PROBLEM PAGE

First of all, here are some new problems.

1. (Suggested by Pat Fitzpatrick). Consider the sequence of digits

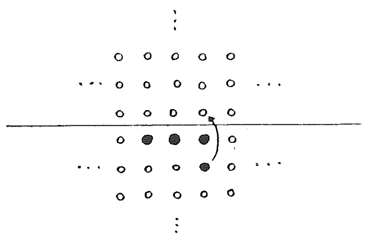
198423768........,

obtained using the following rule.

"After 1984 every digit which appears is the final digit of the sum of the previous four digits."

Does the grouping 1984 appear later in this sequence and, if so, when? What about the grouping 1985?

2. (Due to John Conway and suggested to me by Harold Shapiro.) Imagine playing solitaire on an unlimited board, on which is drawn a horizontal line.



The aim is to lay out pegs below the line in such a way that, with the usual solitaire moves, a single peg can be manoevured as high as possible above the line. The above arrangement of four pegs enables a single peg to reach the second row (above the line).

Find an arrangement which enables a single peg to reach the fourth row and show that there is no arrangement which enables a peg to reach the fifth row.

Now here are the solutions to September's problems.

1. Prove that

$$\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)}{3}, \qquad (1)$$

and deduce that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \qquad (2)$$

This problem was suggested by Finbarr Holland who proves (1) using the finite Fourier transform. The classical proof (found in turn-of-the-century calculus texts) goes as follows.

By De Moivre's theorem,

$$\begin{aligned}\sin(2n+1)\theta &= \mathrm{Im}[(\cos\theta + i\sin\theta)^{2n+1}]\\ &= \sin^{2n+1}\theta\, \mathrm{Im}[(\cot\theta + i)^{2n+1}]\\ &= \sin^{2n+1}\theta\, P(\cot^2\theta),\end{aligned}$$

where P is the polynomial

$$P(x) = \binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \ldots + (-1)^n.$$

For $\theta = k\pi/(2n+1)$, $k = 1,2,\ldots,n$, we have $\sin(2n+1)\theta = \sin k\pi = 0$ and $\sin\theta \neq 0$, so $P(x) = 0$ when $x = \cot^2(k\pi/(2n+1))$, $k = 1,2,\ldots,n$. These n numbers are the roots of P and so their sum is

$$\binom{2n+1}{3}/\binom{2n+1}{1} = \frac{n(2n-1)}{3},$$

which proves (1).

To deduce (2) from (1) rewrite the inequality

$$\sin\theta < \theta < \tan\theta, \qquad 0 < \theta < \pi/2,$$

as
$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta, \qquad 0 < \theta < \pi/2,$$

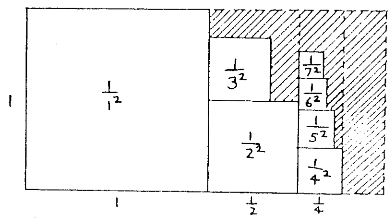
to obtain

$$\sum_{k=1}^{n} \cot^2\left(\frac{k}{2n+1}\right) < \left(\frac{2n+1}{\pi}\right)^2 \sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{n} \left(1 + \cot^2\left(\frac{k\pi}{2n+1}\right)\right).$$

Using (1) and letting $n \to \infty$ gives (2).

It is remarkable that this proof of (2) remained unnoticed until 1953 when it was published by A.M. Yaglom and I.M. Yaglom. It has been rediscovered several times since (*Monthly*, *80* (1973) 424-425).

Before leaving this problem I can't resist including a related 'proof without words'.



2. Prove that, if a_0, a_1 are not both zero then the sequence

$$a_{n+2} = |a_{n+1}| - a_n, \qquad n = 0,1,2,\ldots, \qquad (3)$$

has period 9.

This problem is due to Morton Brown who proposed it in *American Mathematical Monthly* in October 1983. The neat solution which follows was sent to me by Mícéál Ó'Searcóid. It is very similar to one that Morton Brown received from Donald Knuth.

The key observation is that if

$$0 \le 2a_k \le a_{k+1}, \qquad (4)$$

for some k, then the sequence $\{a_n\}$ is

$$\ldots a_k,\ a_{k+1},\ a_{k+1}-a_k,\ -a_k,\ 2a_k-a_{k+1},\ a_{k+1}-a_k,\ 2a_{k+1}-3a_k,$$

$$a_{k+1}-2a_k,\ a_k-a_{k+1}, a_k, a_{k+1}\ldots,$$

which is easily seen to have period 9 (not 1 or 3).

Since we can define

$$a_n = |a_{n+1}| - a_{n+2}, \qquad n = -1,-2,\ldots,$$

the same conclusion holds if $0 \le 2a_{k+1} \le a_k$, for some k.

If a_k, $a_{k+1} \ge 0$ then w.l.o.g. $0 \le a_k \le a_{k+1}$ so either (4) is true or else $a_k \le a_{k+1} \le 2a_k$, in which case $0 \le 2a_{k+2} \le a_{k+1}$. Finally, the sequence must contain neighbouring non-negative terms since $a_n \le 0$ implies that $a_{n+2} \ge 0$.

A number of related questions suggest themselves. For example (a) are there similar sequences with other periods, and (b) what happens if a_0, a_1 are complex. Both these questions have received attention but, as far as I know, only partial answers have been given.

To attack (a) it helps to think of (3) geometrically. The mapping

$$t : (x,y) \longmapsto (y, |y|-x)$$

which takes (a_n, a_{n+1}) to (a_{n+1}, a_{n+2}) can be decomposed as

$$t = q_{\pi/4}\, s\, q_{\pi/2},$$

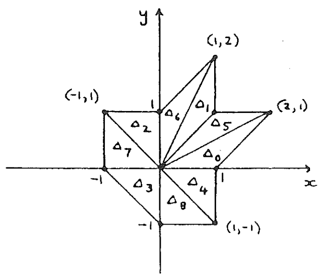
where q_θ denotes reflection in the line making a positive angle θ with the x-axis, and

$$s : (x,y) \longmapsto (x+|y|, y)$$

is a "piecewise shear". Thus

$$t = q_{\pi/4}(s r_{\pi/2}) q_{\pi/4},$$

where $r_{\pi/2} = q_{\pi/2} q_{\pi/4}$ is a positive rotation through $\pi/2$, so that t is conjugate to $u = s r_{\pi/2}$. The effect of u on a triangle Δ_0 is illustrated below. Here $\Delta_n = u^n(\Delta_0)$, $n = 0, 1, \ldots, 8$.



To find other sequences having a similar property to (3) Mike Crampin (Open University) has considered mappings obtained by glueing together a pair of shears each of which fixes the x-axis. In this way he finds, for example, that if a_0, a_1 are not both 0 then the sequence

$$a_{n+2} = \tfrac{1}{2}(a_{n+1} + |a_{n+1}|) - a_n, \qquad n = 0, 1, 2, \ldots,$$

has period 5.

With regard to (b) the main question seems to be "Is $\{a_n\}$ bounded?" Since $\mathrm{Im}[a_{n+2}] = -\mathrm{Im}[a_n]$, this question reduces to "Is $\{\mathrm{Re}[a_n]\}$ bounded?" Dov Aharanov has shown that it is enough to investigate the two cases:

(i) $a_0 = -x + i\varepsilon$, $a_1 = 1 + i\eta$, $0 \leq x \leq 1$,

$\varepsilon > 0$, $\eta > 0$ arbitrarily small, and

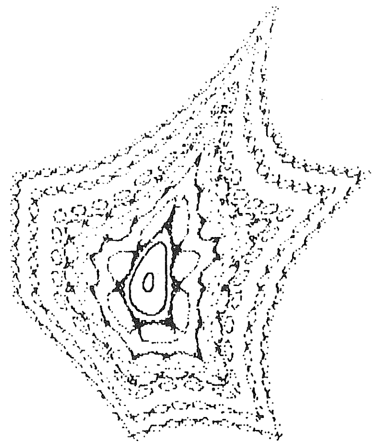
(ii) $a_0 = x + i\varepsilon$, $a_1 = 1 + i\eta$,

where $|x| = O(\max(|\varepsilon|, |\eta|))$.

Taking the special case $a_0 = a \in \mathbb{R}$ and $a_1 = i$, the points $(\mathrm{Re}[a_{2n}], \mathrm{Re}[a_{2n+1}])$, $n = 0, 1, 2, \ldots$, in $\mathbb{R}^2$ form an orbit of the (area-preserving) mapping

$$(u,v) = (\sqrt{y^2+1} - x,\ |u| - y).$$

These orbits are plotted below for $a = \frac{1}{2}, 1, \frac{3}{2}, \ldots, 4$. As $a \to \infty$ the orbits seem more and more to resemble the boundary of the previous figure!



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