IRISH MATHEMATICAL SOCIETY

cumann matamaitice na héireann

BULLETIN

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IRISH MATHEMATICAL SOCIETY BULLETIN

EDITOR: Dr R. Gow
BOOK REVIEW EDITOR: Dr Michael Tuite
PRODUCTION MANAGER: Dr Mícheál Ó Searcóid

The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, at Easter and at Christmas. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files if possible, in order to ensure speedier processing.

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CONTENTS

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THE IRISH MATHEMATICAL SOCIETY

Officers and Committee Members

NOTES ON APPLYING FOR I.M.S. MEMBERSHIP

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society and the Irish Mathematics Teachers Association.

2. The current subscription fees are given below.

Institutional member	IR£50.00
Ordinary member	IR£10.00
Student member	IR£4.00
I.M.T.A. reciprocity member	IR£5.00

   The subscription fees listed above should be paid in Irish pounds (puint) by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than Irish pounds using a cheque drawn on a foreign bank according to the following schedule:

   If paid in United States currency then the subscription fee is US$18.00.
   If paid in sterling then the subscription fee is £10.00 stg.
   If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$18.00.

   The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. The subscription fee for reciprocity membership by members of the American Mathematical Society is US$10.00.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription fee to

The Treasurer, I.M.S.
Department of Mathematical Physics,
University College,
Dublin,
Ireland.

Minutes of the Meeting of the Irish Mathematical Society

Ordinary Meeting
21st December 1994

The Irish Mathematical Society held an ordinary meeting at 12.15pm on Wednesday 21st December 1994 in the Dublin Institute for Advanced Studies, 10 Burlington Road. 20 members were present. The president, B. Goldsmith, was in the chair. Apologies were received from D. Hurley and M. Ó Searcóid.

1. The **minutes** of the 31st March 1994 meeting were approved and signed.

2. **Matters arising**

It was reported that the subcommittee had made a submission to the Science, Technology and Innovation Advisory Council on behalf of the society, as requested. A copy of the submission and a brief report on a follow-up seminar organized by the Council appear in this issue of the bulletin.

3. **Bulletin**

The production of the bulletin is currently on schedule. The editor, R. Gow, asked members to submit articles to the bulletin and also to encourage others to do so.

4. **Treasurer's business**

The treasurer circulated an interim report on the state of the society's finances. It was clear that it is necessary to keep tight control of the finances.

5. **Membership fees**

The president informed the society of the committee's intention to bring before the Easter Meeting a proposal to review the membership fee structure, with a view to implementing fee changes from 1st January 1996.

A brief discussion was held. Most members appeared to agree that a full proposal should be taken to the Easter Meeting.

6. September Meeting

The 1995 September Meeting will take place on 7th and 8th September.

7. EMS/Zürich Meeting

During a brief discussion on collection of EMS fees, it was suggested that the system of "local representatives" of the society be re-instated. It was also suggested that perhaps standing orders for payment of membership fees could be adjusted to include both IMS and EMS membership fees.

S. Dineen gave a brief report on the EMS Zürich Meeting. The EMS are reported to be considering either supporting publications of national mathematical societies or producing their own journal. They appear to favour publication of their own journal.

The EMS are considering supporting mathematical summer schools.

The EMS are now trying to coordinate efforts to prepare for the year 2000, which is to be the Year of Mathematics.

8. Elections

The following were elected, unopposed, to the committee (* denotes re-election to the committee):

Committee Member	
D. Hurley *	**President**
C. Nash *	**Vice-President**
J. Pulé *	**Treasurer †**
B. Goldsmith *	
G. Lessells *	
M. Tuite	
K. Hutchinson	

† As M. Vandyck may be leaving the country, he decided to stand down from the position of treasurer. The committee decided that the position of treasurer should also be filled by election at this meeting.

The following have one more year of office:

E. Gath	
R. Gow	
P. Mellon	**Secretary**
R. Timoney	

The committee is to co-opt an additional member.

The meeting closed at 1.15pm.

Pauline Mellon
University College Dublin.

Conference Announcement

STOKES COMMEMORATION

SLIGO RTC, SLIGO, IRELAND
9TH-10TH JUNE, 1995

A conference to commemorate the life and work of G. G. Stokes has been organized by the Irish Branches of the Institute of Mathematics and the Institute of Physics, under the auspices of the Royal Irish Academy, as part of the Sligo 750 celebrations.

While the mathematical physicist George Gabriel Stokes (1819–1903) has long been associated with the University of Cambridge, where he held the Lucasian Professorship of Mathematics (a chair once occupied by Isaac Newton) from 1849 until his death, it is less generally realized that he was born in Skreen, County Sligo, where his father was rector, and received his early education in Dublin. The aims of this meeting are threefold:

1. To highlight Stokes' contributions to mathematics and physics and explore some of their current ramifications.
2. To understand Stokes' relationships with his family in Ireland, with the University of Cambridge and with the Royal Society.
3. To create awareness of Stokes' life and works in Sligo through the unveiling of a plaque at his birthplace.

The programme will commence at 11.30am on Friday 9th June 1995 in Sligo Regional Technical College. The first day will be devoted to plenary lectures and shorter contributions on mathematics and physics. There will be a Civic Reception in the Town Hall at 7pm, followed by the Conference Dinner in the Silver Swan Hotel at 8pm. On Saturday 10th June, general and historical lectures will take place in Sligo RTC from 10am to noon. After lunch, the conference will move to the townland of Skreen, where a plaque will be unveiled at Stokes' birthplace, followed by afternoon tea in the Rectory grounds (Parochial Hall if wet).

The following have agreed to speak: Prof. Michael Berry, University of Bristol; Prof. Anne Crookshank (a descendant of Stokes), Trinity College Dublin; Prof. Michael Hayes, University College Dublin; Dr Norman McMillan, Carlow RTC; Prof. Frank Olver, University of Maryland; Dr Richard Paris, University of Abertay Dundee; Prof. Denis Weaire, Trinity College Dublin.

Offers of contributed talks should be made, with an abstract, to the Chair of the Organizing Committee, Prof. Alastair Wood, School of Mathematical Sciences, Dublin City University, Dublin 9 [wooda@dcu.ie] or to the Secretary, Dr Eamonn Cunningham, School of Physical Sciences [cunninghame@dcu.ie] at the same address. The deadline for contributed talks is 30th April 1995.

Registration forms for the conference should be returned to:
Ms. Aisling Walsh, Stokes Commemoration, School of Mathematical Sciences, Dublin City University, Dublin 9.

Fax: 01 7045786 (International +353 1 7045786)

Telephone: 01 7045293 (International +353 1 7045293)

email: walshais@dcu.ie

Registration forms must be returned by 30th April 1995, accompanied by a cheque for the appropriate fees, in punts or sterling, made out to Dublin City University.

The registration fee is £40 (includes 2 buffet lunches, coffees, teas and return transport Sligo-Skreen).

Student registration fee is £10 (includes the above, but a letter from Professor/Supervisor is needed).

Accompanying persons fee is £10 (covers buffet lunch Saturday and return transport Sligo-Skreen)

The Conference Dinner costs £20 (Civic Reception is free of charge).

Accommodation is not included. A block of 20 rooms is being held for the night of 9th June for participants at the Southern Hotel, Sligo, telephone 071 62101, international +353 71 62101, at a cost of £30 per person B&B. These are on a first-come first-served basis and immediate booking is advised direct to the hotel. The following bed and breakfast houses have accommodation in the range £16-20:

Ben Wisken Lodge, Donegal Road 071 41088

Treetops, Cleveragh Road 071 60160

Teach Eamonn, Hazelwood 071 43393

Mrs Mugan, Lisnaluag 071 43584.

Student dormitory accommodation is available for £6 at Eden Hill Hostel.

Conference Announcement

IRISH MATHEMATICAL SOCIETY

EIGHTH SEPTEMBER MEETING

UNIVERSITY OF LIMERICK

7TH-8TH SEPTEMBER, 1995

PRELIMINARY ANNOUNCEMENT

The eighth September meeting of the Irish Mathematical Society will take place in University of Limerick on Thursday and Friday, the 7th and 8th of September 1995. Among the principal speakers will be:

Prof. Andrew Fowler, Oxford University,

Prof. Chris Budd, University of Bath,

Prof. John O'Donoghue, University of Limerick,

Prof. Tom Laffey, University College Dublin,

and others to be arranged. There will also be some short 20-minute talks and potential contributors are urged to contact the organizers as soon as possible.

Bed and breakfast accommodation will be available at the Kilmurry student village on campus. There will be a conference dinner on 7th September.

For further information contact Dr Eugene Gath, Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland.

FAX: X-353-61-334927

e-mail: gathe@ul.ie

SCIENCE, TECHNOLOGY AND INNOVATION

Pauline Mellon

In the spring of 1994 the government established the Science, Technology and Innovation Advisory Council (hereafter referred to as the Council) to undertake a review of science and technology in Ireland. The remit of the Council is to:

- determine whether the current policies, objectives, structures and components of the science and technology system are the right ones for achieving economic development through research, technology and innovation;
- determine what mechanisms should be employed to achieve the desired S & T goals in the light of the above;
- provide a report on the findings, with recommendations about changes to improve the effectiveness and efficiency of the national S & T system in contributing to national economic development. This report will be in the form of a draft White Paper on Science, Technology and Innovation for the Minister and Government to consider.

Interested parties were invited to make submissions to the Council for consideration. At the Easter meeting of the Irish Mathematical Society (hereafter called the Society) it was decided that a brief submission should be made by the subcommittee on behalf of the society. A submission was duly sent. In a follow-up action, the Council held a seminar "Learning from International Experience" in Dublin Castle on 19th September 1994, where international perspectives on science and technology were presented. As secretary of the Society, I was invited to attend this seminar. The submission made by the Society to the Council and a summary of the seminar follows.

Irish Mathematical Society
Submission to the
Science, Technology and Innovation Council

The sophisticated technological society that we aspire to relies on an up-to-the-minute knowledge of new scientific, technological and economic developments around the world. As new developments arise at an increasingly rapid rate, not only must these areas of our economy have a highly educated work force but the numeracy requirements and mathematical skills of graduates working in these areas are generally seen to be playing an ever more important role.

Research in these areas relies heavily on mathematical techniques and the pervasive role of scientific computing has meant that mathematical analysis, modelling and interpretation of data is now an everyday requirement. It is interesting to note that, despite recent layoffs in their hardware division in Galway, DEC have actually been recruiting software people including some mathematicians. To continue attracting companies like AST, DELL and INTEL to Ireland we must provide a technologically advanced work force.

Not only are research mathematical scientists needed to keep abreast of current scientific developments, but foresight by active researchers today, can pave the way, and introduce the changes into the teaching of mathematics, which will allow us to produce the numerate graduates we need tomorrow.

- **A strong research community in the mathematical sciences is essential to keep our science and technology abreast of new developments and to continue supplying graduates with up-to-date qualifications.**

In the past, the mathematical sciences have been seen as an essentially "non-experimental" subject with little or no equipment needs. This is no longer appropriate. As with most other scientific disciplines, high-level computing equipment is now an essential requirement.

- **Funding patterns should reflect the "new" equipment**

needs of research in the mathematical sciences.

A steady supply of Ph.D's is necessary to ensure tomorrow's mathematical community. To date, many of our researchers have completed their graduate work in the UK or in the USA. The current under-supply of graduate fellowships in the UK and the hesitations being raised in the USA about funding so many "foreign" Ph.D's spell imminent danger that these routes may become less available. It is therefore critical to put in place a program of support for Ph.D and post-doctoral students.

- **A program of support at the Ph.D and post-doctoral level is required.**

The research environment in our third-level institutions has deteriorated. Increased student numbers and general under funding means that academics have ever expanding teaching and administrative burdens. Satisfactory computing equipment is often not available and our libraries are being denuded by persistent cuts. As such, the time and facilities to do research is constantly being eroded.

- **It is important to acknowledge the difficulties facing research in the present economic climate and to face these challenges in any new framework for science and technology support.**

To implement the above recommendations it is imperative that

- **the mathematical sciences community is represented at every level of the new science council**
- **a separate mathematical sciences budget is installed**
- **a single person on the new science council be given responsibility for the mathematical sciences.**

"Learning from International Experience".
Dublin Castle 19th September 1994.

The seminar was introduced by Mr D. P. Tierney, Chairman of the Science Technology and Innovation Advisory Council. He described the Council's task as the most fundamental and wide ranging review of science and technology ever to be undertaken by the government and reported that the Council had received submissions from 150 interested parties. He called upon the "players in the field of science and technology" first to come to some agreement among themselves as to the important aspects of desired policy before there could be hope of convincing Government, policy makers and the public as to the value of our efforts.

An address by the then Minister for Commerce and Technology, Mr Seamus Brennan, followed. Minister Brennan reiterated the view that there is no consensus as to how spending on science, research, technical development and innovation can best contribute to national development in its broadest sense. **He admitted, however, that not enough money was being spent on science and technology. The issue, as he stated it, "is not whether we can afford to increase such spending but, rather, whether we can afford not to do so".** He also said that it should not be a choice between basic or applied research, as he recognized that both are essential and interlinked. The Minister also announced the setting up of a Single Research Support Fund to include

- the funding for basic research and strategic research;
- support for Ph.D's and M.Sc's and
- the third level / industry cooperative research.

This fund is intended to be administered in full consultation with third level and industry interests.

The minister also expressed his high hopes for Irish participation in the EU Fourth Framework Programme.

The international contribution began with Dr Joseph Clarke, Senior Science Advisor, United States Department of Commerce,

and Chairman of the OECD Innovation Committee.
Dr Clarke began by suggesting that low-unemployment is correlated to higher R & D investment (as a percentage of GDP). Dr Clarke outlined the aims and priorities of the OECD and its Working Group on Innovation and Technology Policy. He claimed that the global economy is increasingly knowledge-based and that the information and communication technologies, in particular, are having a pervasive impact on this economy. He also claimed that specialized skills are being increasingly demanded by new jobs. **His conclusion was that government S & T policies can help in the creation and use of new technologies, as a base for high-productivity, high-wage employment.**

The second speaker from abroad was Dr Harry Beckers, Group Research Coordinator for Royal Dutch/Shell and former chairman of EU Industry Research and Development Advisory Committee.

Dr Beckers spent some time outlining the different natures of industrial and academic research. While industrial research grew out of its academic forefathers, its priorities and therefore its needs are different and this has become increasingly apparent over the last few decades. He claimed that it is therefore not appropriate to evaluate industrial research from an academic viewpoint. He outlined levels of R & D that are appropriate for various industrial sectors and warned that companies should stay close to the average R & D expenditure for their industrial sector. **He agreed that the presence of a good scientific infrastructure and the availability of trained and educated employees within a stable environment is of the greatest importance to the multinationals in seeking attractive locations.**

Perhaps the most relevant international speaker from the Irish perspective was Professor Paolo Fasella, as his comments and observations were mostly about the Irish situation. He referred to the comparative shortage of scientists in Ireland, which is especially worrying as the average age of the Irish scientist is one of the highest in Europe. **He made a direct call on the Irish government to improve their funding for science and technology, saying that it is not adequate or appropriate to rely so heavily on European funding but that this should be strongly supplemented by funds from the Irish government.**

He said that a "neutral" education is necessary to allow long term research benefits and that for a policy to be effective in the long term it must support a wide range of basic research activities.

Professor Paolo Fasella is the Director General of the Science and Technology Directorate of the European Commission.

The final speaker of the session was Mr David Wilkinson, Head of Science and Engineering Base Group of the UK Office of Science and Technology.

Mr Wilkinson informed us of the science policy of the UK government and described the divisions of research funding. The recent White Paper "Realising Our Potential: A Strategy for Science, Engineering and Technology" produced in the UK in May 1993 had as an important tenet the fact that the UK **"government accepts its role as the main funder of basic research. It wishes to sustain within the United Kingdom expertise across the core disciplines of biology, chemistry, mathematics and physics and to provide the climate where centres of international excellence can develop and flourish".**

The presentations were followed by an open forum. It was clear from the questions asked and views offered that many people present agreed with Professor Fasella's viewpoint that the Irish government was sadly lacking in its support for research, and particularly basic research. The open forum was cut short due to an overrun in the time allotted for the presentations. This was rather unfortunate, as there were surely many people present who had strong and relevant points to contribute.

Pauline Mellon,
Department of Mathematics,
University College,
Belfield,
Dublin 4.

PRODUCTS OF GROUP COMMUTATORS

P. Hegarty and D. MacHale

Abstract We show that certain products of group commutators are commutators and derive a number of applications.

1. Introduction

If a and b are elements of a group G, we define the *commutator* of a and b, written as $[a, b]$, to be the group element $a^{-1}b^{-1}ab$. The following facts are immediate from the definition.
(i) The inverse of a commutator is a commutator

$$[a, b]^{-1} = [b, a]. \tag{1}$$

(ii) Any conjugate of a commutator is a commutator

$$x^{-1}[a, b]x = [x^{-1}ax, x^{-1}bx]. \tag{2}$$

(iii) By direct computation

$$\begin{aligned}[a, b] &= [ba, a^{-1}] &(3)\\ &= [b^{-1}, ab] &(4)\\ &= [b^{-1}a, b] &(5)\\ &= [a, ab]. &(6)\end{aligned}$$

However, it is well known that the product of two commutators need not be a commutator. Guralnick, [5], shows that if the commutator subgroup G' satisfies either
(a) G' is abelian and $|G| < 128$ or $|G'| < 16$
or
(b) G' is non-abelian and $|G| < 96$ or $|G'| < 24$,

then the product of any two commutators is a commutator. He also gives examples to show that these two bounds are the best possible. Macdonald, [7], shows that if G has centre $Z(G)$ and satisfies

$$|G : Z(G)|^2 < |G'|,$$

then there is a product of commutators in G which is not a commutator in G, and produces infinitely many examples of this phenomenon.

In this note we investigate certain products of group commutators which can be written as single commutators. We also present analogous results for sums of ring commutators. We then apply the results for group commutators to give elementary proofs of two known group-theoretic results.

2. Products of group commutators

The following commutator identity appears, essentially without motivation, in [9, p.85]:

$$[xy, zt] = y^{-1}[x, t]y[y, t](yt)^{-1}[x, z](yt)t^{-1}[y, z]t. \tag{7}$$

Putting $x = c$, $y = a^{-1}$, $z = d$, $t = b^{-1}$, we immediately obtain

$$[a, b][b, c][c, d][d, a] = (ba)^{-1}[ca^{-1}, db^{-1}](ba). \tag{8}$$

Thus, the expression on the left-hand side of (8) is a single commutator. As special cases, we have

$$[a, b][b, c][c, a] = (ba)^{-1}[ca^{-1}, ab^{-1}](ba) = a^{-1}[b^{-1}ca^{-1}b, b^{-1}a]a \tag{9}$$

by putting $d = a$ in (8), and

$$[a, b][b, c] = (ba)^{-1}[ca^{-1}, b^{-1}](ba) = [a^{-1}ba, a^{-1}c] \tag{10}$$

by putting $d = 1$ in (8) and applying (3)-(6) several times.

Since (10) is fundamental to this paper, and actually appears in [8] and as an exercise in [1], we feel it is instructive to derive (7)-(10) in reverse order, starting from scratch. Firstly, we have

$$\begin{aligned}[a, b][b, c] &= a^{-1}b^{-1}abb^{-1}c^{-1}bc = a^{-1}b^{-1}ac^{-1}bc\\ &= (a^{-1}b^{-1}a)(c^{-1}a)(a^{-1}ba)(a^{-1}c) = [a^{-1}ba, a^{-1}c].\end{aligned}$$

Secondly,

$$\begin{aligned}[a,b][b,c][c,a] &= [a^{-1}ba, a^{-1}c][c,a] \text{ by } (10)\\ &= [a^{-1}ba, a^{-1}c][a^{-1}c, a] \text{ by } (5)\\ &= [a^{-1}b^{-1}ca^{-1}ba, a^{-1}b^{-1}a^2] \text{ by } (10)\\ &= a^{-1}[b^{-1}ca^{-1}b, b^{-1}a]a \text{ by } (2),\end{aligned}$$

which is just (9). Thirdly,

$$\begin{aligned}[a,b][b,c][c,d][d,a] &= [a^{-1}ba, a^{-1}c][c^{-1}dc, c^{-1}a] \text{ by } (10)\\ &= [a^{-1}ba, a^{-1}c][a^{-1}c, c^{-1}da] \text{ by } (4)\\ &= [a^{-1}b^{-1}ca^{-1}ba, a^{-1}b^{-1}ac^{-1}da] \text{ by } (10)\\ &= (ba)^{-1}[ca^{-1}, ac^{-1}db^{-1}](ba) \text{ by } (2)\\ &= (ba)^{-1}[ca^{-1}, db^{-1}](ba) \text{ by } (6),\end{aligned}$$

which is just (8). Equation (7) now follows easily on putting $a = y^{-1}$, $b = t^{-1}$, $c = x$ and $d = z$.

We now ask if either of

$$[a,b][b,c][c,d] \text{ or } [a,b][b,c][c,d][d,e][e,a]$$

can always be written as a single commutator. We show that the answer is no. We need the following results from Liebeck, [6]. Let $G_4 = \langle a_1, a_2, a_3, a_4\rangle$ be the free nilpotent group of class 2 on four generators. Put

$$c_{ij} = [a_i, a_j]$$

for $1 \le i < j \le 4$, so that $[c_{ij}, a_k] = 1$ for $1 \le i < j \le 4$ and all k. An arbitrary commutator in G may be written as

$$[a_1^{\alpha_1}a_2^{\alpha_2}a_3^{\alpha_3}a_4^{\alpha_4}, a_1^{\beta_1}a_2^{\beta_2}a_3^{\beta_3}a_4^{\beta_4}],$$

which simplifies to

$$\prod_{1\le i<j\le 4} c_{ij}^{\delta_{ij}},$$

where $\delta_{ij} = \alpha_i\beta_j - \alpha_j\beta_i$.

The indices δ_{ij} satisfy the relation

$$\delta_{12}\delta_{34} - \delta_{13}\delta_{24} + \delta_{14}\delta_{23} = 0,$$

and this is a necessary and sufficient condition for

$$\prod_{i,j=1}^{4} c_{ij}^{\delta_{ij}}$$

to be a single commutator.

Consider $[a_1, a_2][a_2, a_3][a_3, a_4]$. Here,

$$\delta_{12}\delta_{34} - \delta_{13}\delta_{24} + \delta_{14}\delta_{23} = 1\cdot 1 - 0\cdot 0 + 0\cdot 1 = 1 \neq 0,$$

so $[a_1, a_2][a_2, a_3][a_3, a_4]$ is not a commutator in G_4.

Suppose now that $[a,b][b,c][c,d][d,e][e,a]$ is always a commutator. Put $e = 1$ and we get that $[a,b][b,c][c,d]$ is always a commutator, contradicting the previous result. It is also clear that for any $n \ge 3$, neither $[x_1, x_2]\ldots[x_{n-1}, x_n]$ nor $[x_1, x_2]\ldots[x_n, x_{n+1}][x_{n+1}, x_1]$ can, in general, be written as a single commutator.

Finally, in this section, we mention some ring-theoretic analogues of the results we have presented for groups. If R is a ring and a and b are elements of R, then the ring commutator of a and b, written $[a,b]$, is defined to be the ring element $ab - ba$. It is well known that the sum of two ring commutators need not be a ring commutator and examples are very much easier to construct than the corresponding examples for groups.

The following identities for ring commutators are easily verified:

$$[a,b] + [b,c] = [a-c, b] \tag{11}$$

$$[a,b] + [c,a] = [a, b-c] \tag{12}$$

$$\begin{aligned}[a,b]+[b,c]+[c,a] &= [a-c, b-c]\\ &= [c-b, a-b]\\ &= [b-a, c-a]\end{aligned} \tag{13}$$

$$[a,b]+[b,c]+[c,d]+[d,a] = [a-c, b-d] \tag{14}$$

$$[x,t]+[y,t]+[x,z]+[y,z] = [x+y, z+t]. \tag{15}$$

Again, (11) and (14) cannot be extended to four and five variables, respectively. Examples are easy to construct in the ring of all 3×3 matrices of the form

$$\begin{pmatrix} a & f(x) & h(x,y) \\ 0 & a & g(y) \\ 0 & 0 & a \end{pmatrix},$$

where f, g and h are polynomials in the commuting indeterminates x and y over an arbitrary field F, and $a \in F$ (see [2]).

3. Applications

(**A**) Culler, [3], has shown that $[a,b]^n$ can be written as a product of $[\frac{n}{2}] + 1$ commutators, where $[k]$ denotes the greatest integer contained in k. Culler's methods are highly topological, however, and we now offer a simple proof based on (10).

Firstly, $[a,b][c,a]$ is a single commutator since, by (3) and (4),

$$[a,b][c,a] = [ba, a^{-1}][a^{-1}, ca],$$

which is a single commutator by (10). We use the following well-known identity (which, incidentally, can also be derived using (3)-(6) and (10)):

$$[a,b]^2 = [b^{-1}, a][aba^{-1}b^{-1}a, b] \tag{16}$$

For simplicity, we henceforth denote $aba^{-1}b^{-1}a$ by t. Equation (3) says that $[a,b] = [ba, a^{-1}]$. We may assume that a and b are generators of a free group G on two generators, since the commutator identities we are about to obtain for the free group may then be carried over homomorphically to any other group generated by two elements. Let α be the automorphism of G defined by setting $a\alpha = ba$, $b\alpha = a^{-1}$. Applying α to both sides of (16), we get

$$[a,b]^2 = [a, ba][t\alpha, a^{-1}] = [a, ba][a, t\alpha a^{-1}], \tag{17}$$

where we have used (4) to obtain the last part of the equation. Thus

$$[a,b]^4 = [a, ba][a, t\alpha a^{-1}][b^{-1}, a][t, b], \tag{18}$$

whence $[a,b]^4$ is a product of three commutators since, by the opening remark of this proof, the product of the middle two commutators in (18) is a single commutator. Now apply α to both sides of (18), and post-multiply both sides of the resulting equation by (16). By the same reasoning as above, we thus have $[a,b]^6$ equal to the product of four commutators. It is now a simple induction that $[a,b]^{2n}$ can always be expressed as the product of $n+1$ commutators.

Finally, pre-multiply both sides of (16) by $[a,b]$ to get

$$[a,b]^3 = [a,b][b^{-1}, a][t, b], \tag{19}$$

whence $[a,b]^3$ is a product of two commutators, by the opening remark. By repeating the construction above of $[a,b]^{2n}$ as a product of $n+1$ commutators, we quickly see that $[a,b]^{2n+1}$ can always be expressed as a product of $n+1$ commutators also. This completes the proof of Culler's result.

(**B**) Edmunds, [4], showed that, in any group, any product of n commutators can always be expressed as the product of some $2n+1$ squares. We offer an elementary proof of this result, again based on (10).

Firstly, for any x_1 and x_2 in G,

$$x_1^2 x_2^2 = [x_1^{-1}, x_2^{-1} x_1^{-1}](x_1 x_2)^2. \tag{20}$$

It can now easily be shown by induction that for $k \geq 2$

$$x_1^2 \dots x_k^2 (x_1 \dots x_k)^{-2} = \prod_{i=2}^{k} [z_{i-1}^{-1}, z_i^{-1}], \tag{21}$$

where we have set

$$z_r = x_1 \dots x_r$$

for $r \geq 1$. Put $k = 2n+1$ and use (10) on the right-hand side of (21) to obtain

$$x_1^2 \dots x_{2n+1}^2 (x_1 \dots x_{2n+1})^{-2} = \prod_{i=1}^{n} [z_{2i-1} z_{2i}^{-1} z_{2i-1}^{-1}, z_{2i-1} z_{2i+1}^{-1}] \tag{22}$$

which equates a product of n commutators and a product of $2n+2$ squares. We now show that every product

$$[a_1, a_2] \ldots [a_{2n-1}, a_{2n}]$$

of n commutators in G can be written in the form of the right-hand side of (22).

We simply equate corresponding terms, that is, for $i = 1, \ldots, n$, we put

$$a_{2i-1} = z_{2i-1} z_{2i}^{-1} z_{2i-1}^{-1} \tag{23}$$

$$a_{2i} = z_{2i-1} z_{2i+1}^{-1}, \tag{24}$$

where the z_r are as above. (23) is an equation for a_{2i-1} in terms of $x_1, \ldots, x_{2i}$ in which the variable x_{2i} appears only once and (24) is an equation for a_{2i} in terms of $x_1, \ldots, x_{2i+1}$ in which the variable x_{2i+1} appears only once.

What this means is that x_1 can be fixed arbitrarily and, having found $x_1, \ldots, x_l$, we have an equation for x_{l+1} in terms of $x_1, \ldots, x_l, a_l$, in which the variable x_{l+1} appears only once, so that the equation has a unique solution. It is easy to verify that the following recursion formula for the x_l is consistent with the $2n$ equations contained in (23) and (24).

$$x_1 = a_1$$

$$x_{2i} = z_{2i-1}^{-2} a_{2i-1}^{-1} z_{2i-1}, \quad 1 \le i \le n$$

$$x_{2i+1} = z_{2i}^{-1} a_{2i}^{-1} z_{2i-1}, \quad 1 \le i \le n$$

Hence, every product of n commutators can be written as a product of $2n+2$ squares

$$[a_1, a_2] \ldots [a_{2n-1}, a_{2n}] = x_1^2 \ldots x_{2n+1}^2 (x_1 \ldots x_{2n+1})^{-2},$$

where the x_i are given by the recursion formulae above. However, from these formulae, we see that $x_2 = x_1^{-2} a_1^{-1} x_1 = a_1^{-2}$, which implies that $x_1^2 x_2^2 = a_1^2$ is a square. Thus, every product of n commutators can in fact be written as a product of $2n-1$ squares, as required.

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NON-MEASURABLE SETS AND TRANSLATION INVARIANCE

Eoin Coleman

In this brief note, we prove a simple but quite general fact about translation invariant measures: *if μ is a finite non-trivial measure on a group G, then G has non-measurable subsets.* An immediate very well-known corollary is the existence of a set of reals which is not Lebesgue measurable. The most popular proofs of this latter result leave one with the impression that non-measurable sets of reals are connected with the density of the rationals, [R], the relatively small number of closed sets of reals, [M], or the identification of the reals with infinite binary sequences, [B].

We begin by fixing the familiar terminology.

Definition Suppose that S is a set and F is a σ-algebra of subsets of S. A *measure* over F is a function μ from F into $[0, \infty]$ such that
(1) $\mu(\emptyset) = 0$
and
(2) if $\{ X_n \in F : n \in \mathbf{N} \}$ is a family of pairwise disjoint sets, then

$$\mu\left(\bigcup_{n\in\mathbf{N}} X_n\right) = \sum_{n\in\mathbf{N}} \mu(X_n).$$

The subsets in F are said to be the *measurable* subsets of S. We say that μ is a *total* measure if
(3) $F = P(S)$, i.e. every subset of S is measurable.
A measure μ is *non-trivial* if $\mu(\{x\}) = 0$ for every $x \in S$, and *finite* if $\mu(S)$ is a positive real number. We say loosely that μ is a measure on S when we mean that the domain of μ is a σ-algebra of subsets of S.

Some well-known examples of non-trivial measures are Lebesgue measure on $\mathbf{R}^n$ and the Haar measure on a locally compact group. These measures are also translation invariant, in accordance with the following definition.

Definition Suppose that $G = (G, *)$ is a group. We say that a measure μ on G is (left-) translation invariant if $\mu(g * X) = \mu(X)$ for every g in G and every X in the domain of μ, where $g * X = \{ g * x : x \in X \}$.

Our first observation is a group-theoretic one.

Proposition 1. *Suppose that G is a group, A is a subgroup of G and X is a non-empty subset of G with $A * X \subseteq X$. Then there exists a subset E of X such that the following hold:*

(i) $X = \cup_{a\in A}(a * E)$;

(ii) if a and b are distinct elements of A, then $a * E \cap b * E = \emptyset$.

Proof: Define an equivalence relation R on X as follows: xRy if and only if $x * y^{-1} \in A$. Since A is a subgroup of G, it follows easily that R is an equivalence relation on X. So R partitions X into equivalence classes. Using the Axiom of Choice, choose a representative from each distinct class and let E be the set of these representatives. It is now straightforward to check that E satisfies (i) and (ii).

Corollary If A is a subgroup of G, then there exists a subset E of G such that the following hold:
(i) $G = \cup_{a\in A}(a * E)$;
(ii) if a and b are distinct elements of A, then $a * E \cap b * E = \emptyset$.
In fact, E is just a set of right coset representatives of A in G.

We now derive the main result from Proposition 1.

Theorem 2. *Suppose that μ is a finite non-trivial (left-) translation invariant measure on the group G. Then G has non-measurable subsets (so μ is not total).*

Proof: Since μ is finite, non-trivial and countably additive, it follows that G is an uncountable set. Let A be any countably infinite subgroup of G (just take the subgroup generated by some countably infinite subset of G; model theorists will apply the Downward

Loewenheim Skolem theorem). By the corollary, there is a subset E of G satisfying (i) and (ii). We claim that E is non-measurable. Well, suppose otherwise; then

$$\mu(G) = \mu(\bigcup_{a\in A} a * E) = \sum_{a\in A} \mu(a * E) = \sum_{a\in A} \mu(E),$$

where we have used (i), (ii), countable additivity and translation invariance. This is impossible since μ is finite and A is infinite. Hence E is non-measurable.

Corollary 3. *There exists a set of reals which is not Lebesgue measurable.*

Proof: Let G be the group $(0,1]$ under addition modulo 1. Lebesgue measure restricted to G satisfies the hypotheses of Theorem 2.

Of course, everything goes through for (right-) translation invariant measures if one formulates an appropriate version of Proposition 1.

The use of the Axiom of Choice (AC) in Corollary 3 prompted mathematicians to study whether and how much choice was necessary. In 1970, Solovay, [S], published the following famous theorem:

Theorem. *Suppose that there exists an inaccessible cardinal. Then there is a model of ZF+DC+ "Every set of reals is Lebesgue measurable".*

The Axiom of Dependent Choice (DC) above is equivalent to the Baire Category Theorem, and is strictly weaker than AC. Matters rested here for a while, as logicians worried about the inaccessible cardinal. Then their cares were lifted when Shelah, [Sh], proved (among other things) that if all Σ^1_3 sets of reals are Lebesgue measurable, then the first uncountable cardinal is inaccessible in L, the universe of constructible sets. This, taken in conjunction with Solovay's theorem, established the equivalence of assertions about the consistency of the Lebesgue measurability of classes of reals and the consistency of large cardinal axioms, and

inspired a stream of equiconsistency results. It surprised the wider public to learn that holding unrestrained views about Lebesgue integrability of certain real functions was no different (in terms of consistency) from endorsing set-theoretic universes containing large cardinals.

The complexity of non-measurable sets of reals and their possible whereabouts in the analytic hierarchy of the subsets of $\mathbf{R}$ continue to form the focus of intensive research. The lecture notes, [B], of Bekkali present some of the developments in this area.

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A NOTE ON MINIMAL INFINITE SUBSPACES OF A PRODUCT SPACE

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Abstract Ginsburg and Sands, [1], have identified the five topological spaces which are 'minimal infinite' in the sense that each is homeomorphic to all of its own infinite subspaces. Given a finite family of spaces, and knowledge of which of the five may be embedded in each of them, we show how to obtain the same information concerning their product.

On the set N of positive integers, let τ_1, τ_0, τ_{cf}, $\tau(\uparrow)$ and $\tau(\downarrow)$ denote respectively the discrete, trivial and cofinite topologies, the topology of final segments and that of initial segments. If τ is any one of these five and Y is an infinite subset of N, then clearly the subspace (Y, τ_Y) is homeomorphic to the whole space (N, τ). More significantly, Ginsburg and Sands [1] have demonstrated using Ramsey's theorem that every infinite space contains a homeomorphically embedded copy of at least one of these five, so that they are necessarily the only spaces (up to homeomorphism) enjoying this kind of minimality, and they may be perceived from a certain viewpoint as the 'atoms' of infinite topology. There have been a number of recent applications of this idea: we refer, for example, to Matier and McMaster's uses of it in exploring total negations of properties enjoyed by all finite spaces but not by all countable ones, [3], [4].

For brevity, let us take the phrases 'atom of a space X' to mean a minimal infinite space capable of being embedded into X, and 'atomic structure of X' to mean the list of the atoms of

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X. Once we know the atomic structure of finitely many spaces $X_1, X_2, \ldots, X_n$, then that of their disjoint union is immediately obtainable just by conflation. It might be expected that the same observation would apply to their product, but the truth is rather more interesting. Certainly, each atom of one of the X_i must be an atom of the product space, but the following example shows the converse to fail:

Example The diagonal line $\{(x,x) : x \in \mathrm{N}\}$ in the product space $(\mathrm{N}, \tau(\downarrow)) \times (\mathrm{N}, \tau(\uparrow))$ is discrete: as it also is in $(\mathrm{N}, \tau(\downarrow)) \times (\mathrm{N}, \tau_{cf})$. So (N, τ_1) is a 'surprise atom', occurring in the atomic structures of these two products but absent from the factor spaces. We shall see that there is a uniqueness about these two examples: surprise atoms in finite products are always attributable to one or other of them.

Lemma 1. *Let A be any infinite subset of $\mathrm{N} \times \mathrm{N}$. There is an infinite subset B of A which satifies one of the following conditions*
(a) all elements of B have the same first coordinate
(b) all elements of B have the same second coordinate
(c) no two elements of B have the same first coordinate and the second coordinate is a strictly increasing function of the first.

Proof: This is simple enough to demonstrate directly, but easier still as an application of Ramsey's theorem, [2, p.19]. Classify each two-element subset $\{(x_1,y_1),(x_2,y_2)\}$ of A as red, green, blue or yellow according as $x_1 = x_2$, $y_1 = y_2$, $(x_1 - x_2)(y_1 - y_2) > 0$ or $(x_1 - x_2)(y_1 - y_2) < 0$. Then there is an infinite subset B of A which is monochromatic (that is, every two-element subset of B has the same colour). The red, green and blue cases give (a), (b) and (c) respectively, while yellow leads to a contradiction since each positive integer has only finitely many predecessors.

Lemma 2. *Let τ, τ' be among the five 'minimal' topologies on N and satisfy $\tau \geq \tau'$, and let C be an infinite subset of N. Then every increasing injective map f from the subspace (C, τ_C) to (N, τ') is continuous.*

Proof: It suffices to check the case $\tau = \tau'$. When $\tau = \tau' = \tau_1$ or τ_0 the conclusion is trivial, in the τ_{cf} case continuity follows from injectivity, in the remaining two cases the increasing nature of f is what is needed.

Proposition 1. *Let* $(\mathbf{N},\tau)$ *and* $(\mathbf{N},\tau')$ *be two (not necessarily distinct) of the minimal infinite spaces and suppose that* $\tau \geq \tau'$. *Then there are no 'surprise atoms' in* $(\mathbf{N},\tau) \times (\mathbf{N},\tau')$.

Proof: Suppose that the subset A of $\mathbf{N} \times \mathbf{N}$ is homeomorphic to an atom. Choose B as in Lemma 1; in cases (a) and (b), B will be a homeomorph of an infinite subspace of $(\mathbf{N},\tau')$ or $(\mathbf{N},\tau)$, and therefore via minimality, A itself must be a copy of one of these two. In case (c), B is the graph of an increasing injective function

$$f : (C,\tau_C) \to (\mathbf{N},\tau')$$

whose domain is an infinite subspace of $(\mathbf{N},\tau)$ and whose continuity Lemma 2 establishes. It follows that C and B are homeomorphic, and a further appeal to minimality shows A to be a copy of $(\mathbf{N},\tau)$.

Next we show how the atomic structure of the product of two arbitrary spaces X and Y is determined by those of X and Y separately.

Proposition 2. *Let* M *be an atom of* $X \times Y$ *but neither of* X *nor of* Y. *Then*
(i) M *is discrete, and*
(ii) one of X *and* Y *has* $(\mathbf{N},\tau(\downarrow))$ *as an atom, the other one has either* $(\mathbf{N},\tau(\uparrow))$ *or* $(\mathbf{N},\tau_{cf})$ *as an atom.*

Proof: Take a subspace M' of $X \times Y$ which is a homeomorph of M. Arguing as in the proofs of Lemma 1 and Proposition 1, we find an infinite subset B of M' no two of whose points have either the same first or the same second coordinate. The first projection $\pi_1(B)$ contains a copy E of an atom of X. The second projection $\pi_2(\pi_1^{-1}(E) \cap B)$ of the points of B lying 'vertically above' E remains infinite, and contains a copy F of an atom of Y. So now $E \times F$ is a product of copies of atoms and encloses an infinite subspace of minimal M'. If the topologies of E and F are comparable, Proposition 1 yields a contradiction. If not, we are dealing with the product of $(\mathbf{N},\tau(\downarrow))$ by either $(\mathbf{N},\tau(\uparrow))$

or $(\mathbf{N},\tau_{cf})$; that $(\mathbf{N},\tau_1)$ is a surprise atom here has already been noted in the example, and another argument like that in the proof of Proposition 1 readily confirms that it is the only one.

Using induction, it is now routine to extend this proposition to apply to any finite numbe of factor spaces. We conclude:

Theorem. *The atomic structure of the product of a finite number of spaces is merely the conflation of their individual atomic structures unless:*
$(\mathbf{N},\tau(\downarrow))$ *is an atom of one factor space,*
$(\mathbf{N},\tau(\uparrow))$ *or* $(\mathbf{N},\tau_{cf})$ *is an atom of another, and*
no factor space has $(\mathbf{N},\tau_1)$ *as an atom.*
In the exceptional case, $(\mathbf{N},\tau_1)$ *is to be appended to the conflation of the lists.*

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MATRICES IN PERFECT CONDITION

David W. Lewis

We write $\mathrm{GL}(n, \mathbf{R})$ for the group of all non-singular $n \times n$ matrices with real entries. Let A be an element of $\mathrm{GL}(n, \mathbf{R})$ and let $\| \ \|$ be some norm on the real vector space $\mathbf{R}^n$. We define the *operator norm* of A in the usual way, as the supremum of the bounded set $S_A = \{ \|Av\|/\|v\| : v \in \mathbf{R}^n \text{ and } v \neq 0 \}$, and we denote it by $\|A\|$. The operator norm depends on the underlying norm on $\mathbf{R}^n$.

When the norm on $\mathbf{R}^n$ is the usual *euclidean norm*, that is

$$\|v\| = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2},$$

where $v = (v_i)$, then the corresponding operator norm is the *spectral norm*, so that $\|A\|$ is the square root of the largest eigenvalue of the matrix A^tA. When the norm on $\mathbf{R}^n$ is the *cartesian norm*, that is

$$\|v\| = \max_i |v_i|,$$

then the corresponding operator norm is the maximum absolute row sum norm, given by

$$\|A\| = \max_i (\sum_{j=1}^{n} |a_{ij}|).$$

When the norm on $\mathbf{R}^n$ is the *taxicab norm*, that is

$$\|v\| = \sum_{i=1}^{n} |v_i|,$$

then the corresponding operator norm is the maximum absolute column sum norm, given by

$$\|A\| = \max_j \left(\sum_{i=1}^{n} |a_{ij}| \right).$$

See [3] for proofs.

Definition. The *condition number* of the matrix A in $\mathrm{GL}(n, \mathbf{R})$ with respect to the operator norm $\| \ \|$ is the positive real number $c(A) = \|A\|\|A^{-1}\|$.

Note that $c(A)$ depends on which particular operator norm is in use and $c(A) \geq 1$ for all non-singular matrices A. (This last statement follows from the properties $\|AB\| \leq \|A\| \, \|B\|$ and $\|I\| = 1$, I denoting the identity matrix.)

We remark that condition numbers are important in perturbation theory and yield bounds for errors in numerical methods for solving systems of linear equations, inverting matrices, etc. See [1] and [3].

Definition. The matrix A in $\mathrm{GL}(n, \mathbf{R})$ is said to be *perfectly-conditioned* if $c(A) = 1$.

This definition, of course, depends on which norm is being used.

Most textbooks, including one by the author of this article, [3], say virtually nothing about perfectly-conditioned matrices beyond giving the definition and mentioning that orthogonal matrices are perfectly-conditioned for the spectral norm.

We write $G_{pc} = \{ A \in \mathrm{GL}(n, \mathbf{R}) : c(A) = 1 \}$, so that G_{pc} is the set of all perfectly-conditioned non-singular real $n \times n$ matrices. Here n is a fixed positive integer and our condition numbers are defined with respect to a fixed operator norm.

Lemma. *G_{pc} is a group under the operation of matrix multiplication, so that G_{pc} is a subgroup of* $\mathrm{GL}(n, \mathbf{R})$.

Proof: G_{pc} is a subset of $\mathrm{GL}(n, \mathbf{R})$ which contains I and which is closed under the operation of taking inverses since $c(A) = c(A^{-1})$. Thus it suffices to show that G_{pc} is closed under multiplication.

It is easy to see, via properties of operator norms, that $c(AB) \leq c(A)c(B)$ for any non-singular matrices A and B. Thus $c(AB) = 1$ whenever both $c(A) = 1$ and $c(B) = 1$, since $c(AB) \geq 1$ for all A and B. (If our norm is not an operator norm then G_{pc} need not be a group.)

We will determine the group G_{pc} in general and will specifically describe it for each of the examples of the operator norms given above.

Let $\| \ \|$ be a norm on $\mathbf{R}^n$ and write

$$G_{np} = \{ A \in \mathrm{GL}(n, \mathbf{R}) : \|Av\| = \|v\| \text{ for all } v \in \mathbf{R}^n \},$$

so that G_{np} is the group of all *norm-preserving linear operators* on $\mathbf{R}^n$. (It is an easy exercise to see that G_{np} is a subgroup of $\mathrm{GL}(n, \mathbf{R})$.)

We write $\mathbf{R}_p^*$ for the multiplicative group of all positive real numbers and we will regard $\mathbf{R}_p^*$ as a subgroup of $\mathrm{GL}(n, \mathbf{R})$ by identifying it with the set of all positive scalar multiples of the identity matrix.

Proposition. *Let $\| \ \|$ be a fixed norm on $\mathbf{R}^n$, let A be an element of $\mathrm{GL}(n, \mathbf{R})$, and let $c(A)$ be the condition number of A with respect to this norm. Then $c(A) = 1$ if and only if A is a non-zero scalar multiple of a norm-preserving linear operator on $\mathbf{R}^n$. Indeed the group G_{pc} is isomorphic to the direct product $\mathbf{R}_p^* \times G_{np}$.*

Proof: Consider the set S_A used in the definition of the operator norm $\|A\|$. Note that S_A is a closed and bounded subset of the positive real numbers. It is easy to see that $S_{A^{-1}} = \{ \alpha^{-1} : \alpha \in S_A \}$ because $w = Av$ if and only if $v = A^{-1}w$. Hence $\|A\| = \alpha_1$, where $\alpha_1 = \max S_A$ and $\|A^{-1}\| = \alpha_0^{-1}$ with $\alpha_0 = \min S_A$. It follows that $c(A) = \alpha_1/\alpha_0$, from which we see immediately that $c(A) = 1$ if and only if S_A is a singleton point set. Thus $c(A) = 1$ if and only if there exists a positive real number α such that $\|Av\| = \alpha\|v\|$ for all $v \in \mathbf{R}^n$. Writing $\alpha = \mu^2$ for some positive real number μ, we see that $(\pm\mu^{-1})A$ is a norm-preserving linear operator. It follows easily that G_{pc} is the direct product of the subgroups $\mathbf{R}_p^*$ and G_{np} of $\mathrm{GL}(n, \mathbf{R})$.

Example 1. Using the euclidean norm on $\mathbf{R}^n$, the group G_{np} is well-known to be the orthogonal group $\mathrm{O}(n) = \{ A \in \mathrm{GL}(n, \mathbf{R}) : AA^t = I \}$. Hence $G_{pc} = \mathbf{R}_p^* \times \mathrm{O}(n)$ in this case. Thus the matrices which are perfectly-conditioned with respect to the spectral norm are precisely the positive scalar multiples of the orthogonal matrices.

Example 2. Using the norm on $\mathbf{R}^n$ arising from an inner product given by some positive definite symmetric bilinear form ϕ, the group G_{np} equals $\mathrm{O}(\phi)$, the orthogonal group of ϕ, and $G_{pc} = \mathbf{R}_p^* \times \mathrm{O}(\phi)$ in this case. Note that if ϕ is represented with respect to the standard basis by the matrix B then $\mathrm{O}(\phi) = \{ A \in \mathrm{GL}(n, \mathbf{R}) : A^tBA = B \}$ and also that $\mathrm{O}(\phi)$ is isomorphic to $\mathrm{O}(n)$, because the form ϕ is isometric to the usual dot product on $\mathbf{R}^n$.

Example 3. Using the cartesian norm on $\mathbf{R}^n$, the group G_{np} turns out to be isomorphic to the wreath product $C_2 \wr S_n$, where C_2 is the cyclic group of order 2, and S_n is the symmetric group on n letters. (See [2, p.77] for the definition of wreath product.) We can see this as follows.

If $A \in G_{np}$, then $\|Av\| = \|v\|$ for all $v \in \mathbf{R}^n$. Hence writing $v = (v_i)$ and using the definition of the cartesian norm, the equation $\|Av\| = \|v\|$ becomes

$$\max\left(\left|\sum a_{1j}v_j\right|, \ldots, \left|\sum a_{nj}v_j\right|\right) = \max\left(|v_1|, \ldots, |v_n|\right)$$

for all $(v_1, \ldots, v_n) \in \mathbf{R}^n$. This equality can hold only if each row of A contains exactly one non-zero entry, this non-zero entry being equal to ± 1, and these non-zero entries are all in different columns. (Thus A is a so-called signed permutation matrix.) Examining the multiplication in the group of all such matrices we see that it yields the wreath product $C_2 \wr S_n$, which is the semi-direct product of S_n and C_2^n, where C_2^n is the direct product of n copies of C_2 and S_n acts in the obvious way on C_2^n by permuting factors.

Thus the group of matrices which are perfectly-conditioned with respect to the maximum absolute row sum norm is isomorphic to $\mathbf{R}_p^* \times (C_2 \wr S_n)$. As a set, G_{pc} consists of the positive scalar multiples of the signed permutation matrices.

Example 4. Note that the group G_{pc} in Example 3 is closed under the operation of transposition of matrices. It follows that this same group must also be the group of matrices which are perfectly-conditioned with respect to the maximum absolute column sum norm. ($\|A\|_c = \|A^t\|_r$, where $\| \ \|_c$ and $\| \ \|_r$ denote the maximum absolute column and row sum norms respectively.)

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AN ITERATION RELATED TO EISENSTEIN'S CRITERION

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The following question appeared in the 1994 Irish Mathematical Olympiad, the competition used to select the team to represent Ireland in the International Olympiad:
Let a, b and c be real numbers satisfying the equations:

$$b = a(4-a)$$
$$c = b(4-b)$$
$$a = c(4-c).$$

Find all possible values of $a+b+c$.

A direct approach to this problem is to write c in terms of a, and then obtain an octic polynomial in a:

$$f(a) \equiv -a(4-a)(2-a)^2((2-a)^2-2)+a=0.$$

The octic factorizes over the integers in the form

$$f(a) = a(a-3)(a^3-6a^2+9a-3)(a^3-7a^2+14a-7).$$

Observe that the factors a^3-6a^2+9a-3 and $a^3-7a^2+14a-7$ satisfy Eisenstein's irreducibility criterion for the primes 3 and 7, respectively. This, in our experience, was one of the rare occasions when polynomials satisfying the criterion arose in an uncontrived way, and we decided to investigate why they occurred here.

Put $g(x) \equiv x(4-x)$ and let $g^{(r)}(x)$ be the rth iterate $g(g(\ldots g(x)\ldots))$. Consider the polynomial $h_r(x) = x - g^{(r)}(x)$.

The octic $f(a)$ above is just $h_3(a)$. Observe that if we put $x = 4\sin^2\theta$ (where for definiteness we take $0 \le \theta \le \frac{\pi}{2}$), then $g(x) = 4\sin^2 2\theta$, and thus

$$h_r(x) = 4\sin^2 2^r\theta - 4\sin^2\theta = 2(\cos 2\theta - \cos 2^{r+1}\theta)$$
$$= 4\sin(2^r-1)\theta\sin(2^r+1)\theta.$$

So, if $(2^r \pm 1)\theta = l\pi$ for some positive integer l, we get solutions of the equation $h_r(x) = 0$. The two irreducible cubics dividing $f(x)$ are the irreducible polynomials satisfied by $4\sin^2\frac{\pi}{9}$ and $4\sin^2\frac{\pi}{7}$, respectively. The other factors x and $x-3$ are factors of $h_r(x)$ for all r, corresponding to the choices $\theta = 0$ and $\theta = \frac{\pi}{3}$, $l = \frac{1}{3}(2^r - (-1)^r)$, respectively.

In general, for each $k \ge 1$, 2^k+1 and 2^k-1 are relatively prime and for each divisor d of $(2^k-1)(2^k+1)$, with $1 < d < (2^k-1)(2^k+1)$, $4\sin^2\frac{\pi}{d}$ satisfies a monic irreducible polynomial $\psi_d(x)$ of degree $\varphi(d)/2$, where φ is Euler's function. Also, $\psi_d(x)$ must divide $h_k(x)$ and $x=0$ is a solution, corresponding to $d=1$. Thus

$$x \prod_{1<d|2^k-1} \psi_d(x) \prod_{1<d|2^k+1} \psi_d(x)$$

divides $h_k(x)$. The total degree of these polynomials is

$$\frac{1}{2}\left(\sum_{d|2^k-1}\varphi(d) + \sum_{d|2^k+1}\varphi(d)\right) = 2^k = \text{degree } h_k(x).$$

To calculate the irreducible polynomial satisfied by $4\sin^2\frac{\pi}{n}$, n odd, we use the following identity:

$$\frac{\sin n\phi}{\sin\phi} = \sum_{s=0}^{\frac{n-1}{2}} \frac{(-1)^s n(n+2s-1)(n+2s-3)\cdots(n-2s+1)}{(2s+1)!}\sin^{2s}\phi.$$

This may be written more compactly as

$$\frac{\sin n\phi}{\sin\phi} = \sum_{s=0}^{\frac{n-1}{2}}(-1)^s\frac{n}{2s+1}\binom{\frac{n-1}{2}+s}{2s}(4\sin^2\phi)^s.$$

For example, when $n=5$,

$$\frac{\sin 5\phi}{\sin\phi} = 5 - 5(4\sin^2\phi) + (4\sin^2\phi)^2$$

and when $n=7$,

$$\frac{\sin 7\phi}{\sin\phi} = 7 - 14(4\sin^2\phi) + 7(4\sin^2\phi)^2 - (4\sin^2\phi)^3.$$

Putting $z \equiv 4\sin^2\phi$, then if $\sin\phi \ne 0$ and $\sin n\phi = 0$, we get a monic polynomial $f_n(x)$ with integer coefficients and degree $\frac{n-1}{2}$ with $f_n(z) = 0$. The constant term of $f_n(z)$ is $\pm n$. This is obtained explicitly from the trigonometric identity above, using binomial identities, giving

$$f_n(x) = \sum_{i=0}^{\frac{n-1}{2}}(-1)^i\frac{n}{n-i}\binom{n-i}{i}x^{\frac{n-1}{2}-i}.$$

Suppose now that $n = p^k$, where p is an odd prime and $k \ge 1$. The expression for $f_n(x)$ above shows that all of the coefficients are divisible by p except the coefficient of $x^{\frac{n-1}{2}}$ and the constant term is $\pm p^k$. But the irreducible polynomials satisfied by $4\sin^2\frac{\pi}{p}$, $4\sin^2\frac{\pi}{p^2}, \ldots, 4\sin^2\frac{\pi}{p^k}$ must all divide $f_n(x)$ and the sum of the degrees of these polynomials is $\frac{n-1}{2}$. Thus

$$f_n(x) = \psi_p(x)\psi_{p^2}(x)\cdots\psi_{p^k}(x).$$

We now show by induction on k that ψ_{p^k} satisfies Eisenstein's criterion for the prime p. Since $f_p(x) = \psi_p(x)$, this is clear when $k=1$. The equation

$$f_{p^k}(x) = f_{p^{k-1}}(x)\psi_{p^k}(x)$$

yields

$$x^{\frac{(p^k-1)}{2}} \equiv x^{\frac{(p^{k-1}-1)}{2}}\psi_{p^k}(x) \bmod p,$$

so all coefficients of ψ_{p^k} except its leading coefficient are divisible by p. Furthermore, the constant term of $f_{p^k}(x)$ is $\pm p^k$, and that of $f_{p^{k-1}}(x)$ is $\pm p^{k-1}$, so the constant term of $\psi_{p^k}(x)$ is $\pm p$. Hence $\psi_{p^k}(x)$ satisfies Eisenstein's criterion for p. This explains our initial observations concerning the polynomials

$$\psi_7(x) = x^3 - 7x^2 + 14x - 7$$

and

$$\psi_9(x) = x^3 - 6x^2 + 9x - 3.$$

Let ω be a primitive p^kth root of unity (for definiteness, we can take $\omega = \exp(\frac{2\pi i}{p^k})$) and let $K = \mathbf{Q}(\omega)$ be the corresponding cyclotomic field. Let $L = \mathbf{Q}(\omega+\omega^{-1})$ be the maximal real subfield of K and let A be the ring of algebraic integers in L. One can show that $\mathbf{Z}[4\sin^2(\frac{\pi}{p^k})]$ has finite index p^c in A for some integer $c \geq 0$. But now the fact that the irreducible polynomial $\psi_{p^k}(x)$ satisfied by $4\sin^2(\frac{\pi}{p^k})$ is of Eisenstein type enables us to apply Lemma 2.3 of [1, p.61] to conclude that $c = 0$. So

$$A = \mathbf{Z}[4\sin^2(\frac{\pi}{p^k})] = \mathbf{Z}[2\cos(\frac{2\pi}{p^k})] = \mathbf{Z}[\omega + \omega^{-1}].$$

Finally, we briefly consider the orbit length of the iteration of the map $a \to a(4-a)$, beginning with $a = 4\sin^2(\frac{\pi}{n})$, where n is an odd integer. We obtain successively $4\sin^2(\frac{\pi}{n})$, $4\sin^2(\frac{2\pi}{n})$, $4\sin^2(\frac{2^2\pi}{n})$, ... and the period is r, where r is the least positive integer such that

$$\frac{2^{r+1}\pi}{n} \pm \frac{2\pi}{n}$$

is an integral multiple of 2π. (For example, when $n = 17$, $r = 4$ and when $n = 19$, $r = 9$.) Note that r is the least positive integer such that $2^r \equiv \pm 1 \bmod n$. So, if the equation $2^t \equiv -1 \bmod n$ is solvable, then r is half the order of 2 mod n while, if it is not solvable, r is the order of 2 mod n. If $n = p^k$, where p is an odd prime and k is a positive integer, the equation $2^t \equiv -1 \bmod n$ is solvable if and only if the order of 2 mod n is even, so in particular,

$2^t \equiv -1 \bmod n$ is solvable if 2 is not a quadratic residue modulo p, that is if $p \equiv \pm 3 \bmod 8$.

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JOSEPH WOLSTENHOLME, LESLIE STEPHEN AND 'TO THE LIGHTHOUSE'

Rod Gow

Readers of the *Bulletin* may perhaps recall an earlier article of this author, on the subject of collecting mathematical books, [5]. In that article, we mentioned having acquired copies of *Mathematical Problems* by Joseph Wolstenholme and we described how we had used the *Dictionary of National Biography* (DNB) to find out about Wolstenholme's life and work. Since writing the article, we have discovered how Wolstenholme has acquired a certain literary fame, through the novel *To the Lighthouse* by Virginia Woolf. We thought it may be of interest for mathematicians to read about how Wolstenholme came to be connected with this novel and so we present here what we have read concerning Wolstenholme and Virginia Woolf's father, Leslie Stephen. The information about Wolstenholme that we have used is all taken from published sources, but it may not be well known to the mathematical community.

We begin by quoting from the DNB article on Wolstenholme. Wolstenholme was born on 30 September 1829 in Eccles, Manchester. He entered St John's College, Cambridge in 1846 and graduated as Third Wrangler in 1850. He was elected a fellow of his college in 1852 but then took up a fellowship at Christ's College, Cambridge in the same year. He vacated his fellowship upon his marriage in 1869 to a Swiss woman, Thérèse Kraus. Previous to his marriage, he had served four times as an examiner for the mathematical tripos. In 1871, he was appointed professor of mathematics at the Royal Indian Engineering College at Cooper's Hill, near London, retiring in 1889. He died on 18 November 1891, leaving a widow and four sons.

The Royal Society Catalogue of Scientific Papers lists 23 papers by Wolstenholme, these being mainly on geometric subjects. His name is attached to an elementary result in number theory, known as Wolstenholme's Theorem, which is described in [6, pp.88-90], and may be stated thus. Let $p > 3$ be a prime. Then p^2 divides the numerator of the fraction

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}.$$

However, in a note in [6, p.93], it is observed that a result of this kind already existed in Edward Waring's *Meditationes Algebraicae* of 1782.

The DNB article includes an appraisal of Wolstenholme's work by Andrew Forsyth, Sadlerian Professor of Pure Mathematics at Cambridge University. He writes as follows:

> ...his fame rests chiefly on the wonderful series of original mathematical problems which he constructed upon practically all the subjects that entered into the course of training students twenty-five or thirty years ago. They are a product characteristic of Cambridge, and particularly of Cambridge examinations; he was their most conspicuous producer at a time when their vogue was greatest. When gathered together from many examination papers so as to form a volume, which was considerably amplified in its later edition, they exercised a very real influence upon successive generations of undergraduates; and "Wolstenholme's Problems" have proved a help and a stimulus to many students. A collection of some three thousand problems naturally varies widely in value, but many of them contain important results, which in other places or at other times would not infrequently have been embodied in original papers. As they stand, they form a curious and almost unique monument of ability and industry, active within a restricted range of investigation.

It should be noted that Forsyth campaigned to reform the Cambridge mathematical syllabus, which he saw as outdated and out of touch with the developments in mathematics that had occurred on the continent in the course of the 19th century, and entirely subservient to the competitional aspects of the mathematical tripos examination. Wolstenholme's books may well have typified for him the moribund Cambridge mathematical tradition with its emphasis on the solving of ingenious problems, instead of developing proper structures and theories.

The first edition, [8], of *Mathematical Problems* was published in 1867 and contained 1,628 problems, mainly geometrical. The second edition, [9], appeared in 1878, the book being in a larger format, and it contained 2,815 problems. A third edition of 1891 was largely a reprint with corrections of the second edition. As we stated in our previous article, the problems do not translate well into the modern syllabus and have too great a bias to geometry and sometimes a rather imprecise formulation.

We turn now to the second person mentioned in our title. Leslie Stephen may perhaps be best known nowadays as the father of Virginia Woolf and as the first editor of the DNB. In his own day, he was a major figure in literary circles in London, knowing many of the leading American and British writers and thinkers of the second half of the 19th century. We will briefly provide some details on his life. Leslie Stephen was born on 28 November 1832 and was the son of Sir James Stephen, a politician and Professor of Modern History at Cambridge University (1849-59). His mother was the daughter of the Reverend John Venn and the aunt of John Venn, the logician, whose name is associated with the set-theoretic diagrams. Thus Stephen and Venn were cousins. He entered Trinity Hall, Cambridge in 1850 and studied mathematics. To do well in the mathematical tripos, most students had to undertake intensive coaching to prepare for the rigours of the examination, which required them to repeat numerous sections of bookwork quickly and accurately, before engaging in tricky problems in the later stages of the papers. Stephen's coach was Isaac Todhunter, a famous figure in British mathematics of the 19th century. Todhunter wrote numerous textbooks for schools and universities, which must have sold well, as most went through several editions. His name is associated with four major histories of mathematical subjects, including a history of the theory of probability and a history of the theory of Newtonian gravitational attraction. An enormous amount of effort went into the production of these scholarly works, which tried to describe

virtually every relevant contribution to the subjects up to the time of Laplace. It is interesting to see what Stephen thought of Todhunter, who was clearly something of a character in Cambridge mathematical circles and about whom several anecdotes have survived (see, for example, [3] and [7, pp.87-88]). Stephen wrote as follows about Todhunter, [1, p.27]:

> He lived in a perfect atmosphere of mathematics; his books, all ranged in the neatest order, and covered with uniform brown paper, were mathematical; his talk, to us at any rate, was one round of mathematics; even his chairs and tables strictly limited to the requirements of pupils, and the pattern on his carpet, seemed to breathe mathematics. By what mysterious process it was that he accumulated stores of miscellaneous information and knew all about the events of the time (for such I afterwards discovered to be the fact) I have never been able to guess. Probably he imbibed it through the pores of his skin. Still less can I imagine how it came to pass that he published a whole series of excellent educational works. He probably wrote them in momentary interstices of time between one pupil's entering his sanctum and another leaving it.

Stephen performed reasonably well in the tripos examination of January 1854, achieving the position of Twentieth Wrangler and obtaining a first class degree. The tripos list was of a high quality that year, as the Senior Wrangler was E. J. Routh, later to be author of several mechanics textbooks and the most successful mathematical coach of all time in Cambridge, and the Second Wrangler was J. C. Maxwell. Stephen obtained a fellowship at Trinity Hall in 1854, on the strength of his tripos results, and he remained at Cambridge until 1864. He got to know Joseph Wolstenholme during this time. Annan writes of Stephen in [1, p.54]:

> So far we have seen him as a man with many younger cronies but few intimate friends; deeply attached only to Fawcett or to some odd Cambridge fish such as Joseph Wolstenholme, a mathematician and walker who had the gift of being able to spout thousands of lines of poetry by heart, as the evening fell and the pair of them pounded the last ten miles of the grind back to Cambridge.

In 1864, Stephen left Cambridge to embark on a literary career in London. He wrote articles for reviews and literary magazines and eventually became the editor of the *Cornhill Magazine* in 1871. He published works of literary criticism and wrote on the history of philosophy. A list of his publications may be found in [4]. He was appointed editor of the projected *Dictionary of National Biography* in 1882 and retained the editorship until 1889. The first volume of the DNB appeared in 1885. He died on 22 February 1904.

When Stephen's second wife, Julia, died in 1895, he wrote a long autobiographical letter to his wife's children (three children by her first marriage and four by Stephen). This letter was known by the children as the *Mausoleum Book*. It was published in 1977, [2]. In the *Mausoleum Book*, [2, p.79], he writes:

> I think especially of poor old Wolstenholme, called 'the woolly' by you irreverent children, a man whom I had first known as a brilliant mathematician at Cambridge, whose Bohemian tastes and heterodox opinions had made a Cambridge career unadvisable, who had tried to become a hermit at Wastdale. He had emerged, married an uncongenial and rather vulgar Swiss girl, and obtained a professorship at Cooper's Hill. His four sons were badly brought up; he was despondent and dissatisfied and consoled himself with mathematics and opium. I liked him or rather was very fond of him, partly from old association and partly because feeble and faulty as he was, he was thoroughly amiable and clung to my friendship pathetically. His friends were few and his home life wretched. Julia could not help smiling at him; but she took him under protection, encouraged him and petted him, and had him stay every summer with us in the country. There at least he could be without his wife.

Thus a rather different picture of Wolstenholme emerges from Stephen's own pen, compared with that given in the DNB, Stephen's former undertaking (the article on Wolstenholme was written after Stephen's resignation from the DNB and is unsigned). With regard to the statement above about having Wolstenholme to stay every summer in the country, Stephen had a house in St Ives in Cornwall where his family and guests, sometimes distinguished, would assemble for the holidays. Wolstenholme must have made some sort of impression on

Virginia Stephen (later Woolf), who was born in 1882 and thus quite young when Wolstenholme used to visit, as it seems to be agreed that the character of Mr Augustus Carmichael in *To the Lighthouse*, published in 1927, was based on Wolstenholme. See, for example, the introduction to [10], page x.

Annan wrote of Wolstenholme in [1, p.294]:

> In old age he became something of a bore and Stephen irritated his family by asking the lonely old bachelor (*sic*) to stay in Cornwall with them for the holidays and then, finding his company tedious, leaving wife and daughters to entertain him. Wolstenholme was present on the summer holiday in Cornwall (see *To the Lighthouse* Part III), of which Stephen wrote to C. E. Norton, 21 Sept., 1899, 'I have lost the power of holiday making'.

Of course, the statement that Wolstenholme was a bachelor is incorrect, as we have seen.

The novel draws on Virginia Stephen's experiences of family holidays in St Ives and the tension generated by her father's difficult temperament and moods, and his wife's attempts to placate him. Carmichael (who is not portrayed in any respect as a mathematician) is interested in Persian poetry and has a not insignificant role in the novel. *To the Lighthouse* is one of Virginia Woolf's most highly regarded works and analyses of it certainly mention Carmichael's significance in the story. We feel that it is amusing to see how a mathematician, albeit not of the greatest importance, has achieved some fame as a footnote to an important novel.

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Research Announcement

A UNIFORMLY CONVERGENT METHOD FOR A SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM WITH NONUNIQUE SOLUTIONS

Guangfu Sun and Martin Stynes

We analyse a simple central difference scheme for a singularly perturbed semilinear reaction-diffusion problem that may have nonunique solutions. Asymptotic properties of solutions to this problem are examined. To compute accurate approximations to these solutions, we consider a piecewise equidistant mesh of Shishkin type, which contains $O(N)$ points. On such a mesh, we prove existence of a solution to the discretization and show that it is accurate of order $N^{-2}\ln^2 N$, in the discrete maximum norm, where the constant factor in this error estimate is independent of ε and N. Numerical results are presented which verify this rate of convergence. Full details appear in [1].

Reference

[1] G. Sun and M. Stynes, *A uniformly convergent method for a singularly perturbed semilinear reaction-diffusion problem with nonunique solutions* (1993). (Preprint 1993-11, Mathematics Department, University College Cork.)

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Research Announcement

AN ALMOST FOURTH ORDER UNIFORMLY CONVERGENT DIFFERENCE SCHEME FOR A SEMILINEAR SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM

Guangfu Sun and Martin Stynes

We analyse a high–order convergent discretization for the semilinear reaction–diffusion problem: $-\varepsilon^2 u'' + b(x,u) = 0$, for $x \in (0,1)$, subject to $u(0) = u(1) = 0$, where $\varepsilon \in (0,1]$. We assume that $b_u(x,u) > b_0^2 > 0$ on $[0,1] \times \mathcal{R}^1$, which guarantees uniqueness of a solution to the problem. Asymptotic properties of this solution are discussed. We consider a polynomial–based three–point difference scheme on a simple piecewise equidistant mesh of Shishkin type. Existence and local uniqueness of a solution to the scheme are analysed. The scheme is shown to be almost fourth order accurate in the discrete maximum norm, uniformly in the perturbation parameter ε. Numerical results are presented in support of this result. Full details appear in [1].

Reference

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Research Announcement

FINITE VOLUME METHODS FOR CONVECTION-DIFFUSION PROBLEMS

Martin Stynes

An overview is given of the nature of convection-diffusion problems, and of the use of finite volume methods in their solution. Full details appear in [1].

Reference

[1] M. Stynes, *Finite volume methods for convection-diffusion problems* (Invited plenary lecture at MODELLING 94 Conference, Prague, Czech Republic, 29.8-2.9.1994). (Preprint 1994-4, Mathematics Department, University College Cork.)

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Book Review-Survey

Dynamical Systems VI

Singularity Theory I

Encyclopaedia of Mathematical Sciences Vol. 6

edited by V. I. Arnold

Springer-Verlag 1993, 245 pp.

ISBN 0-387-50583-0

Price DM 141.00.

Dynamical Systems VIII

Singularity Theory II: Classification and Applications

Encyclopaedia of Mathematical Sciences Vol. 39

edited by V. I. Arnold

Springer-Verlag 1993, 235 pp.

ISBN 0-387-53776-1

Price DM 141.00.

Reviewed by Charles Nash

§1. Introduction

I shall review both the above books together since they are parts I and II of a treatment of singularity theory. For brevity I shall also refer to them as part I and part II respectively.

First a few formal preliminaries about the origin of the books, their authors and the nature of their expository methods.

The books are translations from Russian and appeared, in that language, in 1988 and 1989 respectively. They are both *edited by* Vladimir Arnold but are *multi-authored*; however, any given chapter has, in the main, a single author. The same authors wrote parts I and II and are: *V. I. Arnold, V. V. Goryunov, O. V. Lyashko and V. A. Vasil'ev.* The preface of each book gives

the precise authorship details of each individual chapter and also informs us that *B. Z. Shapiro* wrote a little bit of part II. The translators are *A. Iacob* of Mathematical Reviews and *J. S. Joel* respectively. Finally we come to the matter of exposition.

As the phrase *Encyclopaedia of Mathematical Sciences* above indicates, they belong to a mathematical encyclopaedia, being volumes 6 and 39 thereof. This encyclopaedia, which is a translation from a Russian original, is under the general editorship of *R. V. Gamkrelidze.* Its style therefore is expository and the books are a survey of their subject matter. This means that theorems are almost always stated rather than proved; it also means that the books are about 250 pages long instead of being several times that length.

The authors are recognized experts in their fields and so are ideal choices to write such a survey. In addition Arnold, who is the senior author because of his prominent position in singularity theory, has already written many books and so has a good written style. Vasil'ev (Vassiliev) has recently made a big advance in applying singularity theory to knot theory, about which more below. The text of the book is liberally sprinkled with illustrative examples and so the style is not heavy going or turgid; nor is the significance, and relative importance, of the various theorems left totally to the reader to fathom. On the subject of indexes, each volume has an author and a subject index but in both cases the latter is far too short, especially so for reference books belonging to an encyclopaedia. The intended audience is a "student reader" who wishes to learn the subject, be he a mathematician, or a theoretical or mathematical physicist.

Let us place the present two books on singularity theory in context by first discussing dynamical systems themselves—that done we shall move on to singularity theory and the books under review.

§2. Historical background and origins

The founder of the modern theory of dynamical systems was Poincaré, cf. "Les méthodes nouvelles de la mécanique céleste", [1]. Poincaré was interested in answering questions about the *qualit-*

ative behaviour of the orbits of celestial bodies: for example one asks what happens to the planets if their orbits are perturbed slightly? Can the orbits remain stable, change wildly, fall into the Sun or rearrange in some new way? The difficulty of solving even the three body problem analytically meant that methods which could classify the qualitative behaviour were highly desirable.

What emerges is that, for n bodies, with $n \geq 3$, as the initial conditions vary, the orbits can be *chaotic* as well as *regular*: Chaotic motion can be exhibited by an asteroid close to what is known as a Kirkwood gap; for this initial data, its eccentricity can jump in a random manner and, in time, become larger and a fatal collision with a planet can occur. Regular motion is exhibited by a planet such as the the Earth; its initial data is such that its ecliptic plane oscillates a little around a fixed position. For more details cf. [2-4].

Poincaré's pioneering work then gave birth to the present day subject of dynamical systems. In this subject one studies an immense diversity of sophisticated mathematical problems usually no longer connected with celestial or Newtonian mechanics.

A very rough idea of what is involved goes as follows: Recall that the celestial mechanics of n bodies has a motion that is described by a set of differential equations together with their initial data. One then varies the initial data and asks how the motion changes.

§3. Dynamical systems in general

The modern mathematical setting is to view the orbits of the n bodies as integral curves for their associated differential equations. Then one regards the *qualitative study* of the orbits as being a study of the *global geometry* of the space of integral curves as their initial conditions vary smoothly. Integral curves $\gamma(t)$ are associated with vector fields $V(t)$ via the differential equation

$$\frac{d\gamma(t)}{dt} = V(\gamma(t)) \tag{3.1}$$

Hence one is now studying the vastly more general subject of the global geometry of the space of flows of a vector field V on a manifold M.

It turns out that two notions play a distinguished part in the theory of dynamical systems. One fundamental notion that emerges from the example treated below is the existence of a closed integral curve. A second notion, also fundamental, is that of a *singular point* which will be dealt with in the next section.

It is natural to regard two flows on M as *equivalent* if there is a homeomorphism of M which takes one flow into the other; one can also insist that this homeomorphism is smooth, i.e. a diffeomorphism. Finally an equivalence class of flows in the homeomorphic sense is a *topological dynamical system*, and one in the diffeomorphic sense is a *a smooth, or differentiable, dynamical system.*

A further key concept in dynamical systems is that of *structural stability* and to illustrate this we introduce the following example.

Example *The pendulum with friction*

Consider a simple pendulum subject to friction, [5]. One has to solve the second order differential equation

$$\ddot{x} = -x - \mu\dot{x} \tag{3.2}$$

where $\mu \geq 0$ is the coefficient of friction. This is equivalent to solving the pair of first order equations

$$\dot{x} = y, \qquad \dot{y} = -x - \mu y \tag{3.3}$$

A solution to this pair of equations is a curve in the (x, y)-plane and so is also a flow line of the vector field V on $\mathbf{R}^2$ whose components are just $(\dot{x}, \dot{y})$. Thus eq. (3.3) is now of the form (3.1) above with V as just given and $M = \mathbf{R}^2$.

Now it is easy to compute that for μ *strictly positive* the solutions are spirals winding round the origin; but when μ is *zero* the solutions are circles centered at the origin. In other words, a big qualitative change in the trajectories takes place if the pendulum is perturbed, μ increasing μ from zero to some positive value; however if μ is perturbed but stays positive then no qualitative change occurs.

One then says that the simple pendulum with $\mu > 0$ is *structurally stable* but the simple pendulum with $\mu = 0$ is *structurally unstable*.

Thus structurally stability of a dynamical system corresponds to its equivalence under a small perturbation of V.

We now turn to the second fundamental notion of dynamical systems, which is also the subject matter of the books under review, that of singular points.

§4. Singular points and dynamical systems: vector fields

For a vector field V, a singular point is just a point on M where V vanishes. We note that a closed integral curve cannot have a singular point. There are also topological restrictions on the nature and type of singular points of V: Suppose, for simplicity, that M is closed and compact. Then a celebrated and well known result is that the index* $i(V)$ of V is equal to the Euler characteristic $\chi(M)$ of M.

Singular points of V are also closely tied to structural stability, the key point is to study whether they are degenerate or not. The result (loosely) is that a structurally stable system only possesses non-degenerate singular points. The underlying intuition is not too difficult to explain: Consider a vector field V on $\mathbf{R}^2$, say with a non-degenerate zero at $z_0 \in \mathbf{R}^2$ so that, near z_0, V behaves like

$$(z - z_0) \tag{4.1}$$

If we perturb V slightly to a new vector field V_ϵ, then we can write

$$V \longmapsto V_\epsilon = V + \epsilon f(z), \qquad \epsilon \text{ small} \tag{4.2}$$

Clearly V_ϵ also has a non-degenerate zero at the nearby location $z_0 - \epsilon f(z_0)$ (if desired, the implicit function theorem can be used to

* $i(V)$ is the total number of singular points of V, it is an algebraic sum with signs and degeneracies taken into account and assumes that the zeroes are isolated.

create a rigorous version of this argument). Hence non-degenerate singular points perturb to new ones and do not change their total number. By contrast if the zero at z_0 is *degenerate* then, near z_0, V behaves like

$$(z - z_0)^n, \qquad n > 1 \tag{4.3}$$

So the perturbed vector field V_ϵ looks like

$$(z - z_0)^n + \epsilon f(z_0), \qquad \text{near } z_0 \tag{4.4}$$

But, in general,

$$(z - z_0)^n + \epsilon f(z_0) = (z - z_1)(z - z_2)\cdots(z - z_n) \tag{4.5}$$

Hence, on perturbation, the degenerate zero has *bifurcated* into n non-degenerate zeroes. Actually, more generally, degenerate zeroes, can even *disappear altogether* on perturbation because the bifurcation process may produce only complex zeroes which may not belong to the particular M under consideration.

In sum the perturbation of a system with one or more degenerate singular points is structurally unstable, and so we recover the fact that all the singular points of a structurally stable dynamical system are non-degenerate.

§5. Singular functions: the real case

As well as singular points of vector fields the study of dynamical systems requires us to consider singular points of functions. By a singular point of a function f we mean a *critical point*, or extremum, of f.

For example let M be a manifold and f a smooth real valued* function on M

$$f : M \longrightarrow \mathbf{R} \tag{5.1}$$

* As we shall see below both f and M can be generalized considerably: For f we can generalize to complex valued functions $f : M \to \mathbf{C}$ and even maps of the form $f : M \to N$, where N is another manifold. For M we should start with an M which is closed and then generalize to the case where M has a boundary; in fact cases where M is *infinite dimensional* arise naturally and are important, one of these latter is the original problem of Morse cf. § 8.

then, if p is a point in M with local coordinates $(x^1, x^2, \ldots, x^n)$, p is a *critical point* of f if

$$\left.\frac{\partial f}{\partial x^1}\right|_p = \left.\frac{\partial f}{\partial x^2}\right|_p = \cdots = \left.\frac{\partial f}{\partial x^n}\right|_p = 0 \tag{5.2}$$

or, in a more concise notation,

$$df = 0 \quad \text{at } p \tag{5.3}$$

Example *Gradient dynamical systems*

Using such a function $f : M \to \mathbf{R}$ we obtain an important class of dynamical systems known as *gradient dynamical systems*: We require M to have a (Riemannian) metric so that the grad operator is defined and then the flow equation is that of gradient flow

$$\frac{d\gamma(t)}{dt} = \operatorname{grad} f(\gamma(t)) \tag{5.4}$$

so that $V = \operatorname{grad} f$ and f is like a potential function. We see that the flow begins and ends at singular points of f.

We shall now discuss some of the theory of singularities of functions such as f from a qualitative topological viewpoint; for *real valued* functions this is known as Morse theory. The aim in Morse theory is to study the relation between critical points and topology. More specifically one extracts topological information from a study of the critical points of a smooth real valued function

$$f : M \longrightarrow \mathbf{R}, \tag{5.5}$$

where M is an n-dimensional compact manifold, without boundary. For a suitably behaved class of functions f, there exists quite a tight relationship between the number and type of critical points of f and topological invariants of M such as the Euler–Poincaré characteristic, the Betti numbers and other cohomological data. This relationship can then be used in two ways: one can take certain special functions whose critical points are easy to find and

use this information to derive results about the topology of M; on the other hand, if the topology of M is well understood, one can use this topology to infer the existence of critical points of f in cases where f is too complicated, or too abstractly defined, to allow a direct calculation.

We begin with the smooth function $f : M \to \mathbf{R}$ and assume* that all the critical points p of f are distinct and non-degenerate; the non-degeneracy means that the Hessian matrix Hf of second derivatives is invertible at p, or

$$\det Hf(p) \neq 0 \quad \text{where} \quad Hf(p) = \left[\partial^2 f/\partial x^i \partial x^j\big|_p\right]_{n\times n} \tag{5.6}$$

Each critical point p has an index λ_p which is defined to be the number of *negative* eigenvalues of $Hf(p)$. In a neighbourhood of a non-degenerate critical point p of index λ_p we can represent f as

$$f(x) = f(p) - \overbrace{x_1^2 - x_2^2 - \cdots - x_{\lambda_p}^2}^{\lambda_p \text{ terms}} + \underbrace{x_{\lambda_p+1}^2 + \cdots + x_n^2}_{n-\lambda_p \text{ terms}} \tag{5.7}$$

for suitable coordinates $(x_1, \ldots, x_n)$.

We next associate to the function f and its critical points p the Morse series $M_t(f)$ defined by

$$M_t(f) = \sum_{\text{all } p} t^{\lambda_p} = \sum_i m_i t^i. \tag{5.8}$$

The sum will always converge since it only contains a finite number of terms; this is because the non-degeneracy makes the critical points all discrete and the compactness of M permits only a finite number of such discrete points. The topology of M now enters via

* Such functions are called Morse functions and it should be clear from what we have said earlier that when f is not a Morse function one can always perturb it slightly to obtain one.

$P_t(M)$: the Poincaré series of M. This is the following polynomial constructed out of the Betti numbers of M; we have

$$P_t(M) = \sum_{i=0}^{n} \dim H^i(M; \mathbf{R})t^i = \sum_{i=0}^{n} b_i t^i. \tag{5.9}$$

The fundamental result of Morse theory, known as the Morse inequalities, is the statement that

$$M_t(f) - P_t(M) \geq 0 \tag{5.10}$$

This can be refined further to say that

$$M_t(f) - P_t(M) = (1+t)R(t), \tag{5.11}$$

where $R(t)$ is a polynomial with only non-negative coefficients.

We note in passing two facts that can be read off immediately from this pair of statements. If we set $t = 1$ in the first one, we see that any (Morse) function f has at least $\sum_{i=0}^{n} b_i$ critical points. If we set $t = -1$ in the second one then we see that

$$M_{-1}(f) = P_{-1}(M) = \sum_{i=0}^{n} (-1)^i b_i = \chi(M), \tag{5.12}$$

where $\chi(M)$ is the Euler characteristic of M. Note that the first of these facts describes a property of f, while the second is completely independent of f and is only a property of M.

A proof of the Morse inequalities usually uses the level sets of the function f: these are the sets $f^{-1}(c) = \{x \in M : f(x) = c\}$. We shall briefly sketch the part that they play in determining the topology of M. In Morse theory one constructs a half space M_c out of level sets where

$$M_c = \{x \in M : f(x) \leq c\}. \tag{5.13}$$

The topology of M begins to emerge when we consider M_c as a function of c. What happens is that, as c varies, the topology of

M_c is unchanged until c passes through a critical point, when it either acquires or sheds a cell of dimension λ, where λ is the index of the critical point. More precisely we have

Theorem (Bott–Morse–Smale) *M_a is diffeomorphic to M_b if there is no critical point in the interval $[a, b]$. Alternatively, if (a, b) contains just one critical point of index λ then $M_b \simeq M_a \cup e_\lambda$.*

The notation $M_a \cup e_\lambda$ means that a cell of dimension λ has been attached to M_a; also $M_b \simeq M_a \cup e_\lambda$ means that the two spaces have the same homotopy type. Thus, as far as the homotopy type of M is concerned (and this will be sufficient, for example, for computing the cohomology of M) one can think of M as being 'decomposed' into a set of cells

$$M = \bigcup_{\lambda} e_\lambda, \tag{5.14}$$

the number of these cells being equal to the number of critical points and the dimension of the cells being given by the index of the critical points. This decomposition is known as a *stratification* of M.

§6. Singular functions: the complex case

Now suppose that f is *complex valued* instead of real valued i.e. we have

$$f : M \longrightarrow \mathbf{C} \tag{6.1}$$

A corresponding complex analogue of Morse theory exists, known as *Picard-Lefschetz* theory. The content of the theory is quite different: Clearly the complex values of f render it impossible to define the index of a critical point any more; not surprisingly, in view of this, the critical points cease to provide a stratification M using the level sets $f^{-1}(c)$. In fact the level sets no longer undergo a topological change as c passes through a critical point—they are actually all homeomorphic to one another.

In the complex case what one does instead of passing *through* a critical point is to deform one's path to go *round* it; the obvious topology relevant in this setting resides in the winding number of a *closed path*, or cycle, round the singularity or critical point.

This results in integrals round closed cycles which in turn are continuous functions on the parameter space; the analysis of such an object is known as the *monodromy* of the singularity More technically, the level sets over a small circle surrounding a singular point form a fibre bundle (since they are all topologically identical) over S^1, and the monodromy is then the holonomy of the fibre corresponding to going round this circle once.

§7. Singular maps

Next, suppose that we replace $\mathbf{C}$ by a *manifold* N (both M and N are, for the moment, assumed to be real manifolds) giving the map

$$f : M \longrightarrow N, \quad \dim M = n, \dim N = m \tag{7.1}$$

Let us use local coordinates $(f^1, f^2, \ldots, f^m)$ to represent $f(x)$ on N, and $(x^1, x^2, \ldots, x^n)$ to represent x on M. A singularity of f is now defined using its Jacobian matrix

$$J = \left[\frac{\partial f^i}{\partial x^j}\right]_{m\times n} \tag{7.2}$$

rather than the operator d: a singularity of f is a point on M where J has less than its *maximal rank*. In this setting, the topology of the theory involves the Stiefel-Whitney characteristic classes $w_i(M) \in H^i(M; \mathbf{Z}_2)$ of M and the pullback, via f^*, of those of N. Universal polynomials known as *Thom polynomials* provide calculational formulae for these pullbacks. If we generalize to the case where M and N are complex manifolds then the Stiefel-Whitney classes are replaced by the Chern classes $c_i(M) \in H^{2i}(M; \mathbf{Z})$ of M and those of N.

§8. Singular points in infinite dimensions

A brief mention now, as promised, of some examples where M is infinite dimensional. The original problem of Morse, [6], was to study the critical points of the energy functional E defined by

$$E(\gamma) = \int_0^1 \left|\frac{d\gamma(t)}{dt}\right|^2 dt \equiv \int_0^1 g_{ij}\frac{d\gamma^i(t)}{dt}\frac{d\gamma^j(t)}{dt}dt \tag{8.1}$$

where $\gamma(t)$ is a parametrized path on M with end points p and q labelled by 0 and 1, and g_{ij} is the Riemannian metric on M. E is a function or functional of $\gamma(t)$. Hence E is a positive real valued function on the space $PM(p,q)$ of paths on M from p to q. More formally we can represent E as

$$\begin{aligned} E : PM(p,q) &\longrightarrow \mathbf{R} \\ \gamma &\longmapsto E(\gamma) \end{aligned} \tag{8.2}$$

The space $PM(p,q)$ is of course infinite dimensional. The critical points of E are easily seen to be the geodesics joining p to q with the usual equation

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk}\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = 0, \tag{8.3}$$

where Γ^i_{jk} are the components of the Christoffel symbol for the metric g_{ij}. To consider *closed* geodesics we simply require γ to be a loop, on M; this means that we take elements of $Map(S^1, M)$ instead* of $PM(p,q)$. Now we regard E as a functional of the form

$$E : Map(S^1, M) \longrightarrow \mathbf{R} \tag{8.4}$$

Now, for the case where M is the sphere S^k, Morse tackled the infinite dimensionality of $Map(S^1, M)$ by approximating the loops by geodesic polygons with n vertices $p_1, \ldots, p_n$. This makes $E(\gamma)$ a function of the n variables $p_1, \ldots, p_n$ instead of γ, i.e. $E = E(p_1, \ldots, p_n)$. If we denote the space of these $\{p_i\}$ by $Map_n(S^1, S^k)$, then $Map_n(S^1, S^k)$ is to be viewed as a finite dimensional subset of the infinite dimensional $Map(S^1, S^k)$. The idea then is to compute the topology of $Map_n(S^1, S^k)$ and to understand its dependence on n. This allows the passage to the limit $n \to \infty$ where one eventually deduces results such as the existence of an infinite number of closed geodesics on S^k and that E is a *perfect* Morse function; this latter property means that the

* $Map(S^1, M)$ is the space of loops on M, i.e. it is the space of continuous maps from S^1 to M.

Morse inequalities have become *equalities*. For more details cf. Klingenberg, [7].

Much later, the construction of a general Morse theory in infinite dimensions was achieved by Palais, Smale and many others cf. Palais, [8, 9, 10]; still more recently Floer, [11, 12], and Taubes, [13] have successfully tackled infinite dimensional problems which are outside the scope of the Palais-Smale framework. These are problems in Yang-Mills gauge theories but have consequences far outside theoretical physics: for example in Floer's case his work constructs a new class of highly interesting homology for 3-manifolds; for more information cf. Nash, [14].

§9. Classification of singular points of functions and Lie algebras

We come now to a most interesting topic: namely the *classification* of singular points of functions. There is a remarkable correspondence between the classification of singularities of functions and that of simple Lie algebras. There is no space to do justice to it here but some salient features can be mentioned.

Let f be a function with possibly degenerate critical points, with the multiplicity of a critical point being labelled by μ. Now f belongs to the infinite dimensional space $\mathcal{F}$ of functions $\mathcal{F} = Map(M, \mathbf{R})$, say, and, from the abstract viewpoint, the classification of the singular points corresponds to the finding of the orbits of the action of the group $Diff\,(M)$ of diffeomorphisms of M on $\mathcal{F}$.

In practice what happens is that one learns that functions may be transformed, by elements of $Diff\,(M)$, into certain polynomials known as *normal forms*.

The basic idea is to build up a picture of the functions as a subset of $\mathcal{F}$. So first one considers 1 parameter families of functions in $\mathcal{F}$ and analyses their possible singular points, then one considers 2 parameter families and so on.

For example, if $\dim M = n$, then *near* a singular point all 1 parameter families of functions are equivalent under $Diff\,(M)$ (i.e. after a suitable change of variables) to the *normal form*

$$f(x) = x_n^3 + \lambda x_n + q(x) \qquad (9.1)$$

where λ is the parameter and $q(x)$ is a non-degenerate quadratic form in the remaining variables given by

$$q(x) = -x_1^2 - x_2^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_{n-1}^2.$$

In 1 dimension this becomes simply

$$f(x) = x^3 + \lambda x, \qquad (9.2)$$

where such a result is not so hard to prove. If we have 2 parameters λ_1 and λ_2 then the normal form is

$$f(x) = x_n^4 + \lambda_1 x_n^2 + \lambda_2 x_n + q(x) \qquad (9.3)$$

and more generally for k parameters the normal form is

$$f(x) = x_n^{k+1} + \lambda_1 x_n^{k-1} + \lambda_2 x_n^{k-2} + \cdots + \lambda_{k-1} x_n + \lambda_k + q(x) \qquad (9.4)$$

The polynomial

$$x_n^{k+1} + \lambda_1 x_n^{k-1} + \lambda_2 x_n^{k-2} + \cdots + \lambda_{k-1} x_n + \lambda_k \qquad (9.5)$$

that emerges here is recognizable to Lie group experts as being isomorphic to the orbit space of the reflection group known as the Weyl group A_k for the simple Lie algebra $su(k+1)$.

There are also normal forms corresponding to the Weyl groups $D_k \simeq so(2k)$ and the exceptional set E_6, E_7 and E_8 for the exceptional algebras. Thus we have the whole of the so called A, D, E series, [15].

Example *Manifolds with boundary*

If M is a manifold with a non-empty boundary ∂M then $\mathcal{F} = Map(M, \mathbf{R})$ now can contain functions whose singular or critical points are on ∂M itself. These functions turn out, [16], to have normal forms which correspond to the Weyl groups B_k, C_k and F_4 i.e. to the remaining simple Lie algebras $so(2k+1)$, $sp(k)$ and F_4 respectively. Notice, however, that there is precisely one simple Lie algebra missing from the classification above—it is the last exceptional algebra G_2—it too can play a rôle cf. [17].

The indices on the various series A, D, E etc. label the multiplicity of the degenerate singularities of the family, for example all the most degenerate singularities with normal form A_k clearly have the same multiplicity k. Hence the index labels the $\mu =$ constant strata inside $\mathcal{F}$, and, since the value of μ gives the number of parameters of the family, this value of μ also is equal to the codimension of this stratum inside $\mathcal{F}$.

All the classifications above describe singularities which are called *simple*: small perturbations bifurcate them into only finite numbers of new singularities. There are also those which associate a continuum to the singularity: one then says that the singularity has moduli. These are the complement to the discrete series just discussed, i.e. all simple singularities occur in the lists given above.

§11. Applications of dynamical systems

We now give an idea of how diverse the subject is by mentioning some of the problems where ideas from dynamical systems can be applied.

Morse theory provides us with many examples and they are impressive and widespread; a few notable examples are the proof by Morse, [6], that there exist infinitely many geodesics joining a pair of points on a sphere S^n endowed with any Riemannian metric, Bott's proof of his celebrated periodicity theorems on the homotopy of Lie groups, [18], Milnor's construction, [19], of the first exotic spheres, and the proof by Smale of the Poincaré conjecture for $\dim M \geq 5$, [20].

Morse theory has also found a variety of applications in physics; this is not too surprising in view of the central position occupied by the variational principle in both classical and quantum physics. Some of these are described in Nash and Sen, [21].

Gradient dynamical systems were used by Thom, [21, 22, 23], in his work on what is now called *Catastrophe Theory*. Thom took the system

$$\frac{d\gamma(t)}{dt} = \operatorname{grad} V(\gamma(t)) \tag{10.1}$$

where V is a potential function. Next, for families of such V containing up to four parameters, Thom classified the possible critical points into seven types known as the seven elementary catastrophes; he then proposed to use these dynamical systems as models for the behaviour of a large class of physical, chemical and biological systems. In many of these cases the models are not at all adequate; nevertheless, there are some successes. On the mathematical side the classification into seven categories misses some singularities when one has three and four parameter families, cf. part II of the books under review; the seminal nature of Thom's work is clear though, as it is the beginning of the classification theory for singularities.

A vast body of the theory of dynamical systems concerns *Hamiltonian systems*. These of course have their origin in ordinary dynamics but exist now in a much wider context. To have a Hamiltonian system one needs to satisfy some requirements: M must must be even dimensional and must possess a closed non-degenerate 2-form ω known as a symplectic form; a Hamiltonian function

$$H : M \longrightarrow \mathbf{R} \tag{10.2}$$

then provides a vector field V on M via the equation

$$i(V)\,\omega = dH \tag{10.3}$$

where $i(V)$ denotes contraction, or interior product, with the vector V. It is easy to check that H is conserved along the orbits of V and this corresponds to the conservation of energy in the physical cases. The perturbation theory of these systems underwent an enormous development with the work particularly of Kolmogorov, Arnold and Moser resulting in what is now called KAM theory.

The blossoming of ergodic theory also owes some debts here. Ergodic theory originates largely in nineteenth century studies in the kinetic theory of gases. However it has now been axiomatized, expanded, refined and reformulated so that it has links with many parts of mathematics as well as retaining some with physics. Some dynamical systems exhibit ergodic behaviour, a notable class of examples being provided by *geodesic flow* on surfaces of constant negative curvature. This involves too the study of the flows by a discrete encoding known as symbolic dynamics, use of one dimensional interval maps, the zeta functions of Ruelle, the Patterson measure and so on, cf. [25]. Classical and quantum chaos, and the distinction between the two, are also studied in this context.

The last application that we shall mention is that of Vasil'ev to knot theory, [26, 27]. Vasil'ev's work constitutes a big step forward in knot theory but should also be regarded as a big step forward in the tackling of global problems in singularity theory as his methods are not limited just to knot theory.

Vasil'ev constructs a huge new class of knot invariants and we shall now give a sketch of what is involved.

A knot is a smooth embedding of a circle into $\mathbf{R}^3$. Thus a knot gives a map

$$f : S^1 \longrightarrow \mathbf{R}^3, \tag{10.4}$$

so that f belongs to the space $\mathcal{F}$ where $\mathcal{F} = Map(S^1, \mathbf{R}^3)$. Not all elements of $\mathcal{F}$ give knots, since a knot map f is not allowed to self-intersect or be singular. Let Σ be the subspace of $\mathcal{F}$ which contains either self-intersecting or singular maps. Then the subspace of knots is the *complement*

$$\mathcal{F} - \Sigma \tag{10.5}$$

Now any element of Σ can be made smooth by a simple one parameter deformation, hence Σ is a *hypersurface* in $\mathcal{F}$ and is known as the *discriminant.* As the discriminant Σ wanders through $\mathcal{F}$ it skirts along the edge of the complement $\mathcal{F} - \Sigma$ and divides it into many different connected components. Clearly knots in the same connected component can be deformed into each other and so are equivalent (or isotopic).

Now any knot *invariant* is, by the previous sentence, a function which is *constant* on each connected component of $\mathcal{F} - \Sigma$. Hence the task of constructing all (numerical) knot invariants is the same as finding all functions on $\mathcal{F} - \Sigma$ which are constant on each connected component. But topology tells us at once that this is just the 0-cohomology of $\mathcal{F} - \Sigma$. In other words,

$$H^0(\mathcal{F} - \Sigma) = \text{the space of knot invariants.} \tag{10.6}$$

Vasil'ev, [27], provides a method for computing most, and possibly all, of $H^0(\mathcal{F} - \Sigma)$.

Because of the immense importance of this breakthrough we give a brief summary of the steps involved in the construction of [27]. Vasil'ev deals with the infinite dimensionality of $\mathcal{F}$ by approximating its elements by trigonometric polynomials of degree n giving a finite dimensional space $\mathcal{F}^n$ of dimension $3n$. But $\mathcal{F}^n$ is clearly contractible, so Alexander duality gives us

$$H^i(\mathcal{F}^n - \Sigma) \simeq H_{3n-i-1}(\mathcal{F}^n \cap \Sigma). \tag{10.7}$$

Hence the cohomology of the knot space $\mathcal{F}^n - \Sigma$ is computable from the homology of the subsets Σ_n of the discriminant Σ given by

$$\Sigma_n = \mathcal{F}^n \cap \Sigma \tag{10.8}$$

The singularities present in Σ_n give a stratification* of Σ allowing the computation of its homology. This stratification of Σ provides a filtration from which a standard spectral sequence then flows. The spectral sequence is roughly an algebro-topological analogue of a Taylor series and, as for a Taylor series, one must demonstrate convergence and absence of remainder in the limit $n \to \infty$.

The construction then provides us with a hierarchy of knot invariants V_n—the Vasil'ev invariants—which looks like

$$V_0 \subset V_1 \subset \cdots V_n \subset \cdots \subset H^0(\mathcal{F} - \Sigma) \tag{10.9}$$

where each V_n is finite dimensional and already completely constructed for $0 \leq n \leq 8$.

The convergence of the spectral sequence has been conjectured by Vasil'ev and, if proved, would mean that the Vasil'ev invariants distinguish *any* two inequivalent knots. It is already known, Birman [28], that they distinguish more knots than the other well known knot polynomials, namely the Alexander, Jones,

* This is the great advantage of working with Σ_n instead of with $\mathcal{F}^n - \Sigma$; this latter space contains only smooth maps and provides us with no natural way of constructing a stratification.

Homfly and Kaufmann polynomials. Kontsevich, [29], has given a 'universal integral' which associates to each knot an element of an algebra of 'Feynman diagrams' (cf. also Bar-Natan, [30]), from which one calculates the Vasil'ev invariants for the knot; this work uses results of Knizhnik and Zamolodchikov, [31], from the physics literature.

§11. Conclusion

We now wind things up with a return to the books under review. These two volumes certainly cover a wide range of material on singularity theory and, although, they belong to a section of the encyclopaedia on dynamical systems there is much material here for anyone with an interest in singularity theory, not just those who work on dynamical systems.

Part I begins with basic notions concerning singular smooth maps and introduces normal forms. It then moves on to complex functions and Picard-Lefschetz theory to which it devotes a considerable amount of space—about a hundred pages. Next comes a chapter on singularities of smooth maps in general; and the final chapter is on the global singularity theory relevant for maps and deals with the subject of Thom polynomials and related matters.

Part II is a mixture of applications and material on classification of singularities. However part II is largely intended to be independent of part I. The first chapter deals with the singularities and normal forms for functions on a manifold with boundary. This is followed by a chapter on applications including a section on catastrophe theory. Then one moves on to singularities on the boundaries of function spaces. Chapter four is about monodromy and Picard-Lefschetz theory and contains a remarkable early monodromy result of Newton from his *Principia*: For an ellipse, with origin at a focus, this is that the area swept out in time t by the radius vector r is a transcendental function of the tangent of the angle between r and the x-axis. The book then finishes with a chapter on deformation of real singularities and their lacunae, including a discussion of the use of computer algorithms to obtain some of the results.

The style of both volumes is definitely mathematical rather

than physical and so some physicists will find the text heavy going. Cross referencing within the text is done fairly well; and this encyclopaedia does not indulge in the annoying practice of referring one to equations present in other volumes as if one had the desk space, or the money, to have them all at hand; readers of Dieudonné's admirable six volume *Treatise on Analysis* may remember that it continually suffers from that drawback. The bibliography is very good and extremely large in both cases. It is interesting to note, however, that Vasil'ev's paper [27] is in the bibliography but is, unfortunately, not discussed; a comparison of the dates of the Russian original and the English translation is consistent with the fact that the reference entered only at the translation stage.

The price of both books is DM 141 which is about 58 punts and is a little on the expensive side for books of 250 odd pages, though they are produced up to the usual high standards of Springer. Price notwithstanding, I do recommend them both particularly as library purchases, and because they can be read independently of the other volumes of the encyclopaedia.

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Book Review

p-adic Numbers. An Introduction

Universitext Series

Fernando Q. Gouvêa

Springer-Verlag 1993, 284 pp.

ISBN 3-540-56844-1

Price DM 58.00.

Reviewed by Brendan Goldsmith

The role of p-adic numbers and p-adic analysis has become central in many areas of mathematics in the century or so since their introduction by Hensel. Despite this, they are still not part of the 'toolkit' of many mathematicians and are often not talked about (or perhaps just given as a somewhat bizarre example) at undergraduate level. One of the objectives of this book is to make the subject more accessible to an undergraduate audience, 'taking its readers for a short promenade along the p-adic path'. In this it succeeds admirably. I have used parts of the book as background material at a beginning graduate seminar and found that students had little difficulty with it.

There are several standard approaches to p-adic theory with the most popular being either via valuation theory or via absolute values (the two are, of course, intimately related). Here the author has chosen the latter and proceeds to completions and Hensel's Lemma. (Incidentally, there is a nice application of this lemma to the determination of p-adic roots of unity). The initial 'algebraic' approach finishes with some interesting aspects of 'local-global' arguments and applications of the Hasse-Minkowski Theorem, although, not surprisingly, no proof of the theorem is given.

After the introductory appetizer, we are treated to some elementary p-adic analysis focusing on sequences and series. There

is no attempt to develop a p-adic theory of integration but suitable references are given for such a development. Having laid this analytical ground work, the book then switches to p-adic vector spaces and field extensions, finishing off with a nice discussion of normed vector spaces over complete valued fields and showing that the algebraic closure, $\overline{Q}_p$, of the field of p-adic numbers is not complete with respect to the (extended) p-adic absolute value. This leads, rather naturally to a discussion of the completion C_p.

A delightful feature of the book is the large number (329, to be exact!) of exercises which form an integral part of the text. Equally useful is the section 'Hints and Comments on the Problems' where hints of varying degrees of detail are given! The book concludes with a brief discussion on the literature, some comments on software for doing p-adic calculations and a sensible bibliography.

I found the book a pleasure to read and while one might occasionally wish to delve deeper into some topics, one must accept that the author's 'aim is sightseeing, rather than a scientific expedition'. However, this must not be interpreted incorrectly; this is a serious book looking at important mathematics and definitely worthy of a place in the prestigious Springer Universitext series.

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Book Review

Bifurcation and Chaos:
Analysis, Algorithms, Applications

International Series of Numerical Mathematics, Vol.97

Ed. by R. Seydel, F. W. Schneider, T. Küpper, H. Troger

Birkhäuser Verlag (Basel) 1991, 388 pp.

ISBN 3-7643-2593-3

Price SF 118.00.

Reviewed by Eugene Gath

The title of this book was the theme of a conference held in Würzburg, Germany in August 1990, and its contents are the conference proceedings. The title gives the broad thrust of most of the 49 articles, but the variety and breadth of nonlinear phenomena to be found here covers a wide area of science and engineering, including what we normally think of as the nonphysical sciences, chemistry and biology. This diversity makes the book attractive from the perspective of a scientist. The applications of nonlinear dynamics considered here include gas combustion, chemical oscillators, Rayleigh-Bénard convection, elasticity, the rolling of a ship, the stability of a rotating satellite, robot control, climatic modelling, pattern formation, hydrostatics, electronic circuitry and much more! If one ever needs reassurance about the relevance of (applied) mathematics within the world of science and engineering, just browse through this book.

There are several articles which are purely mathematical, and many more which deal with numerical analysis or computational aspects of problems. The attraction for the applied mathematician is the range of different mathematical models employed. The use of nonlinear ordinary and partial differential equations still dominates, and most of the standard examples like the Duffing, reaction-diffusion and nonlinear Laplace equation appear in

various guises. The treatment of many problems combines analytical, numerical and experimental approaches. The language of nonlinear dynamics is assumed throughout, but for most articles, a knowledge at the level of an introductory text, such as Wiggins [1], should suffice.

Without giving undue emphasis to any particular article, for the purpose of illustration of the complexity of some of the models used, I will mention a model, discussed in several articles, namely a nonlinear chemical oscillator called the *Brusselator*, [2], (which has nothing whatsoever to do with EU funding!). This describes a chemical conversion $A + B \to D + E$ via the intermediates X and Y. The latter undergo a trimolecular autocatalysis $2X + Y \to 3X$:

$$\begin{aligned} A &\xrightarrow{k_1} X \\ X + B &\xrightarrow{k_2} Y + D \\ 2X + Y &\xrightarrow{k_3} 3X \\ X &\xrightarrow{k_4} E \end{aligned}$$

There is an influx of species A and B into the reaction

$$\begin{aligned} A^{in} &= A_0^{in} \\ B^{in} &= B_0^{in}(1 + \alpha \cos \omega t) \end{aligned}$$

that is, there is a constant influx of A while the influx of B varies sinusoidally. The system of differential equations describing this reaction is:

$$\begin{aligned} \frac{dA}{dt} &= -k_1 A + k_f(A^{in} - A) \\ \frac{dB}{dt} &= -k_2 BX + k_f(B^{in} - B) \\ \frac{dX}{dt} &= k_1 A - k_2 BX + k_3 X^2 Y - k_4 X - k_f X \\ \frac{dY}{dt} &= k_2 BX - k_3 X^2 Y - k_f Y \end{aligned}$$

where k_f (a control parameter) is the flow rate through the "Continuous Flow Stirred Tank Reactor" described in [3]. The authors of that article then add a 1% Gaussian white noise to each variable and they assign physically realistic values to the parameters. The result of a Runge-Kutta integration of the model demonstrates that the transition between high and low amplitude oscillations is characterized by a secondary periodic Hopf bifurcation, and that the extent of this transition varies with the noise frequency, in qualitative agreement with the sharp pulses of chemiluminescence observed in a luminol oscillator experiment.

The articles in this text, with a few exceptions (primarily those not written with TEX), knit together quite well visually, allowing for the difficulties in imposing uniformity of format for conference proceedings. However, some of the graphics would benefit from the addition of colour to this monochrome text. Another minor criticism relates to the contents, which are given only in alphabetic order, by name of the first author. The editors justify this on the basis that many papers contain overlapping themes, but an extra two or three pages given to a classification by subject or theme would have been a useful addition.

This book undoubtedly deserves a place in a mathematics library, but should also find a place on the shelves of the practitioners of differential equations, numerical analysis and dynamical systems.

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INSTRUCTIONS TO AUTHORS

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