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NEWSLETTER

EDITOR

Donal Hurley

The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

ASSOCIATE EDITOR

Patrick Fitzpatrick

The *Newsletter* also seeks articles of mathematical interest written in and expository manner. All parts of mathematics are welcome, pure and applied, old and new.

Manuscripts should be typewritten and double-spaced on A4 paper. Authors should send two copies and keep one copy as protection against possible loss. Prepare illustrations carefully on separate sheets of paper in black ink, the original without lettering and a copy with lettering added.

Correspondence relating to the *Newsletter* should be sent to:

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IRISH MATHEMATICAL SOCIETY

SECRETARY'S REPORT (January 1983 to August 1984)

As I must now resign as Secretary of the Irish Mathematical Society because I am to go on leave for 1984/1985, I feel it is appropriate to summarise my activities as Secretary.

1. Massera et al.

Shortly before I took up office the Society decided to support the cases of J.L. Massera (an elderly Mathematician and Parliamentarian imprisoned without trial in Uruguay) and V. Kipnis (a Russian Mathematician who had his doctorate revoked following refusal of permission for him to emigrate). As directed by the Society I wrote various letters in support of these two mathematicians to appropriate Ambassadors, to the President of the Academy of Sciences of the USSR, to the Minister for Foreign Affairs in Ireland and most recently (in January 1984) to the United Nations Centre for Human Rights. Most of these letters did not receive a reply but the Minister replied (in a sympathetic vein) to several of my letters and finally the United Nations, the Minister and the International Campaign for Massera wrote to inform the Society of Massera's release in March.

No progress appears to have been made in the case of V. Kipnis, though (in March 1983) the Minister for Foreign Affairs informed us that he was aware of the case. (See also the report by S. Dineen in the Newsletter of March, 1984).

The Society has now been requested to lend its support to a new campaign in support of Yury Orlov (Physicist) and Anatoly Shcharansky (Computer Scientist) both of whom have been refused permission to emigrate from the USSR. The

Society is to decide its attitude to this campaign (which is organised by those who ran the Massera Campaign) at the December 1984 Ordinary Meeting.

Y. Orlov obtained his doctorate in 1963 and worked at a Physics institute in Armenia until 1972, when he moved to Moscow. In 1973 he wrote an open letter in support of Sakharov and became a founder member of the Soviet Branch of Amnesty International. He was dismissed from his Moscow post in January 1974. He continued various kinds of political activity until 1978 when he was arrested and sentenced to seven years hard labour and five years exile, at a suspect trial. He has since been punished for trying to smuggle a scientific article out of his prison.

A. Shcharansky worked at the All-Union Scientific Research Institute of Oil and Gas until 1975, until he was dismissed following repeated unsuccessful applications to emigrate. He then became associated with Sakharov and the Helsinki group. The Soviet press began a campaign accusing him of spying for the USA. He was then convicted at a closed trial and sentenced to 13 years in prison.

These summaries are paraphrased from a May 1980 article by Israel Halperin.

2. Aer Lingus Young Scientists Exhibition

I suggested early in 1983 that the Society might consider sponsoring a prize at this Exhibition. The Society agreed but Aer Lingus eventually replied that they felt more than a prize would be needed to encourage Mathematical projects.

In response to the Aer Lingus request P. Boland drew up a list of topics considered suitable for projects. I have since expanded this into a short article. In addition N. Buttimore (TCD) has written an article on projects concern-

ing software for computer-aided Mathematical instruction and A. Wood (NIHE, Dublin) has written an addendum on Mathematical modelling. These have been circulated to a limited extent, but unfortunately they were not ready in time for the IMTA to circulate them. Aer Lingus are considering including them with other literature on the Exhibition.

3. Joint Meeting with LMS

I wrote to the London Mathematical Society asking them to consider co-operation and reciprocity with the IMS. This has resulted in a proposal (as yet not completely firm) to hold a joint two-day meeting of the LMS and IMS in Ireland in 1986. The meeting, assuming it goes ahead, will probably take place in Dublin on a Friday afternoon and a Saturday morning. There will be six lectures by eminent invited speakers on a coherent set of topics of wide interest. This represents a new departure for the IMS and I hope it will be possible in the future for the Society to organise similar meetings on a smaller scale but with more than one speaker. The proposed joint meeting with the LMS will coincide with the 10th anniversary of the founding of the IMS.

4. Newsletter

As all members are by now well aware the Newsletter has reached an enviable standard of professionalism in recent years due to the efforts of its editor Donal Hurley assisted by Pat Fitzpatrick. The Society and its Committee have repeatedly expressed their appreciation to the Editor and to the Associate Editor. Donal Hurley has indicated that he wishes to step down as Editor after the March 1985 issue, when his term of office expires.

5. Organisation

Some changes in the organisation of the Society have occurred in the past two years. First the system of having local representatives has proved an efficient means of distributing the Newsletter and collecting dues. Second the responsibility for distributing the Newsletter has been given to P. Boland and myself, to relieve the Editor of this unnecessary burden.

This seems an appropriate point to mention the conscientious and herculean efforts of the Treasurer, G. Enright, in keeping the Society's financial affairs in order and in collecting outstanding subscriptions. He has also succeeded in attracting a moderate number of members from abroad.

6. IMTA

Discussions with the Irish Mathematics Teachers Association with a view to establishing reciprocity of membership and closer co-operation have been going on for some time. M. Clancy has represented the IMS. While reciprocity of membership has not yet been established, some benefits have arisen from the discussions.

Richard M. Timoney
26th July, 1984

CONFERENCES

The Irish Mathematical Society is happy to be able to assist with the funding of mathematical conferences. During 1983/1984 we sponsored Professor Everitt's Lecture Tour in December, UCD Matrix Theory Conference in March and UCC Group Theory Conference in May. Applications for support for 1984/1985 should be submitted as early as possible. Forms are available from the Treasurer.

ROYAL IRISH ACADEMY

Proceedings

Section A of the Proceedings of the R.I.A. is available to members of the Irish Mathematical Society at a special price. A discount of one-third of the normal price is allowed on orders placed through the I.M.S. Treasurer.

Institutional Members

Institutional Membership of the Irish Mathematical Society is available for 1984/1985 at IR£35.00. Please encourage your Department, College or other Institution to join and to support I.M.S. activities in this way. Institutional members receive two copies of each issue of the Newsletter and they may nominate up to five students for free membership of the Society. Subscriptions are normally paid in advance in September of each year.

Ordinary Members

Ordinary Membership of the I.M.S. continues to be good value at IR£3.50 (internal) and IR£4.00 (overseas) for the session 1984/1985. Please encourage your colleagues at home and abroad to join. Subscriptions are normally paid in January of each session. Local Representatives should collect from members who do not use Banker's Orders. All subscriptions should be forwarded to the Treasurer. Application forms for new members are available.

PERSONAL ITEMS

Mr Michael Brennan of Our Lady's Bower School has been appointed to a position in the Regional Technical College, Waterford.

Dr Soo Bong Chae of New College (USF), Sarasota, Florida, is visiting the UCD Department of Mathematics from September 1984 to January 1985. Dr Chae is interested in functional Analysis and Set Theoretic Topology.

Dr Stephen Gardiner has taken up a position in the Department of Mathematics at UCD. Dr Gardiner formerly held a joint appointment with the Northern Ireland Department of Agricultural Biometrics at Queen's University, Belfast. His special interests include Classical Potential Theory and the design and analysis of experiments.

Dr Roderick Gow of the Mathematics Department, UCD, is visiting the Department of Mathematics at the University of Wisconsin (Madison) for the academic year 1984-85.

Dr Wolfgang Heusgen is visiting the Mathematics Department, UCD, for the academic year 1984-85. Dr Heusgen comes from Munich and is interested in function Theory, Banach Spaces and Functional Analysis.

Dr Mohammed Kazim Khan of the Mathematics Department, Kent State University, Ohio, will visit the Mathematics Department, UCD, from January 1985 to June 1985. Dr Khan is interested in Statistics and Software Reliability.

Dr Maciej Klimek has been appointed to a position in the Department of Mathematics, UCD. Dr Klimek is from Cracow, Poland, and his interests include Complex Analysis, Polar Sets and Plurisubharmonic Functions.

Dr Gerard J. Murphy has been appointed to a position in the Mathematics Department, UCC. Dr Murphy did his undergraduate studies at TCD, got his Ph.D. at Cambridge University and

comes to UCC from a position at the University of Oregon. His special interests are in Operator Theory.

Dr Paul Robinson who is completing a Ph.D. for Warwick University is replacing Richard Timoney at TCD this year. His interests are in Symplectic Geometry and Geometric Quantization.

Dr Jeanne Mc-Stynes of the Waterford Regional College has been appointed to a position at the Cork Regional Technical College.

Dr Martin Stynes has been appointed to a position in the Mathematics Department, UCC. Dr Stynes did his undergraduate studies at UCC, got his Ph.D. at the University of Oregon and comes to UCC from the Waterford Regional Technical College. His special interest is Numerical Analysis.

Dr Richard Timoney of the School of Mathematics, TCD, is visiting the Mathematics Department, University of North Carolina, Chapel Hill, for the academic year 1984-85.

CSO scoops Nobel mathematics prize

THIS YEAR'S Nobel Prize for advances in Pure Mathematics has been awarded to Ireland's Central Statistics Office.

The Nobel committee, sitting in Stockholm, cited the CSO's "highly imaginative techniques" which led to the discovery of the famous £500 million 'black hole'. The judges also paid tribute to the "truly original" methods employed in assessing the more recent milk quotas.

From *The Phoenix*, 26 October, 1984

AN ELEMENTARY NUMBER THEORY RESULT

*D. Birch and **D. Singmaster

The first two functions introduced in number theory are usually $\pi(n)$, the number of primes less than or equal to n , and $\phi(n)$, Euler's ϕ -function, the number of totitives of n (i.e. positive integers which are $\leq n$ and coprime to n). There is a simple relationship between these functions, namely: $\phi(n) > \pi(n)$ apart from a finite number of exceptional n .

This relationship, despite its simplicity, is generally unknown - it does not occur in Dickson, Hardy and Wright, or Leveque. After our discovery of it, A. Makowski referred us to [M] and [S]. The first appeared in a small unreviewed journal. The second is a proof of Erdos described by Sierpinski in the Polish original edition of his "Elementary Theory of Numbers", but omitted from the English edition. Consequently, we are presenting the result again, in the hope that it will become better known. Our proof is similar to [M].

We need one non-elementary but well known result.

Bertrand's Postulate. [H, p. 343]. For any $x > 1$, there is a prime p such that $2x > p > x$. \square

From this, we obtain the following.

Theorem 1. If n has $r \geq 5$ distinct prime factors then $\pi(\sqrt{n}) \geq 2r$.

Proof. If n has more than 5 distinct prime factors, then $n \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ and hence $\sqrt{n} > 48$. Since 43 is the 14th prime, the result is true for $r = 5, 6$ or 7 . We now prove it generally for $r \geq 7$ by induction. Assume the result is true for every number which has k prime factors, for some

$k \geq 7$, and suppose n has $k+1$ prime factors. Hence $n > 16m$ which implies $\sqrt{n} > 4\sqrt{m}$. By Bertrand's Postulate, there are at least two primes between \sqrt{n} and \sqrt{m} . By the induction hypothesis, $\pi(\sqrt{m}) \geq 2k$, hence $\pi(\sqrt{n}) \geq 2k+2$. \square

Denote the number of composite totitives of n by $c(n)$. The number of prime totitives of n is the number of primes less than or equal to n minus the number of prime factors of n , i.e. $\pi(n) - r(n)$ (where we have used $r(n)$ for what we previously denoted r). Recalling that 1 is neither prime nor composite, we have:

$$(\dagger) \quad \phi(n) = c(n) + (\pi(n) - r(n)) + 1.$$

Lemma. If $n \geq 92$, then $\phi(n) > \pi(n)$.

Proof. From (\dagger) , we see that $\phi(n) > \pi(n)$ if and only if $c(n) \geq r(n)$.

If $r(n) \leq 2$, then at least two of $\{4, 9, 25, 49\}$ are coprime to n and so $c(n) \geq 2 \geq r(n)$ for $n \geq 50$.

If $r(n) = 3$ and $2 \nmid n$, then 4, 8, 16 are coprime to n .

If $r(n) = 3$ and $3 \nmid n$, then 9, 27, 81 are coprime to n .

If $r(n) = 3$ and $n = 2^a 3^b p^c$ for some prime $p > 3$, then at least one of 25 or 49 is coprime to n and at least two of $\{35, 55, 65, 77, 91\}$ are coprime to n . So if $r(n) = 3$ and $n \geq 92$, then $c(n) \geq r(n)$.

If $r(n) = 4$, then $n \geq 2 \cdot 3 \cdot 5 \cdot 7 = 210$ and there are always at least four composite totitives, $q_1^2, q_2^2, q_1 q_2, q_1 q_3$, where $q_1 \leq 11, q_2 \leq 13$ and $q_3 \leq 17$ are prime totitives of n . So $c(n) \geq 4 = r(n)$.

If $r(n) = 5$, then since $\pi(\sqrt{n}) \geq 2r(n)$, there are at least r prime totitives of n , say q_1, q_2, \dots, q_r , all less than \sqrt{n} . Hence $q_1^2, q_2^2, \dots, q_r^2$ are all less than n and so $c(n) \geq r$. (In fact, we have $c(n) \geq r(r+1)/2$.) \square

Either by pursuing the argument a little further or by examining every integer less than 92, we obtain our main result.

Theorem 2.

$\phi(n) < \pi(n)$ if and only if $n = 6, 10, 12, 18, 24, 30, 42$ or 60 ;

$\phi(n) = \pi(n)$ if and only if $n = 2, 3, 4, 8, 10, 14, 20$ or 90 ;

$\phi(n) > \pi(n)$ for all other n . \square

Corollary. $\phi(n) > n/\log n$ except for $n = 1, 2, 3, 4, 6, 10, 12, 18$ or 30 .

Proof. It is well known that $\pi(n) > n/\log n$ for $n \geq 17$ [R, p. 71 or SW, p. 106]. The Corollary follows immediately for $n > 60$ and some calculation yields the complete result. \square

The following two results are immediate corollaries to our work using a little calculation.

Proposition 1. $\phi(n) \geq \sqrt{n}$ except for $n = 2$ or 6 . (Vaidya [V].)

Proposition 2. An integer n has the property that all of its totitives are prime or 1 if and only if $n = 1, 2, 3, 4, 6, 8, 12, 18, 24$ or 30 . (Schatunowsky (1893) and Wolfskel (1900) [D, p. 132, item 73 and p. 134, item 91].)

It is known [SW, 4.1] that

$$\lim_{n \rightarrow \infty} \frac{\phi(n) \log \log n}{n} = e^{-\gamma}, \text{ where } \gamma \text{ is Euler's constant}$$

($= \lim_{r \rightarrow \infty} \left\{ \sum_{i=1}^r 1/i - \log r \right\}$) and $\overline{\lim}_{n \rightarrow \infty} \frac{\phi(n)}{n} = 1$. The first of these is more accurate than our Corollary, but only asymptotically.

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BOUNDED CONJUGACY CONDITIONS

Mark Cartwright

This article surveys the history of the study of groups with finite conjugacy conditions. It presents some new results obtained by the author. It is based on a talk given by the author to the Group Theory Conference held in Galway on 11/12 May 1984.

1. BFC Groups and Derived Groups

A group is said to be a BFC group if there is a finite upper bound on the sizes of its conjugacy classes. B.H. Neumann characterized such groups in 1954 [8].

Theorem 1 (Neumann [8]). If G is a BFC group then the derived group G' is finite. Hence the BFC groups are precisely the finite-by-abelian groups.

The BFC-number of a BFC group is the maximum of the sizes of its conjugacy classes. We write it as $n(G)$ or just n . Already in [8] Neumann wondered whether the order $|G'|$ of a BFC group G could be bounded in terms of $n(G)$. The question was answered affirmatively by Wiegold in [14]; refinement of the argument led to the following result.

Theorem 2 (Wiegold [15]). If G is a BFC group then

$$|G'| \leq n^{\frac{1}{2}n^4 (\log n)^3},$$

where the logarithm is to base 2 (as will be the case throughout, unless specified otherwise).

However, even this bound is much too big. Examples of groups led Wiegold to the conjecture that $|G'| \leq n^{\frac{1}{2}(1+\log n)}$

for any BFC group, and to date no group has been found that disproves this, although there are groups known for which equality holds with arbitrarily large BFC-numbers. For p -groups the conjecture was sharpened by requiring that the logarithm be to base p .

Macdonald improved the bound of Theorem 2 in [7] to $|G'| \leq n^{bn(\log n)^3}$, but the first significant advance came with Shepperd and Wiegold's paper [10] of 1963.

Theorem 3 (Shepperd and Wiegold [10]). Let G be a BFC group.

(i) If G is soluble, then $|G'| \leq n^{q(\log n)}$, where $q(x)$ is a certain quintic polynomial.

(ii) If G is nilpotent of class 2, then $|G'| \leq n^{(\log n)^2}$.

Thus, at the price of restricting attention to special classes of BFC groups, they were able to produce bounds in which the exponent is purely logarithmic, as the Wiegold conjecture requires.

This set the pattern for the next 11 years. By using commutator calculations and Lie ring methods, workers established the Wiegold conjecture first for class 2 p -groups (Bride [2]), then for metabelian p -groups (Vaughan-Lee [11]), and finally for p -groups in general (Vaughan-Lee [12]).

Theorem 4 (Vaughan-Lee [12]). Let G be a p -group with $n(G) = p^b$. Then $|G'| \leq p^{\frac{1}{2}b(b+1)}$.

The fact that this represents the stronger version of the Wiegold conjecture has been important since. It was used in the proof of the following result, by P.M. Neumann and Vaughan-Lee in [9].

Theorem 5 (Neumann and Vaughan-Lee [9]). Let G be a BFC group.

(i) If G is soluble, then $|G'| \leq n^{\frac{1}{2}(5+\log n)}$.

(ii) In any case, $|G'| \leq n^{\frac{1}{2}(3+5\log n)}$.

The result for soluble groups improves Shepperd and Wiegold's, and indeed is just n^2 away from the Wiegold conjecture. This is still the best bound known for soluble groups.

The result for general BFC groups is not quite so good, but it is worth recalling that the previous best bound was Macdonald's, in 1961. Recently the present author has tightened this bound ([3] and [4]). This, however, depends on the Classification Theorem for finite simple groups, for it uses the fact that all such groups may be generated by two elements.

Theorem 6 (Cartwright [3]). If G is a BFC group, then $|G'| \leq n^{\frac{1}{2}(41+\log n)}$.

This is just n^{20} away from the Wiegold conjecture. The proof, as well as the result, parallels Neumann and Vaughan-Lee's bound for soluble groups.

Theorems 4, 5(i) and 6 represent the current state of knowledge.

2. Class of p-Groups

The bound given in Theorem 4 on the size of the derived group of a BFC p-group gives an immediate bound on its class; namely, a p-group with BFC number p^b has class at most $\frac{1}{2}(b^2+b)+1$. This bound is, however, much too big. The 'class-breadth conjecture' for finite p-groups was that such a group with class c and BFC-number p^b satisfies $c \leq b+1$. This would mean that the dihedral groups, for example, are a natural case where the limit is attained. In 1969 the following result was proved, which goes some way towards this.

Theorem 7 (Leedham-Green, P.M. Neumann and Wiegold [6]).

Let G be a finite p-group with BFC number p^b and class c . Then $c \leq (\frac{p}{p-1})b+1$. In particular, if G is nonabelian then $c \leq 2b$.

However, in a series of papers in 1980-81, examples of groups were constructed for each integer k with $c > b+k$. In fact the groups produced in [5] have class approaching $b+b^{\frac{1}{2}}$. These examples are all 2-groups; so far, no-one has found counterexamples for odd primes.

The most recent result in this area (in [4]) improves the bound of Theorem 7 to $c \leq \frac{5}{3}b+1$.

3. Derived Length of Soluble Groups

It is well-known that the derived length of a nilpotent group of class c is at most $1 + \log c$. From Theorem 7 we may therefore deduce the following.

Theorem 8. Let G be a nonabelian finite p-group with BFC-number p^b and derived length d . Then $d \leq 2 + \log b$.

Thus if H is a nilpotent group of BFC-number n and derived length d , we have $d \leq 2 + \log \log n$.

Despite the gap in Theorem 7, this bound is very nearly the best possible. For if H is taken to be a Sylow 2-subgroup of S_{2^k} , the symmetric group of degree 2^k , then H has derived length precisely k and order (and therefore BFC-number) less than 2^{2^k} , so that here we have $d > \log \log n$.

Perhaps more surprising than this is that a similar result can be proved for soluble groups in general, as P.M. Neumann and Vaughan-Lee showed in [9].

Theorem 9 (Neumann and Vaughan-Lee [9]). Let G be a nonabelian soluble BFC group with BFC-number n and derived length d . Then $d < a \log \log n + 8$, where $a = 1 + (5/\log 9) \approx 2.58$.

Among known examples, those with largest derived length relative to their BFC-number are the 2-groups mentioned above.

4. Generalisations

Baer [1] generalised the concept of a BFC group in the following way. Suppose G is a group and $H, K \leq G$. We call (H, K) a BFC pair in G if there are integers m, n such that $|H:C_H(y)| \leq m$ for all $y \in K$ and $|K:C_K(x)| \leq n$ for all $x \in H$. Theorem 1 then generalises as follows.

Theorem 10 (Vaughan-Lee [13]). Suppose (H, K) is a BFC pair in a group G . Let M and N be the normal closures in $\langle H, K \rangle$ of H and K respectively. Then the commutator subgroup $[H, K]$ is finite if and only if both $|M:H|$ and $|N:K|$ are finite.

Moreover, the size of $[H, K]$ may be bounded in a way that parallels Theorems 2-6. Vaughan-Lee in [13] proves the following result, which deals with the case where both H and K are normal in $\langle H, K \rangle$; but it is easy to see how this may be adapted to the more general case. Naturally, the parameters $|M:H|$ and $|N:K|$ must also be used.

Theorem 11 (Vaughan-Lee [13]). Suppose (H, K) is a BFC-pair in G , with $H, K \trianglelefteq \langle H, K \rangle$, and suppose m and n are upper bounds for $\{|H:C_H(y)| : y \in K\}$ and $\{|K:C_K(x)| : x \in H\}$ respectively. Then $|[H, K]| \leq m^{\frac{1}{2}(3+13 \log n)}$.

Corresponding to the Wiegold conjecture for BFC groups, there is a conjecture in this more general situation. It is thought that $|[H, K]| \leq m \log n$ always holds. (This is a symmetric bound, for $m \log n = 2 \log m \cdot \log n$.) Again, cases are known in which this value is attained, but no example has come

to light which disproves the conjecture.

Unlike the situation with BFC groups, the conjecture has been established for relatively few special cases. Vaughan-Lee in [13] verified the conjecture in the case that $[H, K]$ is in the centre of $\langle H, K \rangle$, and improved on Theorem 11 in the case where $[H, K]$ is central in H . The present author has extended this slightly.

Theorem 12 (Cartwright [4]). Let (H, K) be a BFC pair in a p -group G , with $H, K \trianglelefteq \langle H, K \rangle$ and $[H, K] \leq Z(H)$. Let $\max\{|H:C_H(y)| : y \in K\} = p^a$ and $\max\{|K:C_K(x)| : x \in H\} = p^b$. Then $|[H, K]| \leq p^{ab}$.

Also in [4], improved results are obtained for more general cases.

Theorem 13 (Cartwright [4]). Let (H, K) be a BFC pair in G , with $H, K \trianglelefteq \langle H, K \rangle$. Let $\max\{|H:C_H(y)| : y \in K\} = m$ and $\max\{|K:C_K(x)| : x \in H\} = n$.

- (i) If G is a p -group then $|[H, K]| \leq m^{\frac{1}{2} + (7/4) \log_p n}$.
- (ii) In any case, $|[H, K]| \leq m^{(41/2) + (25/8) \log n}$.

This work was done while the author held a Research Studentship under the SERC.

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OBVIOUS

We in the Math. Faculty in the University of Cork were alarmed to discover recently that our students had opened a file on us. Apparently:

'If Professor Barry says it is obvious then it is obvious at once. If Professor Twomey says it is obvious then it will be obvious after a few minutes' thought. If Professor Holland says it is obvious then it will become obvious after about three weeks' intensive study. If Professor Harte says it is obvious then it is probably not true.'

Robin Harte

MATHEMATICAL MODELLING IN PHARMACOLOGY

James J. Grannell

Introduction

The primary objective of Clinical Pharmacology is the effective treatment of *human* disease by drug therapy. The associated problem is the design of *rational* drug administration programs. The resolution of the problem lies in the convergence of many disciplines ranging from Biochemistry and Physiology to Statistics and (more recently) Mathematics. The response of humans to drugs displays considerable variability. Thus, statistical techniques have played an important role in drug administration. Advances in techniques of measurement during the past ten years have made possible the study and reduction of variability by using mathematical (and numerical) modelling.

Classical Basis of Drug Therapy

The basic concepts in drug administration are

- (i) Dose: A single dose is adequate in some circumstances. In other cases it may be necessary to give a sequence of doses, for example, until a disease has been cured or on an extended basis if a disease can only be contained.
- (ii) Response or Pharmacologic effect: There is normally a desired minimum therapeutic effect. Drugs usually have multiple effects, some of which may be toxic (causing, e.g. nausea, vomiting) or lethal.

The problem is to understand the relationship between dose and response. The classical approach is statistical in nature. Consider the single dose administration program as

an illustration. Each member of a sample of the population is given a dose and the fractions of the sample in whom a therapeutic response and in whom a toxic response occurs is measured. The cumulative frequency distribution of response typically (there are well-known exceptions [1]) has the appearance shown in Figure 1.

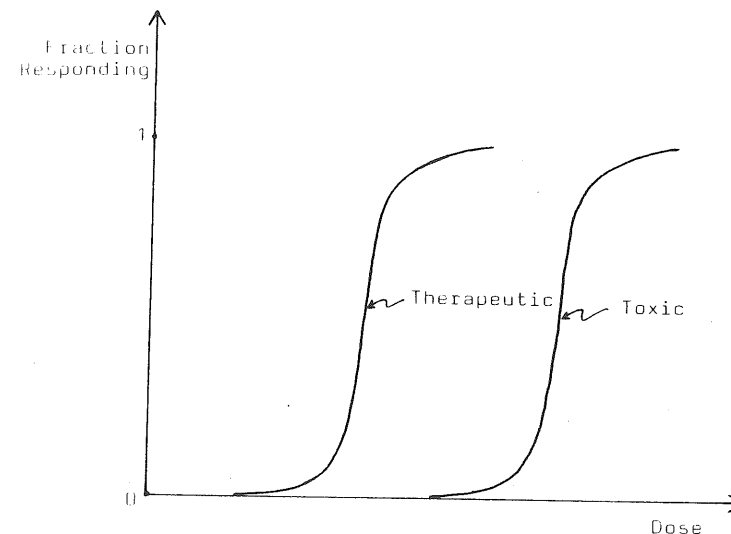


FIGURE 1

The dose at which *n* per cent of the sample shows a therapeutic (toxic) response is commonly denoted by ED_n (resp. TD_n). If ED_{99} is significantly less than TD_1 the drug is considered to be both safe and effective. However, if $TD_1 > ED_{99}$ it is not possible to find a dose which will be both safe and therapeutically effective for the whole population. Unfortunately, many important drugs fall into the latter category and it thus becomes necessary to understand the nature of the variability in order to safely and effectively treat many diseases.

Physiological and Biochemical Considerations

All theories of drug action start from the premise that a drug exerts its effect by interacting chemically with certain molecules, called *receptors*, in the body [1]. The location of the receptors depends on the drug. The identification of receptors is now a major research area in Pharmacology [2,3]. Implicit in such theories is the view that, in order to understand the nature of response to a drug, the *distribution* of the drug throughout the body must be known or inferred. Further, if the interaction is chemical, the *concentration* of the drug at receptor sites will be the key quantity which determines the magnitude of the response.

Drugs are commonly introduced into the body by the oral and intravenous routes (there are many other routes). In the intravenous case the drug is injected directly into the bloodstream. Orally-administered drugs may be absorbed into the blood from any part of the gastro-intestinal tract (mouth-stomach - small intestine - large intestine - rectum). The main site of absorption is the small intestine. In each case the introduction of drug into the blood is important since it transports the drug throughout the body. The other fluid contents of the body are also important for the distribution of drug. Body fluids account for 60% of body weight. The volume of the blood is about 5 litres of which 3 litres consists of *plasma water* and 2 litres is contained in blood cells. A further 26 litres is contained in other cells of the body (*intracellular*) and 10 litres occupies the space between cells (*intercellular*). Most drugs, once in plasma, will distribute throughout the extracellular fluid. The rate of drug penetration into intracellular fluid depends on the degree of perfusion of the various tissues (groups of similar cells) by blood. In addition to convective transport by blood, distribution of drug occurs by various diffusion processes. The known processes are: neutral diffusion in accordance with Fick's law [4], electrodiffusion in accordance with the Nernst-Planck law [5], facilitated diffusion [6] and active transport

[6]. The following list gives a rough categorization of tissue on the basis of *vascularity* (perfusion by blood):

- (i) A highly perfused tissue group; blood cells, heart, lung, liver, kidney, brain and spinal cord.
- (ii) A poorly perfused tissue group; muscle and skin.
- (iii) A fat group; includes bone marrow.
- (iv) A negligibly perfused group; bone, teeth, ligaments, cartilage and hair.

Drugs are removed from the body by the processes of *excretion* and *biotransformation*. The most important vehicle for the excretion of drugs is the kidney. (Volatile gases, which are mainly excreted by the lung, are not considered in this article.) The *renal* (kidney) elimination process is complex and involves three processes: glomerular filtration (driven by a pressure gradient from plasma to urine), tubular reabsorption (electrodiffusion of drug from urine back into plasma) and tubular secretion (removal of drug from plasma by active transport). The relative significance of each process is drug-dependent. Biotransformation corresponds to the de-activation of drug by chemical transformation. Such chemical reactions occur mainly in the liver and are enzyme-mediated. The transformed inactive drug (metabolite) is eliminated by the kidneys [7].

Mathematical Modelling - Pharmacokinetics

The aim of the receptor-theories of drug action is twofold:

- (i) to obtain an understanding of the relationship between dose and response in a manner that disposes of inter-individual variability;
- (ii) to enable the design of rational drug-administration programs.

Since the effect is now postulated to depend on the concentration of drug at receptor sites, the variability must be due to physiological factors which cause a given dose to give rise to different concentrations at the receptor sites of different individuals. Thus, it would seem that prospective mathematical models would have to be able to describe the evolution of drug concentration in the body and also to describe the relationship between concentration at receptor sites (usually unknown, at present) and pharmacologic effect. The approach that has been taken by pharmacologists is pragmatic. Physiological modelling would be a complex process [4,8] and it would probably be quite difficult to develop simple whole body models which could easily be used in clinical practice. The simplest approach would be to administer a drug and see what happens. There is, however, a severe limitation on the nature of measurements that can, ethically, be made at the present time. The measurement of drug levels is limited to those in blood plasma and urine. The concentrations can be very small (of the order of nanograms/millilitre) and are difficult to measure. Considerable effort has been expended on the development of accurate techniques over the past ten years [9].

Compartment Models

Consider the following experiment. A dose of drug is rapidly injected intravenously and plasma concentrations are measured at a number of later times. The results are surprisingly simple. For many drugs, the plasma concentration decays exponentially. More precisely, a decaying exponential seems to fit the data. For other drugs, a linear combination of decaying exponentials will fit the data. The curves are usually fitted using non-linear least square techniques [10]. It could certainly be argued that these results have no great physiological significance. For example, Muntz's theorem [11] tells us that the set of exponentials $\{e^{-\lambda_n t}, 0 < \lambda_1 < \lambda_2 < \dots, \sum \lambda_n^{-1} = \infty\}$ is complete on $L^2(0, \infty)$. However, continuing in a spirit of pragmatism, we could speculate that these expon-

entials could correspond to eigenfunctions of a linear-system of differential equations

$$\frac{dv}{dt} = Av$$

where v is an n -vector, A is an $n \times n$ matrix and n is the number of exponentials. Then the plasma concentration would be one component of a vector. On the basis of this model, pharmacologists consider the body to consist of a number of *compartments* between which drug can transfer reversibly. A compartment does not necessarily correspond to any anatomically or physiologically identifiable part of the body.

The One-Compartment Model

In this case the model equation is

$$\frac{dc}{dt} = -kc, \quad t > 0 \quad (1)$$

where t is time since administration (assumed to be instantaneous) and c is plasma concentration of drug. The time taken for the drug concentration to decrease by a factor of one-half is known as the drug half-life ($= \ln 2/k$). Typical half-lives range from hours to days, so that injection times of the order of a few minutes may be taken to be instantaneous. Both $c(0)$, the initial plasma concentration and k are found from the exponential curve fit. The known initial quantity is the amount of drug administered, D . If the drug were confined to plasma then we would expect

$$V = \frac{D}{c(0)} \approx 3 \text{ litres (the volume of plasma-water)}$$

The quantity, V , bears the unfortunate name of 'volume of distribution' of the drug. Observed values of V can be several orders of magnitude greater than 3. For example, the value of V for quinacrine (an anti-malarial drug with a half-life of 10 days) is of the order of 10^4 litres. In general, only a small fraction of drug is in plasma. The transfer of

drug from plasma to the rest of the body must be rapid in order for the early concentration in plasma to be so small.

This simple model has contributed to the understanding of dose-response variability. When plasma levels of individuals given the same dose are measured it is found that both $c(0)$ and k are variable. The value of k can show considerable variation which is thought to be related to genetic factors. It can also vary significantly with age (e.g. underdeveloped elimination processes in newborn babies), with temperature (which can be raised or depressed in illness and can influence biotransformation processes) and with other factors [7]. Thus, we have a possible explanation of variability in response to a fixed dose. Space does not permit a discussion of the various receptor theories of concentration-effect [1,12]. If the relationship between plasma concentration and that in other parts of the body were simple it might be possible to find a relationship between plasma concentration and effect. Pharmacologists have made the simplest assumption that the concentration in other parts of the body is *proportional* to that in plasma and have gone on to seek relationships between plasma concentration and effect. Relationships of this type which also conform to receptor-theory have been found [13,14].

The model has also been applied to the design of therapeutic drug administration programs. Of the many possible routes for administration we consider only the oral and intravenous cases. Some drugs (e.g. analgesics, hypnotics, neuromuscular blocking agents, bronchodilators and anti-emetics) may be used effectively as a single dose, but drugs are most frequently given on a continuous basis. The following are examples of administration programs.

- (i) Single intravenous injection.
- (ii) Intravenous infusion at a constant rate.
- (iii) A sequence of intravenous injections.

- (iv) Single oral ingestion in tablet or liquid (solution) form
- (v) A sequence of oral ingestions.

As we have seen the one-compartment model can be applied to case (i). It is not *a priori* obvious that this model can be applied to cases (ii) and (iii) but experiment has shown that it can in fact also be used to describe these cases. We proceed now to discuss the administration of drugs according to programs (ii) - (v).

Case (ii)

The model equations are:

$$\frac{dc}{dt} = r - kc \quad (2)$$

$$c(0) = 0$$

where $r = R/V$ and R is the amount of drug infused per unit time. The assumption that V has the same value as in case (i) is verified by experiment [13]. The solution to (2) is

$$c(t) = \frac{r}{k}(1 - e^{-kt})$$

so that the concentration in plasma evolves towards a steady-state or *plateau* level, r/k . Thus it is possible to establish a fixed concentration of drug in plasma (and presumably in the rest of the body). The plateau-effect provides the basis for much of drug therapy. If a range of plasma levels can be identified for which the therapeutic effect is manifested (the *therapeutic range*) then the objective is to get the plasma level into this range and to hold it there for as long as is necessary. The upper limit is the concentration at which toxic effects appear and the lower limit is the minimum effective level. For drugs with large half-lives (e.g. quinacrine) an initial priming dose is essential in order to rap-

idly achieve the therapeutic effect. In this case

$$c(0) = D/V$$

and

$$c(t) = \frac{r}{k} - \left(\frac{r}{k} - c(0)\right)e^{-kt}$$

where D is the amount of priming dose. In order to achieve a rapid effect D should be chosen close to R/k .

Case (iii)

Let the amount of each injection be A and let τ be the time interval between injections. At time $j\tau$, $j = 0, 1, 2, \dots$, the plasma level will jump by an amount $a = A/V$. The model equations are

$$\frac{dc}{dt} = \sum_{j=1}^n a\delta(t - j\tau) - kc$$

$$c(0) = a$$

where δ is the Dirac delta-function. The problem may more easily be formulated as a sequence of initial-value problems. Let $c_n(t)$ denote the plasma-concentration between the $(n-1)$ th and n th injections where t is now measured relative to the time of the $(n-1)$ th injection. Then

$$\frac{dc_n}{dt} = -kc_n, \quad 0 < t < \tau$$

$$c_n(0) = c_{n-1}(\tau) + a, \quad n = 1, 2, \dots$$

where $c_0(\tau)$ is defined to be zero. Then

$$c_n(t) = c_n(0)e^{-kt}.$$

The initial conditions may then be employed to recursively compute:

$$c_{n+1}(0) = ae^{-nk\tau} + a \frac{(1 - e^{-nk\tau})}{(1 - e^{-k\tau})},$$

$$c_\infty(0) = \lim_{n \rightarrow \infty} c_n(0) = \frac{a}{1 - e^{-k\tau}}$$

and

$$c_\infty(\tau) = \lim_{n \rightarrow \infty} c_n(\tau) = \frac{ae^{-k\tau}}{1 - e^{-k\tau}}$$

Hence, a 'plateau' is also established in this case, but now there are oscillations between fixed limits, as indicated in figure 2.

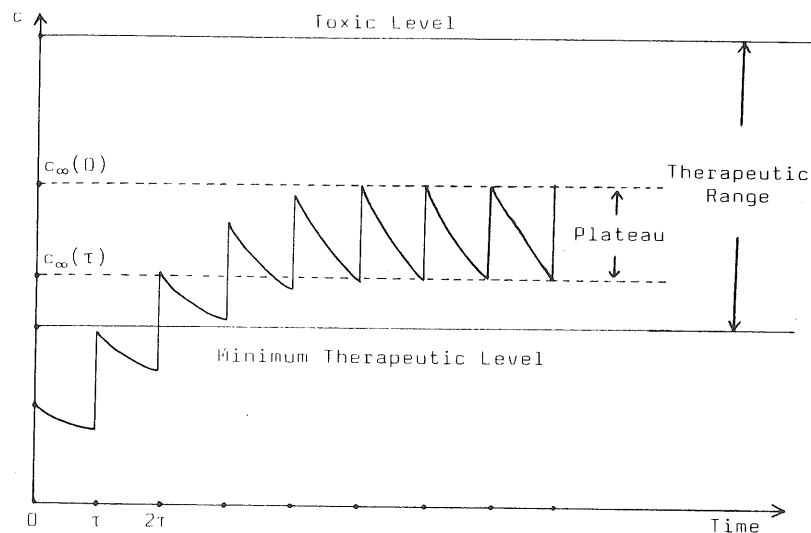


FIGURE 2

If a priming injection equal to $Vc_\infty(0)$ is given the plateau level will be immediately established.

Case (iv)

A tablet or capsule, which dissolves readily in the gastro-intestinal fluid, is the most common form of drug preparation for oral administration. If the plasma concentration of a drug falls in a mono-exponential fashion following intra-

venous injection it will exhibit a bi-exponential behaviour if administered orally [13,15]. The typical plasma concentration curve is indicated in Figure 3. The small intestine is the major site for drug (and nutrient) absorption due to its large surface area (about 200 m²) and high vascularity [7].

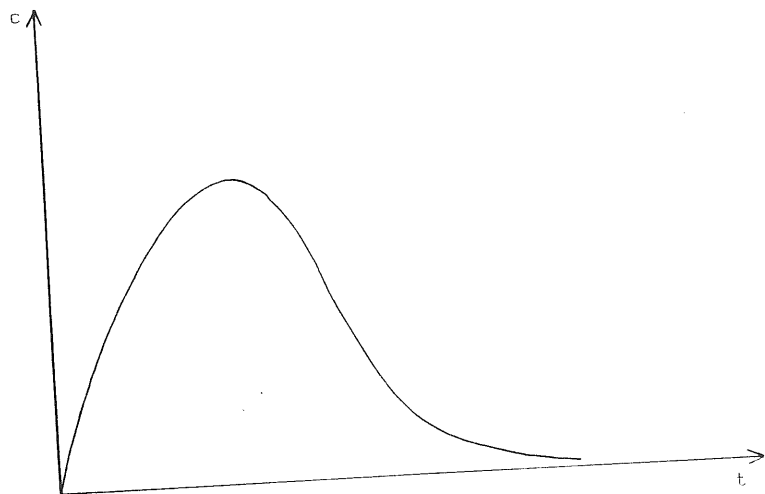


FIGURE 3

Drug which is transported from the intestinal fluid across the cells of the intestinal wall into the blood will be continuously removed by convection. The concentration of drug in blood returning to the intestine will be low (due to distribution) in comparison with that in the intestinal fluid. Hence a high concentration gradient will exist practically throughout the absorption process which implies that drug-transfer is a one-way process [16]. There is a further dimension to compartmental modelling which arises in the multi-compartment case. Identification of coefficients becomes a problem. This is partially circumvented by introducing

mass-balance principles. The models are formulated in terms of the total amount of drug in each compartment, rather than in terms of concentrations, and the assumption is made, as mentioned previously, that concentrations in various parts of a compartment are proportional to the total amount. In the current case, the gastro-intestinal tract may be viewed as one compartment and the remainder of the body as another. The model equations in this case are:

$$\begin{aligned} \frac{dG}{dt} &= -\lambda G \\ \frac{dC}{dt} &= \lambda G - kC \end{aligned} \quad (3)$$

$$\begin{aligned} G(0) &= 0 \\ C(0) &= 0 \end{aligned}$$

where G and C are the amounts of drug in the gastro-intestinal tract and the body, respectively, and D is the amount of the dose. The magnitudes and signs of the system matrix coefficients are chosen to conserve mass and to incorporate directions of mass transfer. Thus, the rate of loss of drug from the gastro-intestinal tract is equal to the rate of entry of drug into the body. Equations (3) are usually expressed in the form:

$$\frac{dC}{dt} = \lambda D e^{-\lambda t} - kC \quad (4)$$

and pharmacologists consider this to be a one-compartment model with time-dependent rate of administration. The solution is

$$C(t) = \frac{\lambda D}{\lambda - k} (e^{-kt} - e^{-\lambda t}) \quad (5)$$

and the plasma concentration is given by

$$c = C/V$$

where V can be found from an intravenous injection experiment. The duration of action of the dose (during which c exceeds the minimum therapeutic level) can be computed (using Newton's method for example) from equations (4) and (5).

Case (v)

The most common form of oral drug-administration program consists of a priming dose followed by a sequence of smaller maintenance doses. The problem is most easily formulated as a sequence of initial-value problems, as in case (iii). Let $C_n(t)$ and $G_n(t)$ denote the amounts of drug in the body and gastro-intestinal tract, respectively, during the interval $((n-1)\tau, n\tau)$. Let D and E denote the amounts of the priming and maintenance doses, respectively. The sequence of initial-value problems is:

$$\frac{dG_n}{dt} = -\lambda G_n, \quad 0 < t < \tau$$

$$\frac{dC_n}{dt} = \lambda G_n - kC_n$$

$$G_n(0) = G_{n-1}(\tau) + E$$

$$C_n(0) = C_{n-1}(\tau)$$

$$G_1(0) = D$$

$$C_1(0) = 0$$

The expression for C_n is complicated and is best computed recursively on a computer. A plateau also occurs in this case and it is easily shown that:

$$C_\infty(t) = \lim_{n \rightarrow \infty} C_n(t) = \frac{\lambda D}{\lambda - k} \left\{ \frac{e^{-kt}}{1 - e^{-k\tau}} - \frac{e^{-\lambda t}}{1 - e^{-\lambda\tau}} \right\}$$

which oscillates between the lower limit $C_\infty(0) = C_\infty(\tau)$ and the upper limit $C_\infty(T)$ where

$$\tau = \frac{1}{\lambda - k} \ln \left\{ \frac{\lambda(1 - e^{-k\tau})}{k(1 - e^{-\lambda\tau})} \right\}$$

The effect of the priming dose is transient and serves only to access the plateau level rapidly. An example, indicating the evolution of the drug level, is given in Figure 4.

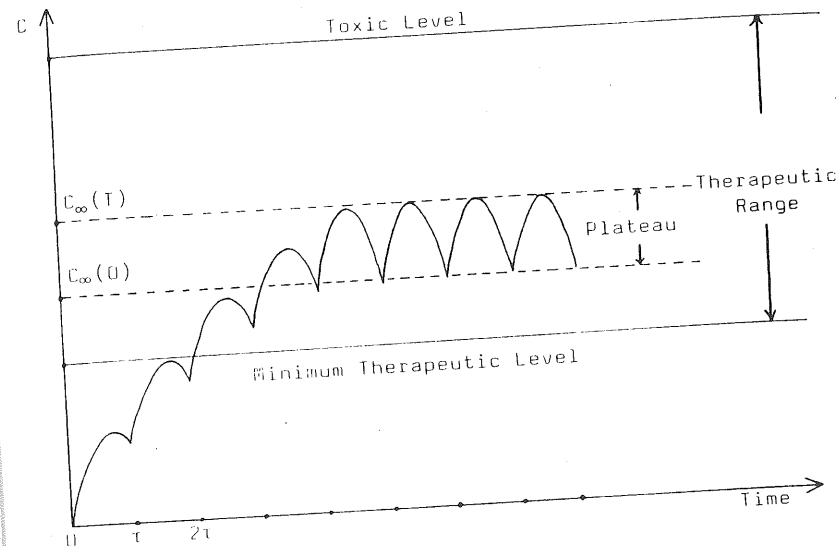


FIGURE 4

The design of the administration program is more complex in this case and requires the use of numerical methods. It is usually further complicated when time intervals between doses are not constant (e.g. during the night).

Multiple-Compartment Models

The one-compartment model is applicable to drugs that distribute very rapidly throughout the body. Many drugs, however, have a distributive phase that lasts for hours or even

days. For such drugs, plasma concentration curves (following intravenous injection) typically display a rapid initial decay [13] as indicated by the X-curve in Figure 5. All multi-compartment models are assumed to contain the blood in a single compartment (known as the *central* compartment). All other compartments are termed *peripheral*. The question of the nature of elimination of drug is important in the construction of model equations. In most cases it is reasonable to assume that elimination takes place from the central compartment alone since it is likely that the highly vascular liver and kidney are contained therein. Other elimination pathways are usually negligible [15]. The equations for the two-compartment model are therefore given by (in the case of intravenous injection):

$$\frac{dX}{dt} = k_1 Y - k_2 X - k_0 X$$

$$\frac{dY}{dt} = -k_1 Y + k_2 X$$

$$X(0) = D$$

$$Y(0) = 0$$

where X and Y denote the *amounts* of drug in the central and peripheral compartments, respectively. The term involving k_0 represents elimination from the central compartment and the magnitudes and signs of the system matrix are based on mass-balance principles. The link with reality is again obtained by postulating that the blood-plasma concentration, c , is proportional to X:

$$c = X/V$$

Data analysis will yield

$$c = Ae^{-\alpha t} + Be^{-\beta t}$$

It is then possible to solve (uniquely) for the coefficients k_0 , k_1 , k_2 and V in terms of A, B, α and β [13]. This would

not be possible if the system matrix was assumed to be a general 2x2 matrix. The value of V obtained in experiments is usually greater than 3 litres which again indicates an early rapid transfer of drug out of plasma into the remainder of the central compartment [13]. Typical curves for X and Y are shown in Figure 5.

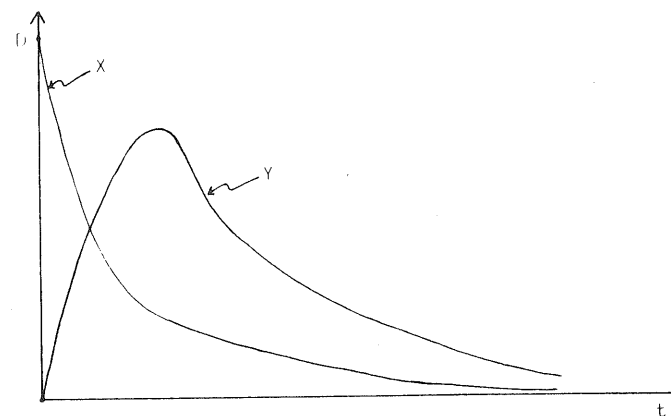


FIGURE 5

If $\alpha \gg \beta$ the amount of drug in the peripheral compartment will peak early and the subsequent decay in both compartments will rapidly become mono-exponential. Since the number of compartments is unknown, *a priori*, it is important to sample plasma levels frequently in the early stage. The single intravenous injection data may be used to design infusion and multiple-injection programs, as in the one-compartment case. Similar considerations apply to oral-dosing programs.

Three-compartment models have been developed for a number of important drugs (e.g. thiopental - a short-acting and widely-used anaesthetic [17] and d-tubocurarine - a neuromuscular blocking agent [18]). There are many possible models that could produce tri-exponential behaviour. It is usually

assumed that no direct drug transfer occurs between the peripheral compartments in addition to the central elimination assumption [13].

If the drug receptors were in a peripheral compartment the relationship between plasma-level and effect would be less than obvious. If it were true that drug concentrations at various sites in a peripheral compartment were proportional to the amount of drug in that compartment it would be possible to relate the pharmacologic effect to the peripheral compartment drug level. Such a relationship was found for the hallucinogenic drug lysergic acid diethylamide (LSD) in a study on five human subjects but further studies are probably warranted to establish the model [18]. It is also known that brain levels of the anaesthetic γ -hydroxybutyric acid in rats are the same when animals fall asleep and when they awaken while the corresponding plasma levels are quite different [19]. However, the author is unaware of any study of this drug that would substantiate a multi-compartment model.

It is possible that therapeutic and toxic effects could occur in different compartments. Such a situation could give rise to interesting constrained optimization problems in the design of administration programs where the objective would be to maximise the therapeutic effect while minimizing, in some appropriate clinical sense, the toxic effects [20].

Non-Linear Models and Problems

The biotransformation and renal tubular secretion pathways of drug elimination are mediated by enzymes (usually proteins). The simplest drug-enzyme reaction is that in which drug and enzyme molecules react to produce a drug-enzyme complex which then dissociates to produce a metabolite (inactive) and the original enzyme. The enzyme concentration is usually very small in comparison with that of drugs. The same enzyme molecules react repeatedly with the drug molecules to gradually

lower the concentration of drug. The reaction is governed by a system of two first-order non-linear differential equations, which can be approximated using a singular-perturbation procedure to yield the single equation:

$$\frac{dc}{dt} = \frac{-Vc}{K + c} \quad (6)$$

for the drug concentration, c , where V and K are constants [6,21]. The solution of (6) is a uniform asymptotic expansion of the true solution with error of order E/D , where E and D are the initial enzyme and drug concentrations respectively. Equation (6) would be valid *in vitro* but it could hardly be expected to model the evolution of plasma concentration *in vivo* (in the case in which biotransformation is the main elimination pathway). However a number of drugs have plasma decay curves which agree qualitatively with the solution of (6). If $c \gg K$ the decay rate is approximately constant (the process is said to *saturate*) and if $c \ll K$ the decay is approximately exponential. The latter result suggests that drugs, which behave in accordance with the earlier one-compartment model and are eliminated by biotransformation, are present in concentrations well below the saturation level. The major anti-epileptic drug, phenytoin, conforms to model (6) [22,23]. Other examples are ethanol (alcohol) [24] and aspirin [7]. In the case of alcohol, $c \gg K$, in the 'therapeutic' range. The therapeutic range for phenytoin is quite narrow (10 - 20 micrograms/millilitre). The minimum desired effect is elimination of epileptic seizures. Toxic effects include ataxia and psychological disturbances. Further the fully-nonlinear behaviour occurs in the therapeutic range. There is a further effect which makes clinical treatment difficult. The drug is usually given orally (one tablet per day) for chronic treatment. Consider, for simplicity, the case of continuous intravenous infusion for which the equations are:

$$\frac{dc}{dt} = r - \frac{Vc}{K + c}, \quad t > 0$$

$$c(0) = 0$$

It may easily be shown that c increases up to the plateau level

$$c = \frac{rK}{V - r}$$

if $r < V$, but that the concentration increases without bound if $r > V$. Thus, the plateau is unstable if r is close to V . This is usually the case in practice. Similar considerations apply in the case of multiple oral-dosing. Further, the design of administration programs for this drug requires numerical integration techniques. Adjustments of dosage would also require care and this model is very useful in such a situation [18,22].

A phenomenon, known by the misleading name of *protein binding*, is considered to be of paramount importance in Pharmacology [15]. All drugs undergo *reversible* chemical reactions with proteins (mainly albumin) in the blood. Proteins and drug-protein complexes cannot diffuse through the cells of arterial and venous walls. Hence, protein binding will influence the distribution of drug. The ratio of the concentration of total drug and that of drug-protein complex appears to be constant over the therapeutic range for the majority of drugs under most conditions. A number of drugs (e.g. phenylbutazone - a powerful anti-inflammatory agent, clofibrate - an inhibitor of cholesterol synthesis, naproxen - used to treat gout) exhibit non-linear effects, in the therapeutic range, which are believed to be attributable to protein-binding. Reduced plasma levels of proteins (which occur in some diseased states and in the newborn) may also cause such effects to be manifested. The subject is surrounded by controversy in the pharmacological literature. Both compartmental [25,26] and physiological [27] models have been considered. Part of the contr-

oversy concerns the effect of protein-binding on elimination and the physiological models deal only with liver or renal function and have not been incorporated into whole-body models. The governing differential equations are non-linear in both cases and have been solved using numerical integration routines. The simplest models contain at least five parameters which makes it difficult to extract a qualitative picture from numerical solutions. The author has recently carried out a singular-perturbation analysis of some of these models which identifies conditions under which the phenomenon may have importance [28]. There seems to be no consensus yet on the status of the various models.

A related, but more complex, problem is that of *drug-interactions*. It is frequently necessary to administer a number of drugs simultaneously. Unfortunately, most drugs interact in a non-linear fashion when co-administered. A number of possible mechanisms of interactive behaviour are understood but are unquantified [7,15]. Mathematical modelling in this area appears to be non-existent.

Some drugs exert their effects, not on the human being directly, but on an invading population of bacteria (which can cause many serious diseases such as tuberculosis and bubonic plague). Compartmental models have been developed for most antibiotics but a satisfactory understanding of the relationship between the nature of the administration program and effect may require incorporation of bacterial population dynamics [14]. An iatrogenic phenomenon known as *superinfection* may occur if a drug disturbs the population balance of micro-organisms in the body. Such a disturbance may allow a species to grow to a size at which it becomes pathogenic. Penicillin, for example, is fatal in guinea pigs for this reason [29]. Mathematical modelling appears not to have been attempted in this area.

Conclusion

Mathematical modelling has contributed significantly to the understanding of many drugs and to rational and safe drug therapy. The subject is recognized as being of increasing importance in Clinical Pharmacology. There are now two medical journals on the subject [30,31]. A need for more physiological modelling has been expressed in recent reviews in Clinical Pharmacology [32,33] which indicates a perception that more advanced mathematical modelling is required for continued progress. There would thus seem to be considerable scope for collaboration between mathematicians and pharmacologists in this area.

The value of statistics has long been appreciated in medical education. A lot of insight can be gained in Pharmacokinetics with a modest background in calculus and differential equations. When one considers the extent of drug therapy in the treatment of disease there seems to be a strong case for the inclusion of mathematical modelling in the medical curriculum.

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MATHEMATICS AND COMPUTING - THE GREAT DIVORCE

Timothy Murphy

Software is a branch of logic, and therefore of mathematics.

A computer program is a sequence of propositions, precisely analogous to a proof in geometry or the differential calculus. It is true that these propositions are metaphors for changes in the internal state of a real or imaginary machine; but equally, a theorem in geometry (say) carries a metaphorical allusion to a real or imagined world.

Computing is the child of mathematics. The fore-runners of computing - Pascal, Babbage, von Neumann - were all drawn from a mathematical background; as were the founders of theoretical computing - Gödel, Turing, Church.

(How amused, incidentally, these patriarchs would have been to hear themselves described as "computer scientists". It would have seemed as absurd to them as speaking of Gauss as a "number scientist".)

The term "science", as has often been remarked, is only attached to a subject if its proponents entertain secret doubts about its scientific respectability. Books are written on "Astrological Science"; no-one speaks of "Astronomical Science" - the word "Astronomy" being, so to speak, its own guarantee. Sociologists are scientists; social scientists, in general, are not. *Lux quod non lucendo.*)

What went wrong? What caused the Great Divorce? And is it too late for a reconciliation?

Certainly the linking of software and hardware, under the name of computer science, cannot be regarded as more than a temporary convenience. The connection between, say, the complexity of algorithms and the technique for depositing ions in a semiconductor is tenuous, to say the least; and it is just as absurd to require the student of one to study the other as it would be to link algebraic geometry and surveying, or to expect a student of fluid mechanics to be *au fait* with the latest advances in plumbing.

There is no doubt that the fault lay, historically, with Mathematics. Most mathematicians affected - and many still do affect - to despise computers and all their works. Only a Turing could see past the feeble machines of his day to the profound theoretical concepts beyond.

Any yet. When all that is said, it remains puzzling why the concept of computability, for example, has not been absorbed into the main-stream of mathematics. After all, it affects every branch of the subject. Most mathematicians are aware, however dimly, that the problems they are sweating over may in fact be insoluble. The ghost of Gödel haunts us all, however far our interests may lie from the computer.

There may be a purely technical explanation (at least in part) why theoretical computing remains so much a "foreign body" in the corpus of mathematics.

Computing is necessarily concerned with "partial" functions

$$f: X \rightarrow Y,$$

ie functions which are defined everywhere on their domain X - and which are therefore not *functions* in the strict mathematical sense. For the computer program may never finish. And what is more, Turing has shown that one cannot simply set, say, $f(x) = 0$ in this case without destroying the computable character of f ; for there is no way (in general) to determine

whether or not the computation will end in a particular case, other than by carrying it out.

Although this use of partial functions may seem a small matter, it did nevertheless set computing on a course away from the mathematical consensus, which was striking a more and more puritanical attitude towards the singularities of functions. If Cantor's insights were to be exploited to their full potential, it seemed essential that the concepts of set and map (or function) should be rigorously defined, and restricted to those definitions. The careless schoolboy notion of function which regarded $\cos(x)$ and $\log(x)$ as siblings, simply caused too many headaches when complex constructs were at issue.

Scott's Data Types

If that is so - and it may be an oversimplification, or at any rate of less significance than is suggested here - then Scott's concept of a Data Type may mark an important step towards reconciliation between Mathematics and Computing [1].

For Scott has, in effect, supplied computing with its own category - the category of Data Types - which is, one might say, the ultimate accolade of mathematical respectability.

We can take Scott's concept in two steps.

Firstly, he sidesteps the partiality of $f: X \rightarrow Y$ by passing to the extended function

$$F: 2^X \rightarrow 2^Y$$

where 2^X denotes the set of subsets of X , and we extend f to F by setting

$$F(S) = \{f(x) : x \in S\}$$

for each subset $S \in X$. If $f(x)$ is undefined for a particular $x \in S$ then no value for $f(x)$ is included on the right-hand side; and in particular $f(\{\}) = \emptyset$.

This has the incidental but important advantage of allowing *non-deterministic* computations. For if we allow that

$$f(x) = F(\{x\})$$

may be empty, then we may equally well go in the opposite direction, by allowing $f(x)$ to be multi-valued, i.e. by allowing

$$|F(\{x\})| > 1,$$

where $|S|$ denotes the cardinal number of S .

This fits in very neatly with the recent preoccupation of both theoretical and practical computing with parallel processing, and the interaction of different computers linked in some way, e.g. across a network. For in that case it is impractical to suppose that the behaviour of a particular computer is entirely predictable. Even if we knew what message would reach the computer, we cannot be certain of the exact instant at which the message will arrive; and so an element of indeterminacy is necessarily injected into our calculations.

However, Scott noted - and this was the second step towards his concept of a data type - that it would go too far to allow *any* function

$$F: 2^X \rightarrow 2^Y.$$

There are simply too many of them. He observed that the functions F arising from (possibly partial) functions $f: X \rightarrow Y$ are order-preserving, i.e.

$$S \subset T \Rightarrow F(S) \subset F(T).$$

In fact they possess a stronger property:

$$S = \cup S_i \Rightarrow F(S) = \cup F(S_i).$$

The neatest way of expressing this condition - and one that leads naturally towards generalization - is to impose a topology on 2^X , and to restrict F to *continuous* functions

$$F: 2^X \rightarrow 2^Y.$$

The appropriate topology turns out to be the Tychonoff product topology on 2^X , where however we take the so-called Sierpinski topology on the factor spaces 2^1 . To be precise, if we take the set $B = \{1, T\}$ in place of 2^1 then the topology is that defined by taking as open sets the 3 subsets

$$0, \{T\}, B.$$

A Data Type D , then, is a topological space, albeit of a rather special kind. It turns out that "special kind" can be most simply defined in the starkest of mathematical terms: a data type D is an *injective object* in the category of T -spaces.

That is a convenient point to break the story. Scott has presented computing with its own category, the category of Data Types (and continuous maps). This gives a new unity and discipline to what has tended to be a fragmentary and disparate subject. Such diverse subjects as Computability, Coding Theory, Algorithmic Complexity and Semantics can at least be discussed in a common language; and that language is one familiar to mathematicians at least.

Computing has come home; it is once again in the mainstream of mathematical thought.

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"CURRENT MATHEMATICAL PUBLICATIONS" AS A RESEARCH TOOL IN THE MATHEMATICAL SCIENCES

Anthony Karel Seda

By the end of this calendar year (1984), it is expected that some 46,000 books and papers will have been reviewed in *Zentralblatt für Mathematik*. The corresponding number for *Mathematical Reviews* is about 40,000. Put in a more accessible way, these figures mean that a research worker in *Partial Differential Equations* (Mathematical Review Code 35), *Numerical Analysis* (65), *Statistics* (62) or *Global Analysis* (58) is faced with a flow of papers and books of about, or in excess of, 50* per week. A worker in *Computer Science* (68), *Functional Analysis* (46), *Operator Theory* (47), *Mathematical Logic and Foundations* (03) or *Combinatorics* (05) is faced with a flow of about 30* papers and books per week. Necessarily, therefore, time being limited, such a worker will be highly selective with regard to his or her choice of material for in-depth study. But making such a choice requires an awareness of the current literature, or "current awareness" in library jargon. Now there are several sources of current awareness, but only one offering anything significantly beyond current awareness, and this is *Current Mathematical Publications* (CMP) published by the American Mathematical Society.

Conversations with several of my colleagues and others have, surprisingly, led me to believe that the usefulness of CMP is not as widely appreciated as it might be. Certainly its value purely from the current awareness point of view is clearly recognised, but it has certain other merits which do not seem to be common knowledge. In this note I want to comment, as a regular and fairly longstanding user, on the effect-

* These figures are based on a simple count of entries appearing in this year's issues of CMP, taking into account cross-referencing and secondary classification.

iveness and cost effectiveness of CMP either in one's own personal library or institutional library.

Coverage

This is pretty well complete. Essentially all material which will be reviewed in Mathematical Reviews is listed in CMP; effectively this means nearly all literature published anywhere in the world. Issues of CMP appear, at present, every three weeks, and typically a given issue carries material dated one month to four or five months prior to the date of that issue. In practice this usually means that my copy of CMP arrives at about the same time as the corresponding journals become available in the library at UCC. But, of course, UCC's library is by no means comprehensive in its coverage.

Accessing Information

All the papers listed in CMP are classified in accordance with the Mathematics Subject Classification of Mathematical Reviews (1980 classification at present). This system is also used by Zentralblatt, and so CMP can easily be used in conjunction with the two major reviewing journals. In addition, each issue of CMP carries an author index and a key index of colloquia, proceedings of symposia, special collections etc. Every half-year complete author and key indexes for that period are included. It is therefore relatively simple to track down any author's work, find references on a given topic or check whether or not preprints (other people's!) have appeared in print. Incidentally, a complete run of five years of CMP occupies considerably less than one metre of shelf space.

Geographical Coding

In an on-going process, institutions are being assigned a geographical code by the American Mathematical Society on a worldwide basis. Each such code contains the postal address of the corresponding institution. From September of last year entries in CMP have carried this code next to the author's name, provided the author's institution has been coded of course. This makes it, or at least will make it in time, a simple matter to send a card request to such an author for off-prints. Granted a little cooperation on the part of authors, this procedure should be cheaper than, and just as effective as, inter-library loan services.

Cost Effectiveness

Whilst Mathfile (the on-line database of Mathematical Reviews) may be inexpensive in the USA, in Ireland its use is fairly expensive. For example, a single journal search using keywords and Mathematical Review codes, and printing, say, about 100 references can cost up to IR£50. This is not all that much less than the cost of an individual's subscription to CMP for one year at current prices. Given the present strength of the US dollar, CMP itself is not particularly cheap but it is relatively so if used in the ways discussed above, and in my view is fully cost effective. I am not of course suggesting that CMP is a viable alternative to Mathfile, if one is seeking hundreds of references, but for the purposes already indicated it is invaluable.

There is one other feature of CMP worth noting. If used regularly, it becomes apparent, to some extent, which subject areas of mathematics are most active and where one's own research specialisation stands in relation to others. This can be quite revealing. For example, on more than one occasion I have heard it said, usually by algebraic topologists, that General Topology is no longer interesting and is practically dead. Now I am not going to comment on the first of these

criticisms, but a glance at a few copies of CMP quickly dispels the second. Currently there is some three to four times as much material, at least, being published in General Topology (54) as in Algebraic Topology (55). So it certainly is not dead - or maybe it just refuses to lie down!

Sophisticated information retrieval systems are certainly a great boon to those researchers to whom they are readily and cheaply available. Their existence and use by some in technologically advanced countries, however, makes it more, rather than less, incumbent on all workers in a competitive environment to be aware by some means or other of what is going on. As in the case of the law of the land, ignorance is not a defensible position. In these days of zero, or even negative, library growth, CMP can go a considerable way towards compensating for funding deficiencies in libraries. If used flexibly, it is an effective and cost effective tool for use by anyone engaged in research in the mathematical sciences.

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MATHEMATICAL EDUCATION

COMPUTERS AND THE TEACHING OF MATHEMATICS TO TECHNICIANS

Paul Barry

This article is a personal reflection on the direction that my teaching, and that of several of my colleagues, has taken in the last few years at Waterford Regional Technical College. The majority of my teaching has been to engineering technicians, and hence a certain bias will not go unnoticed in what follows. Oddly enough, the use of computers in the teaching of mathematics is perhaps not so entrenched in Science or Programming courses as it is on the Engineering side. This is partly due to the syllabi, and partly due to the fact that it has not been seen as part of the mathematics lecturer's role that he/she should get involved on the computing side of things. Nevertheless, given the nature of such courses, it would seem that the sort of mathematics being taught should be very amenable to computer implementation and investigation. My own impression is that personal preferences have been the deciding factor. This is perhaps borne out by the fact that the Physics syllabus on many Science and Programming courses is backed up by a computer-assisted tutorial system which has proved its worth over a number of years.

Before the establishment of the RTCs, the standard level of 'mathematics' that was the daily diet of technical students amounted to little more than mensuration. The fact that nowadays the minimum mathematics qualification is a Pass D in the Leaving Certificate has allowed educators to raise their standards. The presence of professional and degree courses in several RTCs has led to a much more sophisticated type of mathematics being available, and this is certainly bound to have its effect on the Certificate Level courses.

However, the challenge that most lecturers face can still seem daunting. On many technician courses, the majority of students may only have the minimum requirements in mathematics - a Pass D. The fact that a number of these may well have been E's before statistical adjustment means that basic numeracy cannot be taken for granted. To bring a class of such students to a level somewhat beyond Leaving Certificate Honours level in one year can sometimes seem an awesome task especially when many of the topics are so new to them. Experience shows that the first year is the 'make or break' year - students who pass into the second year of their studies seem quite well able to handle levels of mathematical sophistication, often higher than their Leaving Certificate results would seem to indicate possible. (This in itself would seem to raise questions about the way that school mathematics is taught.)

Fortunately, the development and expansion of the RTCs has been paralleled by the increasing use of computers in education. Waterford RTC has well established computer facilities - a VAX 11/780 and micros - which were originally the domain of the courses in computer programming. However, early on, the need for most technicians to have some knowledge of programming was perceived, and hence most courses had a simple programming/computer architecture module attached. Invariably, it was the mathematics lecturer who was charged with these modules. It was natural that these lecturers should seek inspiration for their programming exercises in the mathematics that they were teaching. Hence mathematics was a source of exercises for programming. Gradually, however, the emphasis is changing, and increasingly the computer is being seen as a vehicle for the teaching of mathematics. Computer modelling of mathematics is becoming the bridge that allows the weaker student to come to grips with the concepts that heretofore left him/her baffled. An example of this is the limit process, which has normally been assumed to be beyond the grasp of the majority of first year technician

students. The computer investigation of a few well-behaved limits soon clears that up. The 'delight' on the faces of students as the numbers whizz up the screen is a quiet revolution that is taking place at this level. The limiting process becomes tangible, or at least acceptable.

However, the selling of mathematical concepts is still hard when numerical output is all that is available. Once again, few technician students can make the leap from a set of numbers to a picture of what is going on - especially when a mathematician's superior analogue powers are perhaps what makes him what he is. Fortunately, computer technology (and its falling prices!) provides an answer in the guise of computer graphics. Functions come alive on the graphics screen. Suitably written software allows functions to be added, multiplied, translated, integrated, convoluted - you name it - directly in front of the student. How many mathematical undergraduates ten years ago could 'play' with Bessel functions the way a second year Electronics technician student can do now? We are seeing the Fast Fourier Transform become a friend, multiple regression become routine, stiffness methods a practicality.

The fact that the computer can come up with the 'right' answers once the student has mastered the concepts, and the increasing availability of suitable software (homegrown or bought), poses a profound question on the direction and future of technician mathematics, and by extension, the role of the teacher in this context. Is it more important to be able to differentiate $\exp(-x)\sin(x)\cos(x)$ than to describe what differentiation is? Ideally both are equally important, indeed it can be argued that you cannot have one without the other. However, I have seen classes 'calculate' such derivatives with consummate ease, and yet only a minority could give a satisfactory explanation of what they were doing, and its possible uses. My own fault, of course. Nor do I claim that computers are the answer. More time, better explanations,

better motivation are part of one. But in the absence of at least two of these, the computer surely can help. It frees us to sell the concepts, and lets the students get on with the exploratory modelling that reinforces them. What should we then examine? And what happens when the staff whose courses we are serving find that they have a new breed of student, with vast amounts of (computer-based) mathematical firepower that some at least are only too anxious to use?

At Waterford RTC, the Engineering Department has made a concerted effort to integrate the computer with all aspects of technician learning. The mathematics lecturers who service this Department have been more or less eager to help. The benefits are many - concepts can be taught rather than sterile methods, self study takes on a new meaning, algorithms become important - in short, the student begins to think about what he/she is doing. Students acquire a taste for mathematical modelling, with the inculcation of a 'what if' mentality being reinforced by immediate response in many cases. New functions are investigated with the same (almost!) enthusiasm as new equipment.

A welcome consequence which may not be immediately predictable is that overall numeracy increases. I hesitate to analyse the reasons too closely, but presumably the student feels more in control, and more motivated to get things right. The investigation of an algorithm leads to clearer thinking than the brainless application of a 'method'. The computer interface is more encouraging perhaps than the human one, with all the remembered connotations of correction and humiliation that can go with it. The 'inhumanness' of computer messages leaves them free of judgement, and thus more acceptable and more likely to instigate a fruitful line of thought. Another, more mundane reason is that if the computer is used effectively, concepts are grasped more quickly and more class time can be spent on problems. The computer can draw the sine graph far quicker than I can, and with suitable animation

techniques, phasors can be explained in far less time than a normal chalk and talk session takes. Hence one is left with more time to do the calculations.

Purists, of course, may argue differently - but then, most purists are involved in the formation of pure mathematicians, or at least they should be. It may be argued that nobody who is not familiar with the concrete implementation of their discipline should be let near those who will in fact spend most of their lives on the 'implementing' side. The fact that in universities there are 'pure mathematicians' teaching 'maths methods' courses has more to do with the constraints of timetabling and the need to earn than any pedagogical argument in its favour, and presumably only the greater intelligence of university students buffers them against the damage that this can do. Indeed, voices raised in this Newsletter would seem to indicate that this 'greater intelligence' can no longer seem to be taken for granted, and certain university courses are seeing the same level of student (as regards mathematics, at least) as is the norm in the RTCs. Fortunately, not all purists are Luddites, this being borne out once again by previous articles in this Newsletter. The beneficial effects that take place when the 'pure' meets the 'applied' may then be seen - the software gets better, and the teaching more efficient. Paradoxically, the computer investigation of mathematical methods may yet be our best bet in giving mathematics a 'human face' to those who have been unable to see it.

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BOOK REVIEWS

"MODERN DIMENSION THEORY" (Revised and Extended Edition)

By *J. Nagata*

Published by *Heddermann Verlag*, Berlin, 1983, DM68, pp. 284.
ISBN 3-88538-002-1

From the publisher's description:

"This book is a completely revised and extended edition of "Modern Dimension Theory" published in 1964. It succeeds the old edition in spirit and objective and gives a brief account of modern dimension theory in its present state. Although the book begins with elementary concepts not requiring any knowledge beyond elementary general topology it is not only of interest for the beginner but also for the working mathematician in particular because of its survey character and its rather complete bibliography of approx. 500 titles.

The developments during the last twenty years have been so remarkable and extensive that a thorough revision and a large number of additions had to be made. Especially the chapters on the dimension of non-metrizable spaces and of infinite-dimensional spaces had to be wholly rewritten. New sections on the Pontryagin-Schnirelman Theorem, on "Dimension and Ring", and on "Dimension and Metric Function" were included, to mention just a few. Furthermore, new characterization theorems of dimension are presented.

The reader will easily get a good bird's-eye view of the classical dimension theory, more recent approaches and latest developments, and is thus provided with a starting point for his own research. The working mathematician will appreciate the comprehensive treatment of the subject and the bibliography which together make the book an indispensable source."

"WILLIAM ROWAN HAMILTON. PORTRAIT OF A PRODIGY"

By *Sean O'Donnell*

Published by *Boole Press*, Dublin, 1983, \$24.95, pp. xvi + 224,
ISBN 0-906783-06-2

This latest biography of William Rowan Hamilton deals much more with the man than with his work, but claims that in the end 'even the most recondite achievements' will seem less mysterious in the light of the study of Hamilton's personal development. Now the portrait of an artist as a man may sometimes throw some light on the way in which he came to exercise his imagination; but in the case of scientists, and of mathematicians especially, the roots of personality and invention are far too mysteriously entwined for us to disentangle them with the imperfect insights presently at our command. Whenever the evidence is available, it is good to have biographies of important mathematicians simply because we want to know what kinds of people our heroes were and what kinds of lives they led. Such a biography need not dwell in detail on the subject's achievements, so long as the reader learns enough of these to obtain a true measure of the intellectual stature of the subject, and so long as they are portrayed as vital components of his life (e.g. Mr Reid's biography of Hilbert). As T.L. Hankins writes in what Mr O'Donnell acknowledges as the 'definitive life' of Hamilton (published 1980) "it will not do to write the life of a mathematician excluding his mathematics". Unfortunately, that is what Mr O'Donnell has done - not intentionally, for he attempts to summarize Hamilton's scientific work; but inevitably, for he does not understand any of the mathematics well enough to convey its importance in any way other than by quoting from time to time what others have said of it. Probably Mr O'Donnell wanted this account of Ireland's greatest scientist to reach a much wider audience than did the monumental 'Life' of Graves or the scholarly book of Hankins, and of course he may succeed, for he has a lively style of writing; but if the object was to make

more people aware of Hamilton's greatness, as all readers of this journal would wish, then he has not succeeded. Indeed, there is a sense in which he himself progressively seems to lose enthusiasm for the task he set himself. As the story unfolds, Mr O'Donnell becomes increasingly preoccupied with the defects of Hamilton's personality and chides him regularly, like a Victorian moralist, for having accomplished less than he might have done had he been more practical, punctual, abstemious, modest, sparing in language and decisive in action (also, Mr O'Donnell would have had Hamilton be a better Irishman!). Finally, if Hamilton's mathematics receives short shrift, Hamilton's personality in the end really does little better. A few psychological clichés applied to this or that episode that stand for 'modern' insights do not amount to a serious in-depth study of one of the greatest scientists of the nineteenth century. What survives the moralistic tirades is only a limp cardboard figure.

Mr O'Donnell is best on Hamilton's origins and family background - he makes a good case for the conjecture that Archibald Hamilton Rowan, the wealthy political rebel and Hamilton's godfather, was actually Hamilton's father; and he is good on Hamilton's up-bringing by his devoted Uncle James, the curate of Trim. There is also much of interest in the early chapters on life and education in the Ireland of Hamilton's youth. On the other hand, Mr O'Donnell's idiosyncratic stance manifests itself already at the very beginning: he is excited to dispose of the 'myth' of Hamilton's linguistic powers (as if anyone had ever taken this at its face value!) and refers to this discovery several times. From now on we shall have to be content with the knowledge that, aged thirteen, Hamilton could read easily 'only' in Latin, Greek and Hebrew. Also, we were invited to regret that Uncle James did not think to add Gaelic or Irish to young William's repertoire of languages! We learn of other ways in which Uncle James might have strayed with advantage from the classics-based education he imparted with such ardour, but we learn nothing new about

what must have been happening - the emergence of mathematical genius. And perhaps that's how it will always be regardless of who tries; we are left to wonder, without hope of ever being enlightened, what confluence of ideas and experiences of the young Hamilton could have brought him to the point where, as Hankins reports, on one page of manuscript preserved from student days, Hamilton writes down Newton's laws of motion in an obvious effort to memorize them for an exam, and on the same page he sketches his path-breaking ideas on geometric optics!

And from Hamilton's student days onwards, I find Hankins' description of his character and activities always more mature, serious and balanced. In the matter of Hamilton's drinking habits, Hankins is at once more explicit than Mr O'Donnell and more accurate in judging their impact; Hamilton's phenomenal computational ability never deserted him. When Father Richard Ingram checked the manuscript on Icosian cycles (paper LVIII on p. 623 of our edition of Vol. III of Hamilton's works), dated 1863, he found not a single error! In the case of a mathematician, that's what being able to hold one's drink means! Or take Hamilton's attitude to the Great Famine of '45-'48: Hankins lets Hamilton speak for himself and is altogether more informative on Hamilton in relation to the poverty then prevalent in Ireland; whereas Mr O'Donnell attacks Hamilton in strident terms that would be appropriate only at election hustings.

I could go on, and I could go on much longer in the context of Mr O'Donnell's descriptions of Hamilton's mathematics (but do look at the extraordinary disquisition on p. 146 on the significance of quaternions, ending with the moralist's injunction that 'to be first does not imply being best ...!'). However, perhaps enough has been said. Perhaps the greatest service I can render Mr O'Donnell is to urge readers to buy the book and themselves enjoy finding their own disagreements!

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"COMPLEMENTARY PIVOTING ON A PSEUDOMANIFOLD STRUCTURE WITH APPLICATIONS TO THE DECISION SCIENCES"

By F.J. Gould, Graduate School of Business, University of Chicago, and J.W. Tolle, Dept of Mathematics, University of North Carolina at Chapel Hill.

Published by Hederemann Verlag, Berlin, 1983, DM58, 202 pp.
ISBN 3-88538-402-7.

In these heady days when the influence of computing on mathematics waxes ever more strongly, a natural development is the enhanced status of constructive existence proofs. For example, the standard inf/sup argument used to prove the intermediate value theorem could well be supplanted by the slightly longer (but constructive!) proof which repeatedly bisects the interval under consideration. However, it is often difficult to obtain a constructive proof of a known result. Brouwer [2] published the proof of his famous fixed point theorem in 1912, and several alternative proofs appeared in later years, but not until 1967 did Scarf [5] give an explicitly constructive proof (which enabled one to find "approximate fixed points" z , in the sense that $\|f(z)-z\|_\infty$ is small). Incidentally, Hirsch [3] in 1963 published a proof of (a result equivalent to) the Brouwer theorem, the constructive nature of which was only noticed in 1976 by Kellogg, Li and Yorke [4].

Scarf's paper marks a watershed in the numerical solution of nondifferentiable nonlinear systems of algebraic equations. Its basic algorithmic procedure, that of *complementary pivoting*, was the inspiration for an explosion of research activity in

the 1970s. A description of this procedure follows.

First, an analogy. Consider a house having one entrance. The house consists of a finite number of rooms. All rooms have 0, 1 or 2 doors. All doors link exactly two rooms (the exterior of the house is regarded as a room). Then if one enters the house and obeys the rule that one cannot enter and leave a room through the same door, it is not difficult to see that one's path must terminate in a room inside the house having one door.

Let's translate the analogy into mathematics (I shall cut some technical corners here). Our house becomes a closed, bounded, simply connected subset D of R^n . Each room corresponds to an n -simplex (a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc.: an n -simplex is the closed convex hull of its $n+1$ extreme points or vertices). The set D is the union of these n -simplexes, and moreover the n -simplexes are required to fit together in a geometrically pleasing way: the intersection of each pair S_1, S_2 is either the empty set or an m -simplex, $0 \leq m \leq n$, whose vertices are vertices both of S_1 and of S_2 . Every n -simplex has $n+1$ $(n-1)$ -dimensional faces which are themselves $(n-1)$ -simplexes. Certain of these faces will be designated as doors, in a way that corresponds to the analogy above.

To each vertex of each n -simplex we assign a *label* chosen from the set $T_n = \{0,1,\dots,n\}$. (The way in which this assignment is carried out depends on the nature of the problem being solved.) Thus each n -simplex S has a set of $n+1$ labels associated with it, which may include some repetitions. If this set equals T_n , we say that S is *completely labelled* (cl). Similarly associate a set of n labels with each $(n-1)$ -simplex S' which is a face of some n -simplex. We say that S' is *almost completely labelled* (acl) iff this set equals T_{n-1} .

Observation: if S' is an acl face of S , then either S is $c1$ or S has exactly one other acl face.

Suppose that we choose our labelling so that among all the $(n-1)$ -simplexes making up the boundary of D , exactly one is acl. Returning to our house analogy, this exceptional $(n-1)$ -simplex is our entrance. The acl faces in D are the doors in the house. All the conditions described in the analogy are now seen to hold (the Observation shows that each n -simplex has 0, 1, or 2 acl faces). We can therefore start from the unique acl face on the boundary of D and move (*pivot*) from one n -simplex to another through the acl faces until we terminate at a $c1$ n -simplex.

The labelling should be such that $c1$ n -simplexes yield approximate solutions of the problem under consideration. For example, if $f = (f_1, f_2, \dots, f_n): D \rightarrow \mathbb{R}^n$ is continuous and one wishes to solve $f(x) = (0, 0, \dots, 0)$, an appropriate labelling of each vertex z is

$$0 \text{ if } f_i(z) < 0 \text{ for } i = 1, 2, \dots, n \\ \min\{i: f_i(z) \geq 0\} \text{ otherwise.}$$

It is easy to show that with this labelling every point y in a $c1$ n -simplex S has $\|f(y)\|_\infty$ "small" (the smaller the diameter of S , the smaller the bound on $\|f(y)\|_\infty$). To ensure that exactly one acl $(n-1)$ -simplex lies on the boundary of D , one can for example add extra artificial n -simplexes to D .

This, in essence, is Scarf's algorithm. Many improvements to it have been put forward, and many other algorithms incorporating some form of complementary pivoting have been published since it first appeared.

The objectives of Gould and Tolle's book are (i) "to present a unified framework into which most of these algorithms can fit" and (ii) "to provide instructive examples of their

application". The book is easily divided into two halves reflecting these objectives: Chapters 1-4 cover (i) while Chapters 5-8 examine applications of complementary pivoting.

Although Chapters 1-4 do reach objective (i), the treatment is a little disappointing. It is a determined rigorous abstraction of the way in which our n -simplexes were assembled to form the set D (this generalization is called a *pseudomanifold*) followed by a description of the complementary pivoting algorithm as it operates on this structure. The style of these early chapters is adequate but unexciting. (I note that I was baffled by the "definition" of "basic solutions" on page 24!). The book's choice of examples leans towards operations research, in particular linear programming, and end-of-chapter exercises are good though perhaps too few. My main complaint is that the basic algorithm is not reached until page 70, at the end of Chapter 4. This is surely much too late, considering the publisher's claim that books in this series "introduce the reader to a field of mathematics". A novice should hardly be asked to wait this long to learn about what is, more or less, the algorithm of Scarf we described earlier. One can see the authors' dilemma, caught between the above claim and their own objective (i), but I feel that a page or two in Chapter 2 (where pseudomanifolds are defined) giving a sketch of the algorithm would have been a good investment. Alternatively, Chapter 3 ("examples and constructions of pseudomanifolds") and Chapter 4 ("complementary pivoting") could have been interchanged.

Happily, I can be much more positive about Chapters 5 to 8. These deal with the application of the complementary pivoting algorithm to linear complementarity theory, Brouwer and Kakutani fixed points, unconstrained nondifferentiable minimization, and nondifferentiable programming. This is mostly operations research material which many mathematicians never encounter. The presentation seems somewhat smoother than in the earlier chapters, perhaps because there is less

effort at abstraction. All in all it makes for interesting reading.

Merrill's significant extension of Scarf's algorithm is hidden in section 6.3, "fixed points of upper semicontinuous point-to-set mappings", and deserves a much more prominent place (it isn't even mentioned in the index, which is far too short and selective). Furthermore, there is no description of any of the several new algorithms developed since the mid-70s which now dominate the field; all that is offered is a list of references. This is difficult to justify, and greatly lessens the value of the book to current researchers.

For beginners, my advice is to look first at the attractive review article of Allgower and Georg [1], which is fairly up-to-date and discusses the implementation and efficiency of complementary pivoting algorithms as well as covering nicely much of the book's early material.

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(This review was written while the author was at Waterford Regional Technical College)

"FINITE FIELDS" (Encyclopaedia of Mathematics and its Applications, Volume 20)

By *Rudolf Lidl and Harald Niederreiter*

Published by *Addison-Wesley*, Reading Mass, 1983, Stg £57.80, pp. xx + 755. ISBN 0-201-13519-1

Recently, before breakfast one morning, I received a telephone call from a former student of mine who is now teaching at a Regional Technical College. He wanted to know if a certain polynomial of degree 16 was irreducible over $GF(2)$. This question was motivated by a project one of his students was hoping to carry out in coding theory. For me, this incident epitomises the sudden rush of interest from many sides in the classical theory of finite fields.

It is difficult to know where to begin describing the book in hand. Perhaps some raw statistics may serve to convey something of the scope of this massive work. It has 755 (+xx) pages in all handsomely bound. There is a comprehensive bibliography of 160 pages of detailed references to finite fields

from ABDULLAEV, I, through LAFFEY, T.J. to ZSIGMONDY, K. The author index has 15 pages of entries and there are 21 pages of very useful tables relating to such topics as irreducibility of specific polynomials over fields of small order. In addition there are 634 carefully worded exercises ranging from the pretty:

Let F be a field; if F^* is cyclic show that F is finite, through the purely routine

$$\text{Factor } x^5 + 3x^4 + 2x^3 - 6x^2 + 5 \text{ over } \mathbb{F}_{17}$$

to the difficult, of which there are many.

There are two features of this book that I particularly welcome. All sections, especially those introducing new or difficult material, are explained and clarified by an astonishing number of thoroughly worked examples of varying levels of difficulty. This means that although the book is pitched at a reasonably advanced level it is very suitable for students or those wishing to learn the subject for the first time. The second notable feature is that each chapter is followed by a section of pleasantly readable notes running to twenty pages of text in the case of some chapters. These notes give the historical background and significance of the results proved in the chapter as well as detailed references to the bibliography.

Chapters 1-3 give the standard information on algebraic foundations, structure of finite fields and polynomials over finite fields, carefully presented. The section on the various algorithms for finding the roots of linearised polynomials is particularly readable. Chapter 4 is concerned with the factorisation of polynomials over "small" and "large" finite fields. Chapter 5 on Exponential Sums takes us into the applications of finite fields in number theory and Chapter 6 gives a very thorough treatment of equations over finite

fields. Chapter 7 on Permutation Polynomials gives us further algebraic background for what one suspects is the meat of the book - the many unlikely and exciting applications of the theory of finite fields - linear recurring sequences, linear codes in general, cyclic codes in particular, finite geometries and various aspects of combinatorics such as block designs and latin squares. In the section on finite geometries there is some delightful material on the connection between the theorems of Desargues and Pappus via Wedderburn's Theorem. Overall, the range and scope of the material presented is almost overwhelming.

Any reservations? Well, it is almost churlish to mention them but any book of this length is unlikely to please everyone in every detail. Though there seem to be remarkably few misprints, the language reads a little strangely at times. For example, on page 17 we read "An element $c \in R$ is called a prime element if it is no unit ..." and on page 18 "... the ring $R/(C)$ consists only of one element and is no field". Overall perhaps the book is a bit too clinical with little emphasis on the beauty and elegance of the theory of finite fields. Maybe this is the price to be paid for a definitive work on the subject.

"Finite Fields" is a must for all libraries and, despite its price, probably essential for anyone with a serious interest in algebra. Lidl and Niederreiter have written a superlative book which is likely to be regarded as the last word on finite fields for many years to come.

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PROBLEM PAGE

First of all, here are some new problems.

1. (Suggested by Pat Fitzpatrick). Consider the sequence of digits

198423768.....,

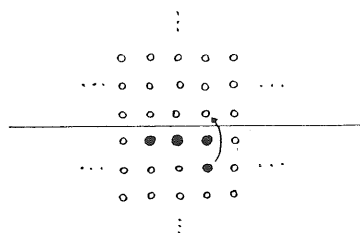
obtained using the following rule.

"After 1984 every digit which appears is the final digit of the sum of the previous four digits."

Does the grouping 1984 appear later in this sequence and, if so, when? What about the grouping 1985?

2. (Due to John Conway and suggested to me by Harold Shapiro.)

Imagine playing solitaire on an unlimited board, on which is drawn a horizontal line.



The aim is to lay out pegs below the line in such a way that, with the usual solitaire moves, a single peg can be manoeuvred as high as possible above the line. The above arrangement of four pegs enables a single peg to reach the second row (above the line).

Find an arrangement which enables a single peg to reach the fourth row and show that there is no arrangement which enables a peg to reach the fifth row.

Now here are the solutions to September's problems.

1. Prove that

$$\sum_{k=1}^n \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)}{3}, \quad (1)$$

and deduce that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \quad (2)$$

This problem was suggested by Finbarr Holland who proves (1) using the finite Fourier transform. The classical proof (found in turn-of-the-century calculus texts) goes as follows.

By De Moivre's theorem,

$$\begin{aligned} \sin(2n+1)\theta &= \text{Im}[(\cos\theta + i\sin\theta)^{2n+1}] \\ &= \sin^{2n+1}\theta \text{Im}[(\cot\theta + i)^{2n+1}] \\ &= \sin^{2n+1}\theta P(\cot^2\theta), \end{aligned}$$

where P is the polynomial

$$P(x) = \binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \dots + (-1)^n.$$

For $\theta = k\pi/(2n+1)$, $k = 1, 2, \dots, n$, we have $\sin(2n+1)\theta = \sin k\pi = 0$ and $\sin\theta \neq 0$, so $P(x) = 0$ when $x = \cot^2(k\pi/(2n+1))$, $k = 1, 2, \dots, n$. These n numbers are the roots of P and so their sum is

$$\binom{2n+1}{3} / \binom{2n+1}{1} = \frac{n(2n-1)}{3},$$

which proves (1).

To deduce (2) from (1) rewrite the inequality

$$\sin\theta < \theta < \tan\theta, \quad 0 < \theta < \pi/2,$$

as
$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta, \quad 0 < \theta < \pi/2,$$

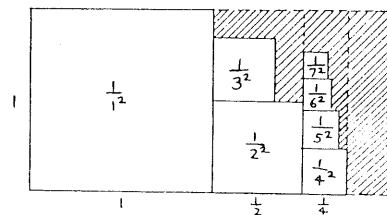
to obtain

$$\sum_{k=1}^n \cot^2\left(\frac{k}{2n+1}\right) < \left(\frac{2n+1}{\pi}\right)^2 \sum_{k=1}^n \frac{1}{k^2} < \sum_{k=1}^n \left(1 + \cot^2\left(\frac{k\pi}{2n+1}\right)\right).$$

Using (1) and letting $n \rightarrow \infty$ gives (2).

It is remarkable that this proof of (2) remained unnoticed until 1953 when it was published by A.M. Yaglom and I.M. Yaglom. It has been rediscovered several times since (*Monthly*, 80 (1973) 424-425).

Before leaving this problem I can't resist including a related 'proof without words'.



2. Prove that, if a_0, a_1 are not both zero then the sequence

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \dots, \quad (3)$$

has period 9.

This problem is due to Morton Brown who proposed it in *American Mathematical Monthly* in October 1983. The neat solution which follows was sent to me by Mícheál Ó'Searcóid. It is very similar to one that Morton Brown received from Donald Knuth.

The key observation is that if

$$0 \leq 2a_k \leq a_{k+1}, \quad (4)$$

for some k , then the sequence $\{a_n\}$ is

$$\dots, a_k, a_{k+1}, a_{k+1}-a_k, -a_k, 2a_k-a_{k+1}, a_{k+1}-a_k, 2a_{k+1}-3a_k, a_{k+1}-2a_k, a_k-a_{k+1}, a_k, a_{k+1}, \dots,$$

which is easily seen to have period 9 (not 1 or 3).

Since we can define

$$a_n = |a_{n+1}| - a_{n+2}, \quad n = -1, -2, \dots,$$

the same conclusion holds if $0 \leq 2a_{k+1} \leq a_k$, for some k .

If $a_k, a_{k+1} \geq 0$ then w.l.o.g. $0 \leq a_k \leq a_{k+1}$ so either (4) is true or else $a_k \leq a_{k+1} \leq 2a_k$, in which case $0 \leq 2a_{k+2} \leq a_{k+1}$. Finally, the sequence must contain neighbouring non-negative terms since $a_n \leq 0$ implies that $a_{n+2} \geq 0$.

A number of related questions suggest themselves. For example (a) are there similar sequences with other periods, and (b) what happens if a_0, a_1 are complex. Both these questions have received attention but, as far as I know, only partial answers have been given.

To attack (a) it helps to think of (3) geometrically. The mapping

$$t : (x, y) \mapsto (y, |y| - x)$$

which takes (a_n, a_{n+1}) to (a_{n+1}, a_{n+2}) can be decomposed as

$$t = q_{\pi/4} s q_{\pi/2},$$

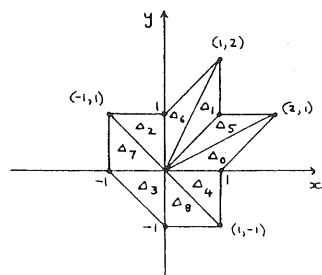
where q_θ denotes reflection in the line making a positive angle θ with the x-axis, and

$$s : (x, y) \mapsto (x + |y|, y)$$

is a "piecewise shear". Thus

$$t = q_{\pi/4} (sr_{\pi/2}) q_{\pi/4},$$

where $r_{\pi/2} = q_{\pi/2} q_{\pi/4}$ is a positive rotation through $\pi/2$, so that t is conjugate to $u = sr_{\pi/2}$. The effect of u on a triangle Δ_0 is illustrated below. Here $\Delta_n = u^n(\Delta_0)$, $n = 0, 1, \dots, 8$.



To find other sequences having a similar property to (3) Mike Crampin (Open University) has considered mappings obtained by glueing together a pair of shears each of which fixes the x-axis. In this way he finds, for example, that if a_0, a_1 are not both 0 then the sequence

$$a_{n+2} = \frac{1}{2}(a_{n+1} + |a_{n+1}|) - a_n, \quad n = 0, 1, 2, \dots,$$

has period 5.

With regard to (b) the main question seems to be "Is $\{a_n\}$ bounded?" Since $\text{Im}[a_{n+2}] = -\text{Im}[a_n]$, this question reduces to "Is $\{\text{Re}[a_n]\}$ bounded?" Dov Aharonov has shown that it is enough to investigate the two cases:

$$(i) \quad a_0 = -x + i\epsilon, \quad a_1 = 1 + i\eta, \quad 0 \leq x \leq 1,$$

$\epsilon > 0, \eta > 0$ arbitrarily small, and

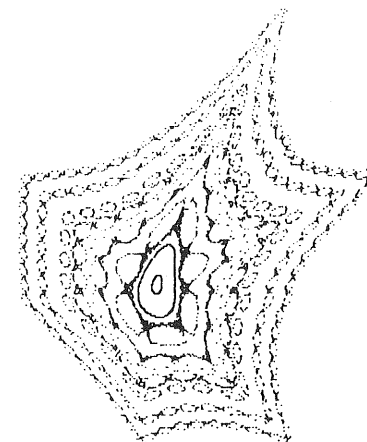
$$(ii) \quad a_0 = x + i\epsilon, \quad a_1 = 1 + i\eta,$$

where $|x| = O(\max(|\epsilon|, |\eta|))$.

Taking the special case $a_0 = a \in \mathbb{R}$ and $a_1 = i$, the points $(\text{Re}[a_{2n}], \text{Re}[a_{2n+1}])$, $n = 0, 1, 2, \dots$, in \mathbb{R}^2 form an orbit of the (area-preserving) mapping

$$(u, v) = (\sqrt{v^2 + 1} - x, |u| - y).$$

These orbits are plotted below for $a = \frac{1}{2}, 1, \frac{3}{2}, \dots, 4$. As $a \rightarrow \infty$ the orbits seem more and more to resemble the boundary of the previous figure!



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ANY QUESTIONS?*

Desmond MacHale

Recently, I attended a mathematical lecture given by a guest speaker where absolutely nobody, except possibly the speaker, had the remotest idea what was going on. Normally, one can absorb at least some of the preliminary definitions and follow, say, the first blackboard full of development of the theory, but on this occasion everyone was completely lost after the first definition. After the speaker had finished over an hour later to an enthusiastic round of applause, the chairman asked for questions, and, of course, there was a deathly and highly embarrassing silence. Then and there I resolved to put together a collection of universal questions for use in such situations. Such questions must sound sensible, but they are designed to cover up the total ignorance of the questioner rather than to elicit information from the speaker. The following is the list I came up with.

1. Can you produce a series of counterexamples to show that if any of the conditions of the main theorem are dropped or weakened, then the theorem no longer holds?
[The speaker can almost always do so - if not you may have presented him with a stronger theorem!]
2. What inadequacies of the classical treatment of this subject are now becoming obvious?
3. Can your results be unified and generalized by expressing them in the language of Category Theory?
[The answer to this question is always NO!]
4. Isn't there a suggestion of Theorem 3 in an early paper of Gauss?
[The answer to this question is almost always YES!]

5. Isn't the constant 4.15 in Theorem 2 suspiciously close to $4\pi/3$?
[This question can clearly be generalized for any constant k - "Isn't k suspiciously close to $(p/q)\pi$ (for suitable integers p and q)?"]
6. I'm not sure I understand the proof of Lemma 3 - could you outline it for us again?
[Lemma 3 should be just a little nontrivial, yet not more than one third of a blackboard in length.]
7. Are you familiar with a joint paper of Besovik and Bombialdi which might explain why the converse of Theorem 5 is false without further assumptions?
[This is a dangerous question to ask unless you like living dangerously. The answer is always "NO" unless the speaker is playing the same game as you are, because Besovik and Bombialdi do not exist, and even if by some unfortunate chance they do exist, it is very unlikely that they have written a joint paper. If the speaker calls your bluff and asks for details and a reference, tell him the paper is available only in Albanian with Portuguese summaries. Promise to mail him a copy but forget to do so.]
8. Why not get a graduate student to perform the horrendous calculations mentioned in Theorem 1 in the case $n = 4$?
[The answer is always "I've a student doing just that at the moment."]
9. Could you draw us a simple diagram to show what the situation looks like for $n = 2$?
[Be careful that he hasn't already done so.]
10. What textbook would you recommend for someone who wishes to get students interested in this area?
[The speaker has almost invariably written such a textbook himself and will be delighted you asked this question. If he hasn't, then you can ask the next question.]

11. When can we expect your definitive textbook on this subject?
12. Why do you think there was such a flurry of activity in this area around the turn of the century and then nothing until your paper of 1979?

[The true answer is that people in the period in between had more sense.]

In general, a good ploy is to stop halfway through a totally meaningless question you are asking and pretend you have suddenly seen the answer yourself. However, never, never

13. What are the applications of these results?

The speaker is probably embarrassed enough already!

* The above article is reprinted from the *American Mathematical Monthly*, Vol. 90, No. 1, p. 48 (January 1983). We are grateful to the Editor of the *American Mathematical Monthly* to reprint it here.

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CONFERENCE REPORT

THE ROLE OF MATHEMATICS IN ENGINEERING EDUCATION

The conference was held on June 7 and 8 at NIHE, Limerick and was attended by over fifty engineers and mathematicians from Universities, Polytechnics, NIHEs, RTCs and Colleges of Education throughout Ireland as well as a number of representatives from industry. The aim of the conference was to discuss whether the importance of a mathematical education for engineers lies in the development of a formal language for the expression of engineering problems or in the provision of a problem solving tool kit. The six sessions were each addressed by a keynote speaker and followed by lengthy discussion periods.

The conference was set in motion by that 'honorary Irishman' Professor Avi Bajpai of Loughborough University of Technology who reviewed the development of engineering education in the last ten years, stressing the greater emphasis on computers which has led to more numerical and statistical techniques, control theory, optimisation and operations research. He suggested that the teaching of modelling by means of case studies could lead to an integrated approach which could be implemented by team teaching. At the conclusion of his talk Professor Bajpai demonstrated some material developed for computer assisted learning in the MIME project (Micros in Mathematical Education).

Professor Eamonn McQuade, NIHE, Limerick, looked into his crystal ball and considered the implications of new technology, e.g. CAD, CAM and expert systems, on the skills required of engineers. He concluded that in the future engineers will need a strong fundamental knowledge of the underlying principles of their technology and insisted that mathematics teaching should concentrate on principles and concepts rather than solution techniques.

11 Dr Joe Morris of UCD inspired a lively discussion with
his talk on the impact of Computer Science in Engineering
Mathematics. He stressed the paramount importance of the
12 ALGORITHM, rather than programming languages, in the teaching
of Computer Science and pointed out that the ALGORITHM should
also be influencing the way mathematics is taught. Some
doubts were also expressed about the introduction of Computer
Science in the secondary school.

13 After a lively conference dinner, participants rose the
next day to hear Professor Sean Scanlon of UCD outline the
roles and interactions of model building and mathematical
analysis in engineering design work. He pointed out that,
as engineers must be equally skilled in both aspects of design,
their mathematical education should be directed towards equip-
ping them with an understanding of the 'fundamental ethos of
the mathematics outlook'.

—
* Tom Power of Waterford RTC addressed the problem of tech-
nician education. He advocated a structured approach to the
teaching of concepts and algorithms with material organised
— so as to be easily referenced by engineering lecturers.

D.
U.
C.
I.
The final session of the conference was addressed by
Dr Peter Lawes from Howmedica. His text was 'We don't know
what we don't know'. He emphasised that only through strong
links between industry and third level institutions can this
information gap be bridged. He invited those in education
to learn from industry so that they could in the long run
influence industry.

A complete account of the proceedings of the conference
will be published in a forthcoming issue of the International
Journal of Mathematical Education in Science and Technology.

The organisers of the conference acknowledge the generous
sponsorship of Howmedica International Inc., Analog Devices BV.

and the Royal Irish Academy.

Gordon S. Lessells

CONFERENCE ANNOUNCEMENT

4th European Meeting of the Psychometric Society
and Classification Societies,
Cambridge, 2-5 July, 1985

Papers on psychometric and classification topics are invited for this meeting. Accommodation will be in Queens' College (full board £44 per day, bed and breakfast £16.50, registration £40, £50 if not a member of a sponsoring society - half price for students).

Enquiries should be sent to:

Dr Ian Nimmo Smith,
MRC Applied Psychology Unit,
15 Chaucer Road,
CAMBRIDGE,
CB2 2EF,
U.K.

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1. Manuscripts should be typed on A4 paper and double-spaced.
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4. Words or phrases to be printed in capitals should be doubly underlined, e.g.
Print these words in capitals → Print THESE WORDS in capitals.
5. Words or phrases to be italicized should be singly underlined, e.g.
Print these words in italics → Print *these words* in italics.
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Print these words in script → Print *these words* in script.
7. Diagrams should be prepared on separate sheets of paper (A4 size) in black ink. Two copies of all diagrams should be submitted: the original without lettering, and a copy with lettering added.
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