Jacobi vector field $Y(t)$ on a geodesic $c(t)$ of M such that $\|Y(t)\|$ is bounded for all $t \in \mathbb{R}$.



## 3.1 Weaker Hyperbolicity

The hyperbolicity condition due to Anosov is the strongest type in the sense that the subspaces $E^+(n)$ and $E^-(n)$ of the tangent space $T_nN$ generate the complement of $Z(n)$ in $T_nN$ (2.1) and the expansion and contraction of the flow are uniform with respect to n. By relaxing either or both of these requirements we get partial rather than complete hyperbolicity (when the subspaces $E^+(n)$ and $E^-(n)$ do not span $T_nN \setminus Z(n)$) and/or nonuniform rather than uniform hyperbolicity.

Pesin studied these various weaker hyperbolicity conditions and gave the connections with Lyapunov exponents [P]. The geodesic flow on manifolds with no focal points satisfying certain geometric conditions are complete nonuniform hyperbolic flows [P], [B]. The theory of the weaker hyperbolicity conditions is much more complex than the Anosov case and is beyond the scope of this article.

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# REMARKS ON 'AN ELEMENTARY NUMBER THEORY RESULT'

*David Singmaster*

In a joint note published in this Newsletter (No. 12, December 1984, pp. 10-13), Peter Birch and I showed that $\phi(n) > n/\log n$ except for $n = 1,2,3,4,6,10,12,18$ or $30$. For convenience, let us set $\Phi(n) = n^{-1}\phi(n)\log n$, so the above says $\Phi(n) > 1$, except for the values given. Our proof used Bertrand's Postulate, so it was not entirely elementary. I have just found that Alan Baker gives an entirely elementary proof that $\Phi(n) > \frac{1}{4}$ for $n > 1$ [1, p. 12]. Further care with his argument shows the asymptotic result $\Phi(n) > \frac{1}{2} - \varepsilon$ for all large enough $n$ and explicit calculation would show $\Phi(n) > 2/5$ for all $n > 2$.

Baker's argument, in more detail, is as follows. First consider $\sigma(n)$, the sum of the divisors of $n$. Then

$$\sigma(n) = \Sigma_{d|n} d = \Sigma_{d|n} n/d = n\Sigma_{d|n} 1/d \leqq n\Sigma_{d|n} d^{-1}, \text{ so}$$

$$\sigma(n) \leqq n(1 + \log n). \qquad (1)$$

Consider now $f(n) = \sigma(n)\phi(n)n^{-2}$. This is multiplicative and $f(p^j) = 1 - p^{-j-1}$. Then

$$f(n) = \Pi_{p^j|n}(1-p^{-j-1}) \geqq \Pi_{p^j \leqq n}(1-p^{-j-1}) \geqq \Pi_{p^2 \leqq n}(1-p^{-2})$$

$$\geqq \Pi_{1 \leqq m^2 \leqq n}(1-m^{-2}) = \tfrac{1}{2}(1+[\sqrt{n}]^{-1}), \text{ for } n \geqq 4, \text{ so that}$$

$$\sigma(n)\phi(n)n^{-2} \geqq \tfrac{1}{2}(1+n^{-\frac{1}{2}}), \qquad (2)$$

and this is seen to hold for $n \geqq 3$.

From (1) and (2), we have

$$\Phi(n) \geqq \tfrac{1}{2}n(1+n^{-\frac{1}{2}})(1+\log n)^{-1}, \text{ for } n \geqq 3. \qquad (3)$$

Baker's argument takes the simpler result $f(n) \geq \frac{1}{2}$ instead of (2) and then uses $1 + \log n < 2\log n$ for $n > 1$ to deduce $\phi(n) > \frac{1}{4}$. But (3) clearly gives the asymptotic result $\Phi(n) > \frac{1}{2} - \varepsilon$, for large enough $n$. Explicit calculation of the ratio of $n/\log n$ to $\frac{1}{2}n(1+n^{-\frac{1}{2}})(1+\log n)^{-1}$ gives a ratio of 0 for $n = 1$, of .349 for $n = 2$ and a ratio $\geq .412$ for $n > 2$, so this elementary method yields $\Phi(n) > 2/5$ for $n > 2$. Further calculation, based on our result, shows that we actually have $\Phi(n) \geq (\log 6)/3 = .59725$ for $n > 2$.

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# MATHEMATICAL EDUCATION

## REPORT ON THE BASIC MATHEMATICAL SKILLS TEST OF FIRST YEAR STUDENTS IN CORK RTC IN 1984


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### 1. INTRODUCTION

A test was given to all first year students in October 1984 to assess their basic mathematical competency. The results of this test show that our student intake have fundamental deficiencies in their basic mathematics.

While the direct remedy of this situation is outside our control, it is hoped that those involved in the teaching and drawing up of mathematical syllabi at primary and secondary level will consider the implications of this report.

### 2. THE TEST: ITS CONSTRUCTION AND PURPOSE

A copy of the test paper is given in Appendix A. It consists of 20 questions which the students had to attempt without the aid of tables or calculators in the allotted time of one hour. The aim of the questions and acceptable answers are also given in Appendix A.

After careful consideration as to what basic mathematical skills students should have after completing their Leaving Certificate (L/C) the pass level for this test was set at 15 or more correct answers.

As well as the answers to the test questions, the sex and best L/C mathematics grade of the students were recorded.

The test was administered in the fourth week of term. Students were told in advance about the test and what sort of questions to expect, but sample papers were not made available to them. They were also advised that the results of the test could be taken into account in assessing their end-of-year grade.

## 3. THE STUDENTS WHO TOOK THE TEST

All first year full-time students who were taking 3rd level courses or their equivalent were required to take this test. The total number involved was 682 all of whom had taken the L/C examination in mathematics, the majority with 1984 Leaving Certificate, repeat and mature students sitting in 1983 and earlier.

It was anticipated that students undertaking degree courses would differ in their capabilities from students undertakses would differ in their capabilities from students undertaking certificate, diploma or professional business qualificating certificate, diploma or professional business qualifications. This was confirmed by the data and the subsequent analysis will distinguish between the 83 degree students and what we term the 599 non-degree students.

## 4. OVERALL RESULTS

An overall pass rate of 27%, Table 4.1, confirmed that there are basic deficiencies in the mathematical skills of first year students.

**TABLE 4.1**

Results for All Students

| Pass (%) | Total |
|---|---|
| 187 (27) | 682 |

However, when these figures are broken down by degree and non-degree students, it becomes clear that the overall pass rate



of 27% is misleading (see Table 4.2).

**TABLE 4.2**

Results for All Students by Degree and Non-Degree

Degree

| Pass (%) | Total |
|---|---|
| 66 (80) | 83 |

Non-Degree

| Pass (%) | Total |
|---|---|
| 121 (20) | 599 |

The radical difference in pass rates of degree and non-degree students is related to the better L/C mathematics grade of the former. This point is considered in Section 7.

## 5. THE EXTENT OF THE PROBLEM

With a pass in the test set conservatively at 15 or more correct answers, the extent of success and failure by the students is of interest. Did all those who failed get 13 or 14 correct answers? and all those who passed 20 correct? The graphs in Fig. 5.1 answer these questions.

For degree students, 20% failed with 14% of students getting 13 or 14 correct answers. However our concern about the 80% of degree students who passed is caused by the fact that only 14% of students managed 19 or 20 correct answers.

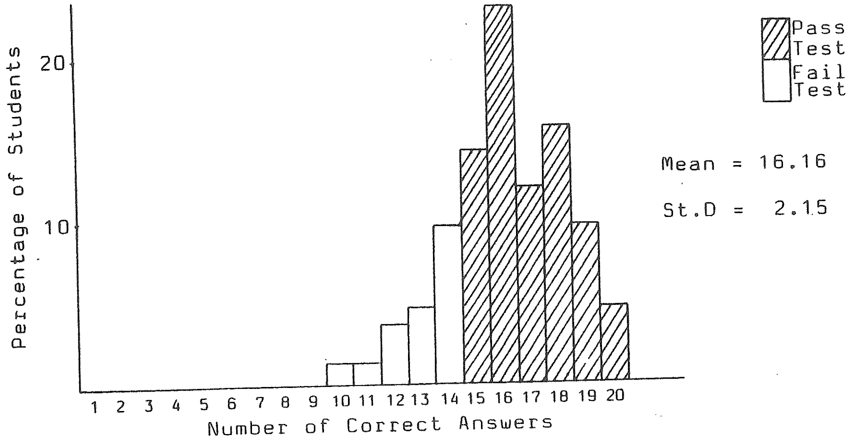
For non-degree students the extent of the failures is alarming, not only did 80% of them fail, 48% of certificate students could only get 10 or less correct answers. Only 1% of non-degree students managed 19 or 20 correct answers.

Areas of difficulty for the student may be seen in the table of Appendix B where a question by question analysis is given.

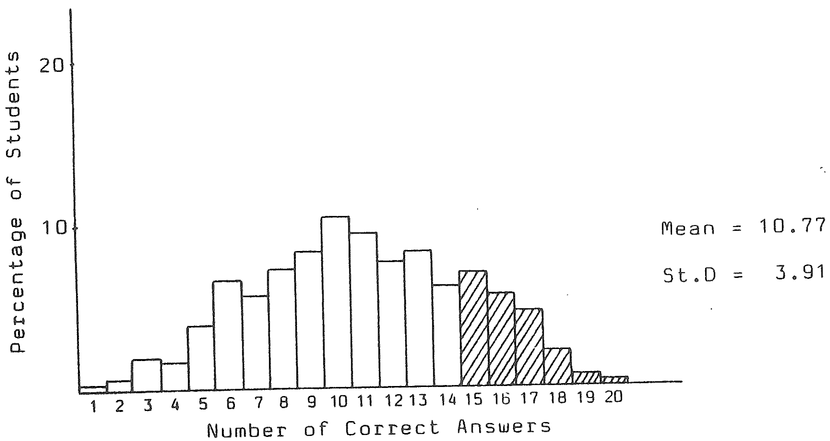
FIGURE 5.1

Percentage of Students by Number of Correct Answers

DEGREE




NON-DEGREE




The Data for these Graphs are given in Appendix C



## 6. STUDENT RESULTS BY SEX

Did the sex of students affect their test performance? The details are given in Table 6.1.

TABLE 6.1

Student Results by Course Type and Sex

| Sex | Pass (%) | Total (%) |
|---|---|---|
| Male | 49 (80) | 61 (73) |
| Female | 17 (77) | 22 (27) |
| Total (%) | 66 (80) | 83 |

| Sex | Pass (%) | Total (%) |
|---|---|---|
| Male | 94 (21) | 444 (74) |
| Female | 27 (17) | 155 (26) |
| Total (%) | 121 (20) | 599 |

Given the results of statistical tests of hypothesis and the awkward theoretic problems raised by the data (e.g. are they a random sample?), we consider that the sex of students is not a significant factor in test performance.

In Table 6.1 we see that the ratio of male to female students in both degree and non-degree courses is 3:1. This ratio does not apply to individual courses as some are exclusively male while in others females predominate.

## 7. RESULTS BY LEAVING CERTIFICATE MATHEMATICS GRADE OF STUDENT

The test results classified by L/C mathematics grade of all students is given in Table 7.1. In Table 7.2 we distinguish between degree and non-degree students. One mature non-degree student whose L/C predates the present grading system was omitted in both tables as being atypical.

## TABLE 7.1

Results of All Students by L/C Mathematics Grade

| | L/C Grade | No. of Students | No. Passed | Pass Rate ± 3S.E.* |
|---|---|---|---|---|
| Higher Course | A | 1 | 1 | 1.000 |
| | B | 32 | 30 | 0.938 ± .129 |
| | C | 68 | 55 | 0.809 ± .143 |
| | D | 52 | 31 | 0.596 ± .204 |
| | E | 5 | 1 | 0.200 |
| Lower Course | A | 37 | 22 | 0.595 ± .241 |
| | B | 204 | 40 | 0.196 ± .084 |
| | C | 197 | 6 | 0.030 ± .036 |
| | D | 85 | 1 | 0.012 ± .035 |
| | | 681 | 187 | |

* S.E. = Standard Error, three standard errors are used to allow multiple comparisons.

From Table 7.1 we see that test performance is closely associated with L/C mathematics grade obtained, with significant difference between students with grade A and those with grades B, C or D on the lower course. The differences between students with grade B and those with grades C or D on the lower course are also significant.

It is tempting from Table 7.1 to order the L/C mathematics grades as follows:

| Higher Course | Lower Course |
|---|---|
| A | |
| B | |
| C | |
| D | A |
| E | B |
| | C |
| | D |



but this is only a tentative ordering since in some instances (grades A and E, higher course) there are too few observations and in others (grades B, C and D, higher course) the distinction is not clear.

## TABLE 7.2

Results of Degree and Non-Degree Students by L/C Mathematics Grade

| | | DEGREE | | | NON-DEGREE | | |
|---|---|---|---|---|---|---|---|
| | L/C Grade | No. of Students | No. Passed | Pass Rate | No. of Students | No. Passed | Pass Rate |
| Higher Course | A | - | - | - | 1 | 1 | 1.000 |
| | B | 19 | 18 | 0.947 | 13 | 12 | 0.923 |
| | C | 38 | 31 | 0.816 | 30 | 24 | 0.800 |
| | D | 16 | 12 | 0.750 | 36 | 19 | 0.528 |
| | E | - | - | - | 5 | 1 | 0.200 |
| Lower Course | A | 6 | 4 | 0.667 | 31 | 18 | 0.581 |
| | B | 4 | 1 | 0.250 | 200 | 39 | 0.195 |
| | C | - | - | - | 197 | 6 | 0.030 |
| | D | - | - | - | 85 | 1 | 0.012 |
| | | 83 | 66 | | 598 | 121 | |

From Table 7.2 we see that the better performance of degree to non-degree students is indeed related to their superior L/C grade. We also note that non-degree students with similar L/C grade to degree students (viz. grades B and C, higher course) are able to perform as well.

Finally in Table 7.2 we see that 81% of non-degree students intake had grade B, C or D on the lower course mathematics paper. Their subsequent poor test performance indicates a lack of basic mathematical skills in the majority of non-degree first year students.

## APPENDIX A

By the term "basic mathematical skills" we mean a facility in handling the topics given in the following list:

(i) addition, subtraction, multiplication, division, use of brackets,

(ii) indices and logs,

(iii) ratio and proportion, percentages,

(iv) approximations,

(v) units,

(vi) factorisations,

(vii) transposition and evaluation of formulae,

(viii) simple equations,

(ix) elementary geometry,

(x) elementary trigonometry.



## CORK REGIONAL TECHNICAL COLLEGE

## BASIC MATHEMATICAL SKILLS TEST 1

NAME: ______________________ CLASS: __________

TIME: 1 Hour

Instructions: Answer ALL questions. All questions carry equal marks.
Write answers clearly in boxes provided.
Use of calculators and mathematical tables not allowed.
Roughwork paper provided

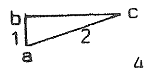
---

| | | | |
|---|---|---|---|
| 1. | Evaluate $\dfrac{\frac{2}{3}-\frac{1}{5}}{\frac{2}{3}+\frac{1}{5}}$. | 1. | 7/13 |
| 2. | What value would you assign to 31 + 47 x 10 - 3? | 2. | 498 |
| 3. | Find the value of $\sqrt{6.4 \times 10^5}$. | 3. | $8 \times 10^2$ |
| 4. | Evaluate $3.21 \times 10^{-2} - 6.71 \times 10^{-4}$. | 4. | $3.1429 \times 10^{-2}$ |
| 5. | If log x = 2, what is $\log(x^3)$? | 5. | 6 |
| 6. | Solve for x: log 3 + log 9 - log 12 = log x. | 6. | 2.25 |
| 7. | Express $4^4.4 \div 4^{-2}$ in the form $4^a$. | 7. | $4^7$ |
| 8. | If $5^x = \frac{1}{25^2}$, find x. | 8. | -4 |
| 9. | The price of an article is £32.50 including 25% VAT. What is the price excluding VAT? | 9. | £26 |
| 10. | Divide 72 in the ratio 1:5. | 10. | 12:60 |
| 11. | The approximate value of $\dfrac{0.077 \times \sqrt{120}}{(0.38)^3 \times (2.19)^2}$ is (a) 32, (b) 3.2, (c) 320, (d) 21.3, (e) none of these. | 11. | (b) or 3.2 |
| 12. | Express 0.01 $m^3$ in $cm^3$. | 12. | $10^4$ $cm^3$ |
| 13. | By using factors or otherwise, find the value of $221^2 - 220^2$. | 13. | 441 |
| 14. | Factorise $2x^2 - x - 3$. | 14. | (x+1)(2x-3) |
| 15. | Re-arrange the formula x = y(1+at) to give t in terms of the other quantities. | 15. | $t = \frac{x}{ay} - \frac{1}{a}$ or equivalent |

16. Evaluate s in the formula $s = ut + \frac{1}{2}at^2$ when $u = 20$, $a = \frac{1}{4}$, $t = 8$.    16. 168

17. Find x if $3(x-2) = 12 + (5x-7)$.    17. -11/2

18. Find the solutions of the equation $(x-1)^2 - 4 = 12$.    18. 5, -3

19. 

Δabc is a right angled triangle. Find |bc|.    19. $\sqrt{3}$

20. If $\cos A = \frac{4}{5}$, find the value of $1 - \sin^2 A$.    20. 16/25

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## APPENDIX B

### Question by Question Test Performance

TABLE B.1

Number and Percentage of Students Who Answered Each Question Correctly

| Question No. | Degree 83 (%) | Non-Degree 599 (%) | All Students 682 (%) |
|---|---|---|---|
| 1 | 79 (95) | 439 (73) | 518 (76) |
| 2 | 39 (47) | 189 (32) | 228 (33) |
| 3 | 73 (88) | 358 (60) | 431 (63) |
| 4 | 64 (77) | 272 (45) | 336 (49) |
| 5 | 67 (81) | 234 (39) | 301 (44) |
| 6 | 44 (53) | 125 (21) | 169 (25) |
| 7 | 73 (88) | 307 (51) | 380 (56) |
| 8 | 63 (76) | 153 (26) | 316 (46) |
| 9 | 74 (89) | 317 (53) | 391 (57) |
| 10 | 79 (95) | 521 (87) | 598 (88) |
| 11 | 41 (49) | 243 (41) | 284 (42) |
| 12 | 36 (43) | 116 (19) | 152 (22) |
| 13 | 74 (89) | 466 (78) | 540 (79) |
| 14 | 68 (82) | 397 (66) | 465 (68) |
| 15 | 79 (95) | 338 (56) | 417 (61) |
| 16 | 81 (98) | 417 (70) | 498 (73) |
| 17 | 78 (94) | 412 (69) | 490 (72) |
| 18 | 72 (87) | 322 (54) | 394 (58) |
| 19 | 78 (94) | 468 (78) | 546 (80) |
| 20 | 83 (100) | 211 (35) | 294 (43) |

Spearman's rank correlation between degree and non-degree students is .692. This correlation is significant at the 1% level and indicates that the areas of difficulty are common to both categories of student.



## APPENDIX C

TABLE C.1

Number of Correct Answers by Number and Percentage of Students

DEGREE

| No. of Answers Correct | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| No. of Students | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 3 | 4 | 8 | 12 | 19 | 10 | 13 | 8 | 4 | 83 |
| % of Students | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.2 | 1.2 | 3.6 | 4.8 | 9.6 | 14.5 | 22.9 | 12.0 | 15.7 | 9.6 | 4.8 | 100 |

NON-DEGREE

| No. of Answers Correct | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| No. of Students | 3 | 5 | 12 | 11 | 25 | 41 | 34 | 43 | 51 | 62 | 57 | 47 | 49 | 38 | 41 | 34 | 28 | 13 | 4 | 1 | 599 |
| % of Students | 0.5 | 0.8 | 2.0 | 1.8 | 4.2 | 6.8 | 5.7 | 7.2 | 8.5 | 10.4 | 9.5 | 7.8 | 8.2 | 6.3 | 6.8 | 5.7 | 4.7 | 2.2 | 0.7 | 0.2 | 100 |

OVERALL

| No. of Answers Correct | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| No. of Students | 3 | 5 | 12 | 11 | 25 | 41 | 34 | 43 | 51 | 63 | 58 | 50 | 53 | 46 | 53 | 53 | 38 | 26 | 12 | 5 | 682 |
| % of Students | 0.4 | 0.7 | 1.8 | 1.6 | 3.7 | 6.0 | 5.0 | 6.3 | 7.5 | 9.2 | 8.5 | 7.3 | 7.8 | 6.7 | 7.8 | 7.8 | 5.6 | 3.8 | 1.7 | 0.7 | 100 |

# UNDERGRADUATE PROJECTS IN GROUP THEORY: COMMUTATIVITY RATIOS

*T. Porter*

Many universities have tried undergraduate projects in mathematics with varying success, but often one hears that although in applied mathematics, the students' work can be creative and, to some limited extent original, in pure mathematics, the project is often to write an account of some theory which the student has searched for in "the literature". Can one do better than this? Can one provide practical and "creative" material for a project in pure mathematics? I want to suggest that one can, by describ[illegible] two projects in Group Theory with which I have been involved.

Group theory like many other branches of pure mathematics taught at university level can tend to be too much in the definition-theorem-proof tradition. Students can finish up with apparently good knowledge of Sylow subgroups and the finer points of soluble groups but faced with an actual group they may not have the faintest idea where to start if required to analyse the subgroups, conjugacy classes, quotients groups etc., i.e. they do not know how to handle the more elementary theory, so their knowledge of the general deeper parts of group theory consists of a collection of statements about ill understood concepts. (If in doubt, set a group of students to work out from scratch a complete list of isomorphism types of groups up to some small given order. Can they do it?)

The situation on potential project material is similar to that on the traditional group theory course work. Of course, there are courses in group theory which have a reasonable, even an adequate, supply of examples in them and similarly there are several different solutions to the problem of designing "creative" projects in the subject. The three main approaches one can take would seem to be (i) use presentations, (ii) use rep-



resentations, or (iii) concentrate on geometric symmetry groups. In the two projects that I supervised, presentations of groups were used as the basic tool since these were being treated in a 3rd year course on Knot Theory running at the same time.

In this note I will describe briefly the subject matter of a 3rd year project equivalent to a half paper in the final exams. I will describe in a further note the content of a full-paper-equivalent project taken the following year by a different student.

First a word of caution, the group theory involved is not deep, or complicated. The prerequisites were an intuitive idea of presentations and a reasonable ability to handle modular arithmetic. No claim is made for originality of the results nor for elegance of the method; what is important is that the student, once the main idea was outlined, completed the calculations by themselves. Certain pieces of theory had to be sketched out for them, but details of proofs were to be provided by them. This was not always done successfully, but the end result was some very good work by a student who was not one of the "high flyers".

For the non-group theorist, let me recall the idea of a presentation. I will give an example. The dihedral group, $D_4$, of order 8 is the group of symmetries of a square and has presentation

$$\langle x, y \ : \ x^4 = y^2 = (xy)^2 = e \rangle$$

That is the elements x and y generate $D_4$ and the relations $x^4 = e$, $y^2 = e$ and $xyxy = e$ are sufficient to give all relationships between products of powers of x's and y's in $D_4$.

The idea of the project was to calculate commutativity ratios for various families of groups. The commutativity ratio is the probability that two elements taken at random in a group G will commute (see D. MacHale [3]). This ratio R(G) can be calculated by the equation

$$R(G) = \frac{\text{number of commuting pairs}}{(\text{order of } G)^2}$$

and is closely linked to the number of conjugacy classes of G. It is however a more intuitive invariant than the latter. The families studied were the dihedral groups and generalised quaternion groups; an attempt was made at general metacyclic groups. I will give the calculations for dihedral groups and give the results for the other families.

$D_n$, the dihedral group of order 2n has presentation

$$D_n = \langle x,y : x^n = y^2 = (xy)^2 = e \rangle \quad (\text{for } n \geq 3)$$

First note that $xyxy = e$ implies $yx = x^{-1}y^{-1} = x^{n-1}y$ so a simple argument shows that any element of $D_n$ has a unique normal form $x^iy^j$ for $0 \leq i \leq n-1$, $0 \leq j \leq 1$. In this normal form, multiplication is given by

$$(x^iy^j)(x^ky^\ell) = x^ry^s$$

where

$$r \equiv i + k + jk(n-2) \pmod n$$

$$s \equiv j + \ell \pmod 2$$

(This formula and the existence and uniqueness of the normal form had to be proved by the student. Although fairly simple inductive proofs, they demand care in their presentation.)

It is now clear that $(x^iy^j)$, $(x^ky^\ell)$ is a commuting pair if and only if

$$jk(n-2) \equiv \ell i(n-2) \mod n$$

or

$$2jk \equiv 2\ell i \pmod n$$

As should come as no surprise, the cases n odd and n even are different.

If n is odd, $2jk \equiv 2\ell i$ if and only if $jk \equiv \ell i$. An attack case by case follows:



If $j = 0$ and $\ell = 0$, this works for all i and k. (This, of course, corresponds to $x^ix^k = x^kx^i$ - not surprising!)

This gives $n^2$ commuting pairs.

Similarly $j = 0$, $\ell = 1$ gives n more.

If $j = 1$ and $\ell = 1$, then $i = k$ giving another n.

Thus for n odd

$$R(D_n) = \frac{n^2 + 3n}{4n^2}$$

For n even, one gets some additional solutions, namely when

$$jk - \ell i \equiv \frac{n}{2} \mod n.$$

As is easily checked, this gives 3n more commuting pairs and

$$R(D_n) = \frac{n^2 + 6n}{4n^2}$$

if n is even.

It should be noted that the student using group tables for small values of n found the patterns for n odd and n even by themselves. I then pointed out that the presentation should give one those patterns in general. They then went away and produced the calculation summarised above.

For the dicyclic group of order 4n,

$$\langle 2,2,n \rangle = \langle x,y : x^n = y^2, y^{-1}xy = x^{-1} \rangle$$

the calculations are similar, giving

$$R(\langle 2,2,n \rangle) = \frac{n^2 + 3n}{4n^2}$$

As both dihedral and dicyclic groups are metacyclic groups, I then suggested that the same methods would perhaps work for all metacyclic groups. For the non-group theorist a metacyclic group G is a group with a cyclic normal subgroup, whose corresponding quotient group is also cyclic. One easily checks that such a group must have presentation

$$G = \langle x,y: x^m = e,\ y^{-1}xy = x^r,\ y^n = x^s \rangle$$

where m, n, r, s are positive integers, $r,s \leq m$ and $r^n \equiv 1$ and $rs \equiv s \pmod m$. (A discussion of this can be found in [2], p. 65.) Any element in G can be written uniquely in the form $y^i x^j$ (the reverse order being adopted to accord with [2]). Multiplication in this form gives

$$(i,j)(k,\ell) = \begin{cases} (i+k,\ \ell + jr^k) & \text{if } i+k < n \\ (i+k,\ \ell + jr^k + s) & \text{if } i+k \geq n \end{cases}$$

The condition for commutativity between (i,j) and (k,ℓ) is

$$j(r^k - 1) \equiv \ell(r^i - 1) \pmod m$$

This is as far as I can go. The student, in fact, failed to get to this point due to a slip earlier in their final calculations. I had hoped for some indication of the number of solutions, at least for special values of r and s as this is exactly what happens for the $D_n$ and <2,2,n>, but apart from obvious cases such as s = 0, r = 1 ($G \simeq C_n \times C_m$) or the dihedral and dicyclic families themselves, no particularly nice families were apparent. I did not look very far into this and in retrospect I should have looked at some of the other families of metacyclic groups such as Coxeter and Moser's ZS-metacyclic groups (see [1]). Perhaps someone would like to set this as an undergraduate project on modular arithmetic and group theory.

My own view of the project was that the student obtained a remarkable feeling for the calculations involved, their sense of enjoyment was obvious and the benefit to their general understanding of other group theory based courses: "Groups and Knots" (2 joined units), "Rings, Modules, and Linear Algebra" (1 unit), and "Group Theory" (1 unit) was considerable even though the use of presentations as such was only a part of the Groups and Knots course and none of the material in the



project was directly useful in that.

The point may be that presentations provide one means by which students can "do" group theory. "Doability" would seem to be a useful concept in teaching mathematics. You only really learn mathematics by "doing", i.e. by handling examples until you feel what a theorem says, by recreating in some small way the original *raison d'etre* of a concept and so on. The problem is that one must balance such ideas with a need to cover a reasonable amount of ground so as to satisfy the external examiner. In a project one can sometimes avoid this pressure to some extent, since the process of discovery, the accuracy of calculation and, that which is of great importance, the presentation, are what will be looked for by the examiner. Perhaps one should hope that some step in a similar direction might be made in the conventional exam. setting as well.

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# HISTORY OF MATHEMATICS

## J. MacCULLAGH AND W.R. HAMILTON

## THE TRIUMPH OF IRISH MATHEMATICS 1827-1865

*N.D. McMillan*

### INTRODUCTION

In the first article in this series [*Newsletter*, No. 10, pp. 61-75] the reform of the mathematical component of the Dublin University curriculum in the period 1790 to 1831 was described. Ireland took the lead in this period internationally in that Trinity was the first university to introduce into its curriculum the new mathematical methods developed in revolutionary France. The Irish reform which culminated in the election of Bartholomew Lloyd to the Provostship of Dublin University in 1831 preceded the British reform movement in mathematics.

The University produced two outstanding mathematicians in this second period, James MacCullagh and William Rowan Hamilton. The former established a powerful geometrical research tradition by introducing a new mathematical methodology and by his inspirational teaching methods. Hamilton on the other hand introduced some of the most original and innovative mathematical concepts in the history of the science of algebra. The development of their respective contributions are described chronologically.

The objective of this article is to reveal the nature of the two mathematical methodologies in this Dublin Mathematical School, because it was from the synthesis of these two methodologies that G.F. Fitzgerald was able to found the science of Relativity.



### JAMES MacCULLAGH AND THE GEOMETRIC TRADITION: THE QUEST FOR THE CARTESIAN SYNTHESIS

The geometric reform of the Dublin University curriculum [1] may be dated from 1758 when Euclid was first introduced into the curriculum, but the research tradition was established by James MacCullagh [2]. MacCullagh's geometric ability was unrivalled in his day. His publications are marked by their lucid economic style and their supreme elegance. It is known that he was very conscious of the educational effect of his writings on the minds of his contemporaries, especially his students in Trinity.

MacCullagh's first major paper, on the *Rectification of the Conic Sections*, was read at the Royal Irish Academy on 21st June 1830. It was a critique of the mathematical methods used by Fresnel in his theoretical studies on the laws of double refraction of light in crystals. MacCullagh presented a series of conic theorems aimed at providing the mathematical tools which would enable the theory of light to be placed on a firm geometrical base.

On the 28th May 1832 at the Royal Irish Academy he read a paper on *The Theory of Attractions* in which he opened up another research interest. In this paper MacCullagh resolved a long-standing dispute over the correctness of Laplace's theory of attraction which had been at issue for a number of years between three of the greatest mathematicians of the period: Laplace, Lagrange and Sir James Ivory. His geometric approach resolved this dispute in favour of Laplace.

After three further years of struggle with his geometrical conceptions, MacCullagh read on the 24th June 1835 another paper: *Geometrical Propositions Applied to the Wave Theory of Light*. In this he explained internal and external conical refraction using geometrical principles to replace Fresnel's three principles of conservation of *vis viva* (energy), the uniformity of elasticity of the ether and the continuity of displacement *parallel* to the refracting interface.

On the 9th January 1837 he produced *On the Crystalline Reflexion and Refraction* in which he proposed a theory of "great geometric simplicity which was compatible with all previous physical notions" [2]. He predicted results from his theory and compared the results with experimentally known data. The continuity condition was replaced by a principle of equivalent vibrations, which supposed that vibrations in two media are equivalent and these considerations were extended to both the parallel and perpendicular components.

MacCullagh's mathematical description of light propagation was quite remarkable. Despite the difference of mathematical representation today, it is clear that the Maxwellian Equations of Electromagnetism are only MacCullagh's equations with the addition of a single term, the famous displacement current [3].

In MacCullagh's next paper, *The Dynamical Theory of Crystalline Reflexion and Refraction*, read at the Academy on the 9th December 1839 he deduced all his previous geometrical results from "a single physical hypothesis and from strictly mechanical principles". The full extent of this dynamical theory was presented in *The Dynamical Theory of Light* published posthumously by two of his devoted followers [4]. In this theory can be seen the modern mathematical description of light and in fact the equations of electromagnetic propagation are compatible with MacCullagh's equations.

The contemporaries of MacCullagh accepted a generally held view that his work had established a consistent theory of light. His Royal Irish Academy obituary claimed for him a position of eminence above Fresnel, because it was believed that he had established

> "the general equations of the motion of the propagation of light, not only in all known media, but also for all media which could ever be discovered, or even conceived."



His honours were of the highest order. The Royal Society in 1842 awarded him the Copley medal, but the recognition by the Royal Irish Academy in awarding him the Cunningham Medal in 1837 was his most treasured honour because of his Irish nationalism.

## WILLIAM ROWAN HAMILTON

A revival of interest in Hamilton can be seen from two recent biographical studies [5, 5a] both of which emphasize his isolated life at Dunsink Observatory and his idiosyncrasies. Hamilton was a child prodigy and in particular was noted for his early ability to calculate. He was essentially self-taught in mathematics and probably therefore benefited more from the atmosphere of Trinity under Bartholomew Lloyd than from the actual teaching, since by the time he came to college he was already an accomplished mathematician.

He was by nature an algebraist and like all truly great mathematicians his work revolved around one great idea, in Hamilton's case the *Characteristic Function* V, which he claimed was the most complete and simple definition that could be given of the application of analysis to optics. This function for him contains the whole of mathematical optics and in this he reveals himself clearly as a follower of Lagrange and Laplace. Hamilton's early work on *Caustics* [6] contained the germ of the idea, but this idea was first properly exploited in his classic series of memoirs *Theory of Systems of Rays*. These memoirs appeared between 1827 and 1833 and developed optics merely as an aspect of the calculus of variation. Fermat's idea of least time and Mapertuis' least action principles are in effect exploited by considering all possible curves between two points A' $(x',y',z')$ and A $(x,y,z)$ and selecting a curve giving the smallest value of the integral of the type

$$V = \int (x,y,z \frac{dx}{du},\frac{dy}{du},\frac{dz}{du})du$$

The stationary value of this curve is called the *ray* in optics but is referred to as the *extremal* in the calculus of variation. The theory is both mathematically and physically noteworthy because it was independent of the physical hypothesis about the nature of light. The wavefronts of the Huygen theory are derived from the direction cosines α, β, γ of the ray, where these are the partial differential coefficients of the principal function V, so that

$$\alpha = \frac{\partial V}{\partial x}, \quad \beta = \frac{\partial V}{\partial y} \quad \text{and} \quad \gamma = \frac{\partial V}{\partial z}.$$

The differential form

$$\alpha dx + \beta dy + \gamma dz$$

has to be derived to determine the wavefront, and in optics the solution is simply then obtained by making V the length of the ray.

This method was pregnant with possibilities. Hamilton showed its power initially when he applied it to a special crystal problem and predicted that in place of a double refraction of light the ray would be refracted into a cone. This startling prediction was made in his classic series of memoirs in 1832 and experimentally confirmed by Humphrey Lloyd [7].

In the development of the properties of extremals, he made significant discoveries and Synge [8] has pointed out that because of the extreme difficulty of this work it was ignored only to be rediscovered by later mathematicians: Kummer in 1860 on the general theory of rectilinear congruences; Bruns in 1895 rediscovered and renamed the third characteristic function as the image function; and Jacobi developed the theory of infinitesimal contact transformations using only one of Hamilton's equations now known as the Hamilton-Jacobi



equation.

The next step in Hamilton's mathematical odyssey was to extend his calculus to dynamics. He began this work in 1833 with a paper *On the General Method of Expressing the Paths of Light and of Planets by the Coefficients of a Characteristic Function* which generalized and extended this optically developed theory into mechanics, and extended his treatment from two to three bodies. Other works on this topic were his paper in 1834 *On a General Method in Dynamics*, and an essay in 1835. He demonstrated that his method when applied to the then known solar system of ten planets, simplified the problem of solving the sixty differential equations of Lagrange, to the search for a single function which satisfies two partial differential equations of the first order and the second degree. Hamilton proposed a general treatment for an attracting system of bodies by his reduction of the mathematics "to the study of one central function". His method was as in his ray theory to reduce the problem to one based on the initial and final co-ordinates of the body, which resulted in the characteristic function V being a function of the 6n co-ordinates of initial and final positions and the Hamiltonian H. This energy operator was constant along any real path, but would vary if the initial and final points were varied.

In the first essay he considered methods of approximating the characteristic function as applied to planets and comets and introduced a new auxiliary function called the principal function S. In the second essay he introduced his famous canonical equation of motion and deduced that S was equal to the time integral of the Lagrangian between fixed points. The statement that the variation of this integral must be equal to zero is now referred to as Hamilton's principle.

Hamilton focused on the fundamentals of algebra in a paper *Algebra of Pure Time* that once again had at its heart "the great idea" of his calculus of variation. In this study he attempted to place the algebraic notions of negative and imaginary numbers on a firm foundation which was for him to be found in "the ordinal character of numbers". He believed these must be ordered on an intuitive basis in time and that this ordering was more deep-seated in the human psyche than the intuition of order in space.

Hamilton was philosophically determined to replace the Cartesian Algebra of the Point by one based on the intersection of two lines. Four elements were necessary to determine the relation of one line in space to the other:

(i) The relation which the length of the line bears to the length of the other line;

(ii) The angle through which one line must be turned to coincide with the direction of the other;

(iii) The plane in which the two lines lie;

(iv) The determination of this plane with respect to some reference plane.

The combination of the four elements then forms the Calculus of Quaternions. He developed this theory, appropriately one might say, in stages; firstly he dealt with Couples, then Triplets and finally Quaternions. The value of the couple (a,b) depends on the *order* as well as the magnitude of its constituent step and in this study he identified the *operator* i to change a real number a, into an imaginary number $\sqrt{-1}$a, by rotating this on the Argand Diagram by 90°, and so on, where $i^2 = -1$. He concluded that i was equivalent to $\sqrt{-1}$ and that this was "a perfectly real operation". Given this result he moved on to consider the triplet (a,b,c) and he was motivated by a desire to connect in some new way "calculation with geometry.

The "triple algebra", so called by de Morgan, led directly to the Quaternions, since Hamilton discovered that these operators were *non-commutative*, and it was his genius to grasp that it was possible to develop a *meaningful and consistent algebra* which is not commutative, the first in the history of mathematics. The date of inspiration is indelibly recorded in the annals of history by Hamilton writing his famous equation on Brougham Bridge on the Royal Canal on the 16th October 1843. Boole subsequently demonstrated (based on Hamilton's suggestion) that there is not one algebra but many with a wide range of fundamental postulates [9].



Hamilton introduced the *dot product* and *vector product* in his algebra which contained within the one algebra a total description of three-dimensional space. He introduced the two well known operators in modern mathematical physics

$$\Delta = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

and

$$\Delta^2 = (\frac{d}{dx})^2 + (\frac{d}{dy})^2 + (\frac{d}{dz})^2$$

the latter named 'del' subsequently by Gibbs. This work led directly through the mediation of FitzGerald to the theory of Relativity developed by Einstein who essentially extended this theory to four dimensions.

The importance of Hamilton's work for these later developments can be judged from a large number of mathematical papers on related topics, such as his 1861 memoir *On Geometrical Nets in Space*. It has been pointed out that Pauli's spin matrices introduced in 1927 are simply Hamilton's i,j,k [10] and that Hamilton first made the discovery of the distinction between group and wave velocities.

His diverse contributions have been recognised by a number of significant scholars. The work of Hankins [5] has however provided the most detailed overview of his life; while that of his contemporary Graves [11] gives the best appreciation of his standing in his own day. His honours were rather limited, and although he was President of the Royal Irish

Academy from 1837-1845, won the Royal Society's Medal in 1836, and had the enviable reputation in his own day of being the leading man of science in Ireland, he never was elected a Fellow of the Royal Society.

## THE DYNAMICS OF COMPETITION

The mathematical competition between MacCullagh and Hamilton was an essential component in propelling these men to the greatness they both achieved. The personal rivalry between the two men has been well documented [3] and was produced principally by MacCullagh's jealous reactions to a series of discoveries by Hamilton. Both men were attacking the same problems by different methods and perhaps it was not therefore surprising that MacCullagh should feel that his studies had already anticipated some of Hamilton's discoveries, most notably conical refraction. While this rivalry became very bitter and personal it was a conflict engendered by the international controversy over the nature of light and the importance given at the time to unification of the sciences of mechanics and light [12].

In this age of great international competition mathematics was the acknowledged battle centre for the sciences and the ultimate theory of the mechanical philosophy. For MacCullagh this golden grail was to be won by providing a geometrical solution to the problem based on a mechanical model of the ether, the long sought Cartesian synthesis. For Hamilton on the other hand a new algebra was seen as the solution and his theoretical foundation of the physical sciences would have required no mechanical model or ether modelling.

The conflict in the two men's methodologies was between that of the applied mathematician and the pure mathematician, the materialist and the idealist, and the Newtonian and continental schools.



MacCullagh was in the last analysis a disciple of Bartholomew Lloyd and a natural philosopher in the Newtonian sense of the word, while Hamilton the self-taught mathematician was unequivocally a supporter of French mathematical "physique". This difference can be seen in their views on the central question under debate internationally at the time, the wave-corpuscular controversy [13] over the nature of light. MacCullagh was a "faint hearted supporter" of the wave theory with Newtonian doubts that the wave theory's physical basis had not been rigorously established. Hamilton was a true "supporter" of the wave theory [14]. Dublin also boasted in its midst at this time Dionysius Lardner, a collaborator of the Edinburgh Newtonians who formed the central core of the "objectors" to the wave theory [15].

Dublin in the 1830s was opened up to the maelstrom of this international research controversy and this provided the dynamic for what were probably the most significant developments in the history of Irish mathematics. The fact that Dublin mathematicians were able to move to the forefront of science at this time was because Bartholomew Lloyd [16] had reformed the curriculum of Dublin University and made it possible for the young lions of the emerging school to base their research on the most advanced paradigms of the day. However, from this period on, the research traditions in Dublin mathematics were based on its own bifocated traditions of geometry and algebra and not merely on foreign inspiration.

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13. For a contemporary view on this debate with a special reference to Dublin, see Lloyd, H., *'Report on Optics'*, British Association for the Advancement of Science, (Dublin, 1835).

14. The views of MacCullagh and Hamilton on this question were revealed in the debate at the 1843 British Association meeting at Cork.

15. Lardner was appointed the first professor at the new London University College in 1827 by Lord Brougham, the leading Newtonian who was based in Edinburgh and associated there with David Brewster, the other leading supporter of the corpuscular theory.

16. The reform of the curriculum in Dublin University was dealt with in Part 1 of this article.

*School of Science,*
*Regional Technical College,*
*Carlow.*

# BOOK REVIEW

"GEORGE BOOLE. HIS LIFE AND WORK" (Profiles of genius, No.2)

By *Desmond MacHale*

*Boole Press*, 1985. xv + 304 pp. £25.00

I The Boole Press achieves a notable publication in this second book in their biography series. The name of the house itself reflects the esteem in which George Boole (1815-1864) is held in Ireland; and this book meets the expectations that the reader might hold, for Dr MacHale has produced a fine and rounded portrait of one of the great thinkers of the 19th century. Boole is a remarkable example of a man who mastered his disadvantaged origins to take the founder chair in mathematics at Queen's College Cork and, in his research, to lay down a main line of study in logic and also systematise and extend knowledge of differential equations. MacHale's book is a major achievement, unlikely to be eclipsed as a general biography, and deserves to be a best-seller.

In addition to a general description of the works, MacHale provides much information, often little-known, on Boole's life and career: birth in Lincoln to a poor but intelligent cobbler (Ch. 1); proprietor of his own schools in the area from his 20th to his 35th years (Ch. 2); and then extensive accounts, on which hitherto virtually nothing was known, on the period at Cork (mainly Chs 5,6,8,11,14). He also pays due attention to Boole's attempts to write poetry (Ch. 12), and the extremely important religious components of his thought (Ch. 14). A variety of portraits and photographs adorn the text, which ends with a full bibliography of Boole's writings (a summary description of the *Nachlass* in the archives of the Royal Society would have been welcome[1]), a selection of related writings on Boole (supplemented in the bibliography attached to this review), and an excellent index.



In the final chapter of the book, which might have been better called an appendix, MacHale elaborates upon the family tree of p. 4 to describe Boole's family. They comprise his wife Mary Everest Boole (1832-1916), a notable figure in (Boolean) educational psychology in her own right; her five daughters, who include a mathematician, a chemist, and a novelist; and various descendants, who count among their number the mathematician C.H. Hinton[2], his grandson the entomologist H.E. Hinton and niece the mathematician Joan Hinton, and the applied mathematician Sir G.I. Taylor.

II On the prehistory of Boole's achievements, MacHale is less strong, making Boole appear more isolated than was the case and even attributing to him some achievements of others. The principal main source of the mathematical traditions to which he contributed is the work of Lagrange, who tried to algebrise mathematics, and in the context of the calculus initiated the study of differential operators and functional equations. MacHale notes this tradition but mishandles it by attributing the operator form (d/dt)x to Leibniz (p. 45): in fact, with Leibniz it is d which is the operator, working on x to produce dx, and 'dx/dt' means there 'dx ÷ dt'.

Lagrange's approach was developed in the early years of the 19th century by several minor but interesting French figures (Arbogast, the Français brothers, Brisson, Servois), but died out there in the 1820s when the methods of Cauchy took over. However, they became a major attraction in England, especially the differential operators, when the general situation in mathematics improved after the reforms of the 1810s and 1820s occurred at Cambridge University (Becher 1980). In addition, logic was also emphasised there, giving it a prominence new to English thought[3].

III Boole's researches belong to both these areas. In his spare time from school-teaching he worked especially on differential operators, systematising the theory in his great paper of 1844 'On a general method in analysis' (pp. 61-66). Here he stressed the laws of commutativity and distributivity, and also the index law

$$\pi^m\pi^n(\underline{u}) = \pi^{m+n}(\underline{u}) \qquad (1)$$

where $\pi$ was some differential (or difference) operator. Then, a few years later, inspired by a disagreement on logic between Sir William Hamilton and Augustus de Morgan[5], he outlined his programme for 'a mathematical analysis of logic' (pp. 68-72), giving again the laws of commutativity and distributivity but replacing (1) by the index law

$$x^2 = x, \text{ or } x(1-x) = 0. \qquad (2)$$

I am very surprised that MacHale endorses Russell's quite mistaken view that Boole's Boolean algebra made him the 'discoverer of pure mathematics'[6]; for it is clear even from the titles of his books that Boole saw his work as mathematics *applied* to logic, and especially to mental processes. Thus *The laws of thought* (1854) means exactly what it says: a mathematical psychology[7]. The processes described in (2) and their consequences could apply to any area of thought, including mathematics itself: MacHale rather misses the importance for Boole of the example of singular solutions to differential equations (pp. 220-222), since they (as a 1) have the dual properties of solving the equations (like an x) but lying outside the general solution (*à la* 1-x).

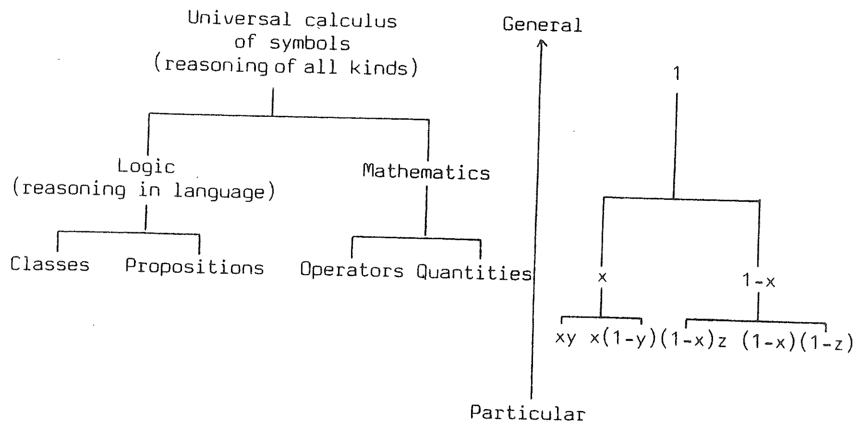
Indeed, mathematics and logic themselves seem to be complementary parts of a 'universal calculus of symbols', to use the happy phrase of Laita (1977). On its basis I presented in (1982, 37) the following theme for Boole's ideas, and one which moreover has a Boolean structure, as self-reference would demand (see Fig. 1). The view that the mind has the power to pass from the particular to the general, marked by the central arrow in the diagram, is crucial to his theory: it seems



to have informed the rather rigid style of his school-teaching (pp. 41-43), was stressed explicitly in a lecture of 1851 (quoted on p. 99) and may have motivated his practice of working at night in the dark (p. 166, repeated on p. 228), freeing the mind from the geometrical and visual and allowing symbols free rein.

FIGURE 1

Representation of Boole's System, together with the corresponding Boolean structure




In addition, Boole's logic carried with it a strong religious connotation, in that the universe 1 was reflected in the ecumenical views of the time, especially of F.D. Maurice, Boole's hero in his later years (p. 206), whose portrait was laid before him as he died (pp. 240-241). These sides of Boole's idea rapidly died with him, despite their advocacy by his widow (or perhaps, considering the eccentricity of her style, because of it).

During his last years Boole concentrated once again on mathematics, producing important text books on differential and on difference equations, in which both differential operators and functional equations were prominent; in addition, he wrote papers in these areas, and also in probability, where he continued a concern launched in the later chapters of *The laws of thought*. MacHale devotes his Ch. 15 to these areas, perhaps a little lightly; for example, the significance of his contributions to probability is not easy to assess. He notes Mrs Boole's involvement in the textbooks (pp. 219-220), but his later general judgement that 'she had very little knowledge of mathematics and little more than a superficial understanding of her husband's work on logic' (p. 258) seems grossly unfair: Laita (1980) argues persuasively for the *general* correctness of her testimony. In many ways we are indebted to her for the prosecution of his later studies, as well as for the line of genius which she and her husband bequeathed to the world.

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FOOTNOTES

1. Hesse (1952), not in the bibliography, describes some materials in the *Nachlass*; see also footnote 6. G.C. Smith is working on the correspondence between Boole and W. Thomson (later Lord Kelvin), held at Kelvin's end in the Cambridge and Glasgow University Libraries.

2. MacHale discusses on pp. 259-260 aspects of C.H. Hinton's book *The fourth Dimension* (Hinton 1904). However, Hinton's geometrical treatment of syllogistic reasoning - in which he drew on sister-in-law Alicia Stott Boole's ideas (Hinton, p. 90) - presents a symmetry which surely should not obtain (p. 102, where the box AEO should be dropped from the scheme).



3. See Van Evra (1984). In a curious detail of non-transmission of thought, the French 'logique' of Lagrange's time did not come over. A form of semiotics (to us) in the hands of men such as Condorcet and Condillac, it linked specifically to algebra (Albury 1980, Auroux, 1981), and to other things, such as the education of the deaf. However, the general concern with signs is evident in England especially with Babbage, and to some extent in Boole.

4. These words were introduced, in connection with functional equations, by Servois (1814), one of several French figures whose work refutes MacHale's claim that 'Boole was the first person to define clearly the concept of an operator' (p. 65). Further, I know of no information to back his claim on p. 218 that 'Boole's premature death alone prevented him from being enrolled by the French Academy of Sciences'. Finally, it is disheartening to see the mathematician V.A. Lebesgue called 'Lebesque' on pp. 46-48, 54, and even indexed on p. 301 as 'Lebesque, Henri', which misidentifies the mathematician as well as mis-spells the name.

5. On this influence, see Laita (1979). MacHale shows himself on p. 285, n.6 not to be abreast of the current interest in de Morgan: see especially Joan Richards (1980) on the lead up to de Morgan's position on algebra, Pycior (1983) on its development, and Merrill (1978) on his contributions to logic.

6. See pp. 130, 217; and also p. 224. In an improbable move, Mrs Boole hoped around 1905 that Russell might edit Boole's manuscripts for publication; he declined, and recommended Couturat, but nothing was done (see my (1977), 137).

7. On Boole as a 'psychologist logician' as opposed to Mill as a 'logical psychologist', see John Richards (1980). The apologies for Boole's psychologism given in Musgrave (1972) are based on misunderstandings.

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# BOOKS RECEIVED

"OPTIMAL SEQUENTIAL BLOCK SEARCH"

By *Li Weixuan*

Research and Exposition in Mathematics 5. Berlin: *Heldermann Verlag* 1984, viii + 209 p., soft cover, 38.00 DM.
ISBN 3-88538-205-9

Sequential block search is a mathematical method to search for the optimal value of a unimodal function. It has become an important method in operations research not only because it has extensive practical applications but also because it is a basic technique in multi-variate optimization.

Sequential search was first studied by J. Kiefer in 1953 and has since been developed by many mathematicians working in various fields. In this book, the author presents a comprehensive account of the mathematical aspects of this topic, and the emphasis is on results by Chinese researchers which appear here for the first time in English language.

"MODAL THEORY. AN ALGEBRAIC APPROACH TO ORDER, GEOMETRY AND CONVEXITY"

By *A.B. Romanowska and J.D.H. Smith*

Research and Exposition in Mathematics 9. Berlin: *Heldermann Verlag* 1985, xii + 158 p., soft cover, 38.00 DM.
ISBN 3-88538-209-1

Modal theory is a new algebraic discipline, comparable in scope and intention with the well-established disciplines of linear algebra, lattice theory, and semigroup theory. The topics of modal theory, belonging mostly to universal algebra, have broad connections to the theory of semigroups, semilattices, lattices, convex sets and geometry; also included are interesting applications to computer science. The work summarizes various recent results in a new, unifying manner. The value of the work lies in the investigation of general properties of modes and modals (idempotent entropic algebras that are not necessarily associative) and in the analysis of various special cases.

## PROBLEM PAGE

Both the new problems this time are inequalities.

1. If $1 < p \leq 2$ and $\alpha = \frac{\pi}{2p}$ show that

$$\left(\frac{\cos\theta}{\cos\alpha}\right)^p \geq 1 + (\tan\alpha)\cos(p\theta), \quad \text{for } 0 \leq \theta \leq \alpha.$$

This was shown to me by Matts Essen of Uppsala. It can be done by elementary calculus, but function theorists may like to speculate on how the inequality arises 'naturally'.

The other problem was submitted by Bob Grove of Auburn University, Alabama.

2. Suppose that $0 = \phi_1 \leq \ldots \leq \phi_n < \pi$, that $A=[\sin(|\phi_i-\phi_j|)]$, and that $||A|| = \max\{||Ax|| : ||x|| = 1\}$. Show that

$$||A|| \leq \cot(\tfrac{\pi}{2n}),$$

and characterize the case of equality.

Now for the solutions of some previous problems.

1. Consider the sequence of digits

198423768........

obtained using the rule:

"after 1984 every digit which appears is the final digit of the sum of the previous four digits."

Does 1984 appear later in the sequence and, if so, when? What about 1985?

This problem was suggested by Pat Fitzpatrick who says that it originated in a Hungarian mathematical magazine.



First note that if we reduce all digits mod 2 then the sequence is

1100011000 ..... ,

which is periodic with period 5. Hence 1985, which reduces to 1101 mod 2, can never appear.

To see that 1984 must reappear note that there are only $10^4$ four digit numbers. Hence, some block of four digits must repeat, say abcd. Since the sequence of digits can be generated backwards in a unique manner from any given block of four digits, we can arrive at a second occurrence of 1984 by working backwards from the second occurrence of abcd.

Working with a computer one finds that 1984 reappears after 1560 steps. However, Pat points out that this fact can be ascertained even in the event of a power failure, using a little algebra to reduce the effort. Here is the idea.

The problem can be written in the form

$$a_{n+4} = a_{n+3} + a_{n+2} + a_{n+1} + a_n \pmod{10}$$

where $a_0 = 1$, $a_1 = 9$, $a_2 = 8$, $a_3 = 4$. We know that this sequence has period 5 when reduced mod 2 and so it is enough (since 2 and 5 are coprime) to find the period n when the sequence is reduced mod 5. The original sequence will then have period 5n.

Recasting the problem in matrix form we have

$$\underline{u}_{n+1} = A\underline{u}_n \pmod{10}, \quad \text{for } n = 0,1,2, \ldots,$$

where

$$\underline{u}_n = \begin{pmatrix} a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We are then seeking the smallest n such that

$$A^n \underline{u}_0 \equiv \underline{u}_0 \pmod 5.$$

Now the vectors $\underline{u}_0$, $\underline{u}_1$, $\underline{u}_2$, $\underline{u}_3$ are linearly independent over $\mathbb{Z}_5$ since

$$\begin{vmatrix} 1 & 9 & 8 & 4 \\ 9 & 8 & 4 & 2 \\ 8 & 4 & 2 & 3 \\ 4 & 2 & 3 & 7 \end{vmatrix} = -149 \equiv -1 \pmod 5,$$

As

$$A^n \underline{u}_i \equiv \underline{u}_i \pmod 5, \quad \text{for } i = 0,1,2,3,$$

this shows that

$$A^n \equiv I \pmod 5.$$

The matrix A satisfies its own characteristic equation, that is,

$$x^4 - x^3 - x^2 - x - 1 = 0 \qquad (*)$$

and, since this polynomial is irreducible over $\mathbb{Z}_5$, the smallest field containing $\mathbb{Z}_5$ and A is $GF(5^4)$. (I am grateful to Bob Margolis at the Open University for a short refresher course on Galois theory!) This means that the multiplicative order of A in $GF(5^4)$ is a divisor of $5^4-1 = 624 = 2^4.3.13$. It is now a tedious but elementary exercise to check (with the aid of (*)) that the multiplicative order of A in $GF(5^4)$ is 312.

So the answer to the original problem is indeed 1560=5x312.

**Remarks** 1. It is easy to check that $A^5 \equiv I \pmod 2$ and so the above discussion shows that the period is 1560 whenever $\det(\underline{u}_0\,\underline{u}_1\,\underline{u}_2\,\underline{u}_3)$ is relatively prime to 10.



2. Tim Lister at the Open University noticed that 9126 appears in 1984 ... after exactly 780 steps (half a period). In fact

$$A^{156} \equiv -I \pmod 5,$$

since the non-zero elements of $GF(5^4)$ form a *cyclic* multiplicative group, and so

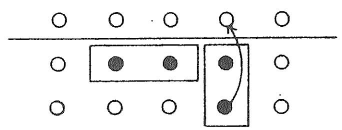
$$A^{780} \equiv -I \pmod{10}.$$

So this is no coincidence!

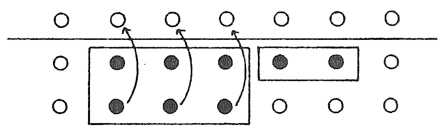
3. In his book "Geometry", Coxeter credits Lagrange as the first to notice that the final digits of the Fibonacci numbers form a periodic sequence with period 60. However, he gives no algebraic discussion of this fact.

1. Playing solitaire on an unlimited board, on which is drawn a horizontal line, you are required to lay out pegs below the line in such a way that a single peg can be manoeuvured as high as possible above the line.

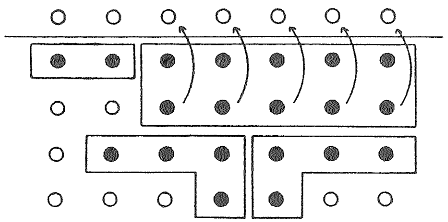
The diagrams below illustrate positions which enable a peg to reach the second, third and fourth rows, respectively. The blocking of the pegs indicates, informally, the order of play.



Position 1



Position 2

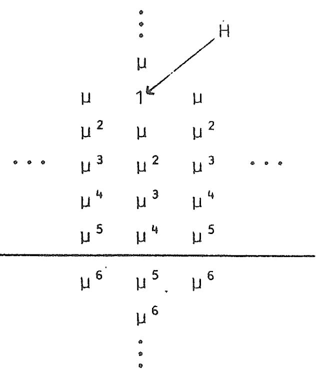


**Position 3**

Notice that, while playing position 2, we obtain position 1 moved up one row. Similarly, while playing position 3, we obtain position 2 moved up one row.

However, there is no arrangement of pegs below the line which enables a single peg to reach the fifth row above the line. Here is the beautiful proof of this fact given in volume 2 of "Winning Ways for your Mathematical Plays" by Berlekamp, Conway and Guy (warning: these books are addictive!).

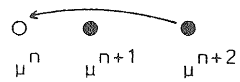
Choose a particular hole H on the fifth row. It is enough to show that H cannot be reached. Assign to H the value 1 and then assign to any other hole the value $\mu^n$, where n is the length of the shortest path (parallel to the axes) to H. Here $\mu$ is a number between 0 and 1 to be chosen in a moment.






Any position can now be assigned a value by adding the values of the pegs.

We choose $\mu$ in such a way that a move of the following type




leaves the value of the position unchanged. Evidently we require that

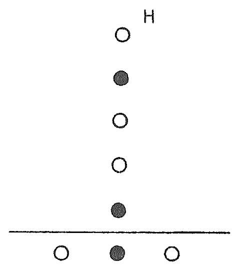
$$\mu^n = \mu^{n+1} + \mu^{n+2}, \text{ that is, } \mu^2 + \mu - 1 = 0,$$

and so $\mu = \frac{1}{2}(\sqrt{5} - 1) \simeq 0.618$, a not-unfamiliar number!

It is now easy to check that no move can increase the value of a position, and so to reach H it is necessary to start with a position whose value is at least 1. If such a position exists, with all pegs below the line, we may assume that it contains only finitely many pegs.

However, a straightforward calculation shows that the total value of *all* the holes lying below the line is 1. Hence no such position exists.

<u>Remarks</u> 1. This valuation of positions makes the attainment of row 4 look rather a modest achievement, since the hole below H has value only $\mu$. It is much harder, for example, to reach the following position



which has value

$$\mu + \mu^4 + \mu^5 = 1 - \mu^4 \approx 0.854.$$

It seems unlikely that one can reach every position (below H) which has value less than 1, but I don't know of a counter-example.

2. A similar calculation reveals that for the analogous problem in three dimensions it is impossible for a peg to reach the eighth row. The seventh row can be reached, however.

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# CONFERENCE REPORT

## GROUPS IN GALWAY 10-11 MAY 1985

A Conference on Group Theory, sponsored by University College, Galway, the Royal Irish Academy, and the Irish Mathematical Society was again held at University College, Galway, on Friday and Saturday 10th and 11th May 1985. The invited speakers were T.O. Hawkes (Warwick), G.D. James (Cambridge), T.J. Laffey (UCD), P.D. MacHale (UCC), T.G. Murphy (TCD), and S.J. Tobin (UCG). Lectures were also given by D.W. Lewis (UCD) and M. Ó'Searcóid (UCC).

Seán Tobin opened the Conference with his talk entitled 'Razmyslov Algebras' (see *I.M.S. Newsletter* 13 (March 1985) pp. 57-65). He explained Razmyslov's construction of a non-soluble group of exponent $p^2$, for odd primes p.

Tim Murphy, in his lecture on 'Tensor Groups', discussed the duality between linear groups and the sets of tensors which they fix. He also illustrated the use of Penrose's notation for tensors, and his hand-out demonstrated the qualities of the TEX computer typesetting system.

Gordon James described 'A q-Analogue of the Symmetric Group Algebra'. Given a field F of characteristic p, and a nonzero element q of F, he defined an F-algebra H, whose properties (for example the number of its simple right ideals, and the dimension of its centre) can be obtained from those of the group algebra over F of the symmetric group by substituting q for p.

Tom Laffey's title was 'Some Maximal Subgroups of the General Linear Group'. He began by presenting a short proof (due to Radjavi) of the fact that every element of $SL_n(F)$ is a commutator in $GL_n(F)$, except when $n = |F| = 2$. He then proved that the invertible monomial matrices form a maximal subgroup of $GL_n(F)$, provided $|F| > 5$, and he showed how this, and related results, have been used to classify certain linear maps from a space of square matrices to itself.

Des MacHale spoke on 'The Relationship Between $|G|$ and $|Aut\ G|$ for a Finite Group'. He reported on functions f(n) such that if $p^{f(n)}$ divides $|G|$, then $p^n$ divides $|Aut\ G|$, and also on results related to the conjecture that if G is a non-cyclic p-group of order $p^3$ or more, then $|G|$ divides $|Aut\ G|$. He finished by mentioning a number of other open questions.

In the last lecture of the Conference, Trevor Hawkes dealt with three topics involving 'Linear Methods in Soluble Groups'. He first proved that if H is an F-projector of G, where F is a saturated formation, and if every irreducible character of H can be extended to G, then H has a normal complement. He then showed that the Fitting length of G is bounded in terms of the composition length of a Fischer subgroup (nilpotent injector). Finally he reported on groups with a fixed-point-free operator group.

Labhair Mícheál Ó'Searcóid faoi 'Comhaireamh p-Fhoghrúpaí'. David Lewis gave a talk on 'Sums of Squares in Division Algebras'.

We would like to thank the lecturers, the sponsors, and the participants for their continued support.

*Rex Dark*



# CONFERENCE ANNOUNCEMENT

## BAIL IV

**The Fourth International Conference on Boundary and Interior Layers-Computational and Asymptotic Methods**

7-11 July 1986
in
Novosibirsk, USSR

Hosted by the Siberian Branch of the USSR Academy of Sciences

**Chairman: Professor S.K. Godunov**
*Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR*

**Co-Chairman: Professor J.J.H. Miller**
*Numerical Analysis Group, Trinity College, University of Dublin, Ireland*

A formal First Announcement and Call for Papers will be made in due course by the Siberian Branch of the USSR Academy of Sciences in Novosibirsk.

Registration for participants from all non-socialist countries will be handled by the BAIL Secretariat in Dublin. Attendance from non-socialist countries is limited to a maximum of 80 participants, so early registration is advisable. If you are interested in attending, please contact the address below without delay. Participants from socialist countries should contact Novosibirsk directly.

Before BAIL IV you may attend the 10th ICNMFD in BEIJING and then travel to NOVOSIBIRSK by rail via MONGOLIA. For more details read on:

The dates of the BAIL IV Conference have been harmonised with those of the 10th International Conference on Numerical Methods of Fluid Dynamics, 23-27 June 1986, in Beijing. Participants are thus enabled to enjoy a short post-conference tour of China and then travel by rail from Beijing to Novosibirsk, departing Wednesday, 2nd July and arriving Saturday, 5th July. It is proposed that participants from Western Europe fly as a group to Moscow and then onwards to Novosibirsk in order to avail of economical air fares. In addition, it is expected that another group will travel by train from Beijing to Novosibirsk. If sufficient interest is shown, a group may also be organised to fly from Western Europe to Beijing. Further information about these travel arrangements may be obtained from the BAIL Secretariat in Dublin.

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